The Proof of the Riemann Hypothesis

Zuhair Alahmad

Faculty of Science - Damascus University

Hechamalahmad@yahoo.com

ABSTRACT

The problem of finding the zeros of functions is one of the important issues in mathematics, and I would not be exaggerating if I said that all of mathematics is based on such problems. Here, curiosity struck me to understand the mechanism or the secret behind these functions. I never expected that this curiosity would lead me to encounter an important function like the Riemann zeta function, starting from the Taylor and Maclaurin series, which at least enabled me to find a function that links the point belonging to a certain domain and the values of this domain with an exponential function, as demonstrated in the proof.

In conclusion, I believe that if this function cannot find the zeros of the Riemann zeta function, it will at least allow us to look at the zeta function from another perspective that is easier to deal with.

Before starting with the solution for the Riemann zeta function, we want to introduce a function known to everyone, but whose mathematical logic we may not fully understand

Let us consider the infinite series: $f_n(x)$ where $n \to \infty$

$$f_n(x) = a^n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots \quad (1)$$

We observe that the value of the function $f_n(x)$ at any point (t) is :

$$f_n(t) = a_n t^n + a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \dots \quad (2)$$

However, we can rewrite each $t \in \mathbb{R}$ in the form $t = x + b$ where $x = x_0$

And according to the Taylar series, we study the variation of the function $f_n(t)$ around the point x. from that we conclude:

$$f(t) = f(x) + \frac{f'(x)}{1!}(t-x) + \frac{f''(x)}{2!}(t-x)^2 + \dots \quad (3)$$

by substituting (t) with the value x+b, we find:

$$f(x+b) = f(x) + \frac{f'(x)}{1!}(x+b-x) + \frac{f''(x)}{2!}(x+b-x)^2 + \dots \quad (4)$$

$$f(x+b) = f(x) + b\frac{f'(x)}{1!} + b^2\frac{f''(x)}{2!} + b^3\frac{f^{(3)}(x)}{3!} + \ldots \tag{5}$$

now, let's subtract $f(x)$ from both sides of equation (5)

$$f(x+b) - f(x) = b\frac{f'(x)}{1!} + b^2\frac{f''(x)}{2!} + b^3\frac{f^{(3)}(x)}{3!} + \ldots \tag{6}$$

we divide both sides of equation (6) by b

$$\frac{f(x+b)-f(x)}{b} = f'(x) + b\frac{f''(x)}{2!} + b^2\frac{f^{(3)}(x)}{3!} + \ldots \tag{7}$$

We observe in equation (7) that $\frac{f(x+b)-f(x)}{b}$ represents the value of the derivative at the point c. where c ∈]x,x+b[This is according to the finite increments theorem.

Thus, we can rewrite equation (7) in the following form:

$$f'(c) = f'(x) + b\frac{f''(x)}{2!} + b^2\frac{f^{(3)}(x)}{3!} + \ldots \tag{8}$$

At the point x=x_0=0 , we write equation (8) in the following form:

$$f'(c) = f'(0) + b\frac{f''(0)}{2!} + b^2\frac{f^{(3)}(0)}{3!} + \ldots \tag{9}$$

Thus, we can rewrite equation (9) in the following form:

$$f'(c) = f'(0) + \frac{f''(0)}{1!}.c + \frac{f^{(3)}(0)}{2!}.c^2 + \ldots \tag{10}$$

By comparing equations (9) and (10), we conclude that:

$$\begin{aligned} &1=1 \\ &c = \frac{1}{2}b \\ &c^2 = \frac{1}{3}b^2 \\ &c^3 = \frac{1}{4}b^3 \\ &. \\ &. \end{aligned} \tag{11}$$

.

By making some adjustments, we can rewrite equation (11) in the following form:

1=1

C= $\frac{b}{2}$

$\frac{c^2}{2!} = \frac{b^2}{3!}$ **(12)**

$\frac{c^3}{3!} = \frac{b^3}{4!}$

.

.

.

By adding both sides of equation (12), we conclude:

$$\mathbf{1+ c+\frac{c^2}{2!}+\frac{c^2}{3!}+\ldots = 1 + \frac{b}{2!}+\frac{b^2}{3!}+\frac{b^3}{4!}} \quad \textbf{(13)}$$

we observe on the left side of equation (13) that (1+$\mathbf{c+\frac{c^2}{2!}+\frac{c^3}{3!}+\ldots}$) is the expansion of the function (e^c).

$$\mathbf{=> \quad e^c == 1 + \frac{b}{2!}+\frac{b^2}{3!}+\frac{b^3}{4!}} \quad \textbf{(14)}$$

We multiply both sides of equation (14) by (b) , then add +1 to both sides. We obtain the equation in the following form

$$\mathbf{1+ be^c= 1 +b +\frac{b^2}{2!} + \frac{b^3}{3!}+\ldots} \quad \textbf{(15)}$$

The right side of equation (15) is the expansion of the function (e^b).therewhere we conclude the following:

$$\mathbf{1 +be^c =e^b \quad => \quad e^c =\frac{e^b-1}{b}} \quad \textbf{(16)}$$

And this is the mathematical logic behind the limit of the function ($\frac{e^x-1}{x}$) as ($x\to 0$)

$$\lim_{x\to 0}\frac{e^x-1}{x} =1 \quad ; \quad 0 < c < x$$

"Returning to the Riemann zeta function, which is written in the following form:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots \tag{17}$$

We will write the function $f(z)$ in the following form:

$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z + \frac{z^2}{2^s} + \frac{z^3}{3^s} + \ldots \tag{18}$$

$$f(0) = 0 \qquad \& \qquad f(1) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \ldots = \zeta(s) \tag{19}$$

Using the same method as in the beginning of the solution, we obtain the equation:

$$e^z = \frac{e^b - 1}{b} \tag{20}$$

Where (z) is a complex number and is written in the form ($z=x+yi$) . We choose($b=1$)

Then we conclude that: $e^{x+yi} = e-1$ (21)

The task is to determine the values of x and y in equation (21)

$$e^x[\cos y + i \sin y] = e-1 \quad \Rightarrow \quad e^x \cos y + i e^x \sin y = e-1 \tag{22}$$

$\Rightarrow e^x \sin y = 0 \quad ; \quad e^x > 0 \qquad \forall \; x \in \mathbb{R}$

$\Rightarrow \sin y = 0 \quad \Rightarrow \quad y = n\pi \qquad ; \quad n \in \mathbb{Z}$ (23)

Substitute the value of (y) into equation (22) :

$e^x \cos(n\pi) = e-1 \qquad ; \cos(n\pi) = (-1)^n$

$\Rightarrow e^x (-1)^n = e-1$

Since (e-1) is strictly positive, then ($n=2m$)

$\Rightarrow e^x = e-1 \quad \Rightarrow \quad x = \ln(e-1)$ (24)

Thus, we obtain the complex number $z = x+yi$

$Z = \ln(e-1) + 2m\pi i \qquad ; \; m \in \mathbb{Z}$ (25)

To clarify further on deriving the equation for ($e^z = \frac{e^b - 1}{1}$), and based on the initial steps of the solution, and considering that b=1, we write:

$$\frac{f(z+1)-f(z)}{z+1-z} = f'(z) + \frac{f''(z)}{2!} + \frac{f^{(3)}(z)}{3!} + \ldots \qquad \textbf{(26)}$$

But:

$$\left.\frac{f(z+1)-f(z)}{z+1-z}\right|_{Z=0} = f'(0) + \frac{f''(0)}{2!} + \frac{f^{(3)}(0)}{3!} + \ldots = \zeta(s) \qquad \textbf{(27)}$$

We can also express $f'(z)$ in the form:

$$f'(z) = f'(0) + f''(0).z + \frac{f^{(3)}(0)}{2!}.z^2 + \ldots \qquad \textbf{(28)}$$

By comparing Equation 27 with Equation 28, we conclude that:

$$f'(z) = \zeta(s) \quad \Longrightarrow \quad e^z = e-1 \qquad \textbf{(29)}$$

Thus, (z=ln$(e-1) + 2m\pi i$) that we obtained in solving Equation ($e^z = e - 1$) is the set of zeros of the Riemann zeta function.

References:

1- E. C. Titchmarsh (1986). The Theory of the Riemann Zeta Function, Second revised (Heath-Brown) edition. Oxford University Press.
2- Kawagoe K., Wakayama M.,Yamasaki Y. The q-Analogues of the Riemann zeta, Dirichlet L-functions, and a crystal zeta-function. Forum Math, 2008, vol. 1, p. 126.
3- Guy, Richard K., Unsolved problems in number theory, (3rd edition). Springer-Verlag (2004).
4- From Euclid to Present: A Collection of Proofs regarding the Infinitude of Primes, Lindsey Harrison, December 14, 2013.
5- (Dun) W. Dunham,"Journey Through Genius: The Great Theorems of Mathematics," Penguin Books, John Wiley and Sons, 1990.