GROUP INVERSES AND GENERALIZED GROUP INVERSES OF ANTI-TRIANGULAR MATRICES

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Abstract. We provide representations for the group inverse and generalized group inverse of an anti-triangular operator matrix of the form $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ under the conditions $ca = c$ or $ab = b$. Specifically, we present the weak group inverse for these types of anti-triangular block operator matrices.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra equipped with an involution denoted by $*$. An element $a \in \mathcal{A}$ is said to have a group inverse if there exists an element $x \in \mathcal{A}$ such that

$$xa^2 = a, ax^2 = x, \text{and } ax = xa.$$

If such an x exists, it is unique and is denoted by $a^{\#}$, referred to as the group inverse of a. Consider $\mathbb{C}^{n\times n}$, the Banach algebra of all $n \times n$ complex matrices, where the involution is given by the conjugate transpose. It is clear that a matrix $A \in \mathbb{C}^{n\times n}$ has a group inverse if and only if $rank(A) = rank(A^2)$.

The involution $*$ is proper if $x^*x = 0 \Longrightarrow x = 0$ for any $x \in \mathcal{A}$, e.g., in a C^*-algebra, the involution is always proper. An element a in a Banach algebra with proper involution $*$ has weak group inverse provided that there exist $x \in \mathcal{A}$ and $k \in \mathbb{N}$ such that

$$x = ax^2, (a^*a^2x)^* = a^*a^2x, a^k = xa^{k+1}.$$

If such x exists, it is unique, and denote it by $a^{\textcircled{w}}$. As is well known, $A \in \mathbb{C}^{n\times n}$ has weak group inverse X if and only if it satisfies the equations

$$AX^2 = X, AX = A^{\textcircled{D}}A.$$

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Here, A has core-EP inverse $A^{\textcircled{D}}$ if it satisfies the equations

$$A(A^{\textcircled{D}})^2 = A^{\textcircled{D}}, (AA^{\textcircled{D}})^* = AA^{\textcircled{D}}, A^{\textcircled{D}}A^{k+1} = A^k,$$

where $k = ind(A)$ is the Drazin index of A. The group inverse and weak group inverse are valuable tool in functional analysis, particularly in the study of linear operators and their properties. It allows for the analysis of elements that may not possess a classical inverse while still retaining useful algebraic properties. We refer the reader for group inverse and weak group inverse in [1, 3, 8, 9, 10, 13, 14, 15, 16, 18, 21, 22].

An element $a \in \mathcal{A}$ has generalized Drazin inverse if there exists $x \in \mathcal{A}$ such that

$$ax^2 = x, ax = xa, a - xa^2 \in \mathcal{A}^{qnil}.$$

Such x is unique, if exists, and denote it by a^d. Here, $\mathcal{A}^{qnil} = \{x \in \mathcal{A} \mid 1+\lambda x \in \mathcal{A}$ is invertible$\}$. Evidently, $x \in \mathcal{A}^{qnil}$ if and only if $\lim\limits_{n\to\infty} ||x^n||^{\frac{1}{n}} = 0$. Replacing $\mathcal{A}^{qnil}$ of quasinilpotents by the set of all nilpotents, we call the unique x the Drazin inverse of a, and denote it by a^D. The Drazin and generalized Drazin inverses play important roles in ring and matrix theory (see [2, 4, 10, 17, 19, 23]).

An element $a \in \mathcal{A}$ has generalized group inverse if there exists $x \in \mathcal{A}$ such that

$$x = ax^2, (a^*a^2x)^* = a^*a^2x, \lim_{n\to\infty} ||a^n - xa^{n+1}||^{\frac{1}{n}} = 0.$$

Such x is unique if it exists. We call the preceding x the generalized group inverse of a, and denote it by $a^{\textcircled{g}}$. The generalized group inverse is a natural generalization of weak group inverse. For a square complex matrix, they coincide with each other. We list some characterizations of generalized group inverse.

Theorem 1.1. *(see [6, Theorem 2.2, Theorem 4.1 and Theorem 5.1]) Let $\mathcal{A}$ be a Banach *-algebra, and let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^{\textcircled{g}}$.
(2) *There exist $x, y \in \mathcal{A}$ such that*

$$a = x + y, x^*y = yx = 0, x \in \mathcal{A}^{\#}, y \in \mathcal{A}^{qnil}.$$

(3) $a \in \mathcal{A}^d$ and there exists $x \in \mathcal{A}$ such that

$$x = ax^2, (a^d)^*a^2x = (a^d)^*a, \lim_{n\to\infty} ||a^n - xa^{n+1}||^{\frac{1}{n}} = 0.$$

(4) *There exists an idempotent* $p \in \mathcal{A}$ *such that*

$$a + p \in \mathcal{A} \text{ is invertible}, (a^*ap)^* = a^*ap \text{ and } pa = pap \in \mathcal{A}^{qnil}.$$

The generalized inverses of an anti-triangular matrix are very tools in the context of differential equations. It is hard to find the representations of generalized inverses for an anti-triangular matrix. Let $M = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$, where $ca = c$ or $ab = b$. Castro-González and Dopazo considered the Drazin inverse of M with $a = b = 1$ (see [4]). Bu et al. studied the Drazin inverse of M with $a = b = a^2$ (see [2]). Liu studied the group inverse of M with additional condition $a^2 = a$ for complex matrices (see [12]). In [23], Zou discussed the same problem for an idempotent a in a ring. In [3], Cao et al. extended Liu's results and investigate the group inverse of M over a right Ore domain.

The aim of this paper is to present representations for the group inverse and generalized group inverse of an anti-triangular operator matrix M. In Section 2, we explore the generalized Drazin invertibility of M and derive the representation of its group inverse. Section 3 focuses on establishing the existence and representation of the group inverse of M. Finally, in Section 4, we provide the weak group inverse for these types of anti-triangular block operator matrices.

Throughout the paper, all Banach algebras are complex with a proper involution $*$. We use $\mathcal{A}^{\#}, \mathcal{A}^{D}, \mathcal{A}^{d}, \mathcal{A}^{Ⓦ}$ and $\mathcal{A}^{ⓖ}$ to stand for the sets of all group invertible, Drazin invertible, g-Drazin invertible, weak group invertible and generalized group invertible elements in $\mathcal{A}$, respectively.

2. GENERALIZED DRAZIN AND GROUP INVERSES

To facilitate our discussion, we will now examine the existence of the generalized Drazin inverse of M. The following lemma is essential for this investigation.

Lemma 2.1. *Let* $\mathcal{A}$ *be a Banach algebra and* $a \in \mathcal{A}^{-1}, b \in \mathcal{A}^{qnil}$. *Then If* $ba = b$, *then the equation* $xa + 1 = xbx$ *has a solution* x *such that* $x, a - bx \in \mathcal{A}^{-1}, bx \in \mathcal{A}^{qnil}$.

Proof. Let $x = \sum\limits_{i=0}^{\infty} c_i a^{\alpha_i} b^i$, where $c_i \in \mathbb{C}, \alpha_i \in \mathbb{Z}$. Since $ba = b$, we have

$$\begin{aligned} xbx &= [\sum_{i=0}^{\infty} c_i a^{\alpha_i} b^i] b [\sum_{i=0}^{\infty} c_i a^{\alpha_i} b^i] = [\sum_{i=0}^{\infty} c_i a^{\alpha_i} b^i][\sum_{i=0}^{\infty} c_i (b a^{\alpha_i}) b^i] \\ &= [\sum_{i=0}^{\infty} c_i a^{\alpha_i} b^i][\sum_{i=0}^{\infty} c_i b^{i+1}] = \sum_{k=0}^{\infty} d_k b^{k+1}, \end{aligned}$$

where $d_k=\sum\limits_{i=0}^{k}c_ic_{k-i}a^{\alpha_i}$. Also we have

$$\begin{array}{lll} xa+1 & = & [\sum\limits_{i=0}^{\infty}c_ia^{\alpha_i}b^i]a+1\\ & = & 1+c_0a^{\alpha_0+1}+\sum\limits_{i=1}^{\infty}c_ia^{\alpha_i}(b^ia)\\ & = & 1+c_0a^{\alpha_0+1}+\sum\limits_{k=0}^{\infty}c_{k+1}a^{\alpha_{k+1}}b^{k+1}. \end{array}$$

Then $xa+1=xbx$ implies that

$$1+c_0a^{\alpha_0+1}=0, d_k=c_{k+1}a^{\alpha_{k+1}}.$$

That is,

$$\begin{array}{c} 1+c_0a^{\alpha_0+1}=0,\\ c_0^2a^{\alpha_0}=c_1a^{\alpha_1},\\ c_0c_1a^{\alpha_0}+c_1c_0a^{\alpha_1}=c_2a^{\alpha_2},\\ c_0c_2a^{\alpha_0}+c_1c_1a^{\alpha_1}+c_2c_1a^{\alpha_2}=c_3a^{\alpha_3},\\ \vdots \end{array}$$

Then

$$\begin{array}{c} c_0=-1, c_1=1, c_2=-2, c_3=5, c_4=-14, \cdots,\\ c_{k+1}=\sum\limits_{i=0}^{k}c_ic_{k-i}, \alpha_k=-1. \end{array}$$

Let $\{C_n\}$ be the series of Catalan numbers, i.e., $C_0=1, C_{n+1}=C_0C_n+\cdots+C_nC_0 (n\geq 0)$. Then $c_0=-C_0, c_1=C_1$. By induction, we claim that $c_{2n}=-C_{2n}, c_{2n+1}=C_{2n+1}$. Hence, $|c_n|=C_n (n\geq 0)$. By using the asymptotic expression of the Catalan numbers C_n, we have

$$\lim\limits_{n\to\infty}C_n/\big(\frac{4^n}{\sqrt{\pi}n^{\frac{3}{2}}}\big)=1.$$

Therefore

$$\lim\limits_{n\to\infty}\sqrt[n]{|c_n|}=\lim\limits_{n\to\infty}\frac{4}{\big(\pi^{\frac{1}{2n}}(\sqrt[n]{n})^{\frac{3}{2}}\big)}=4.$$

As $b\in\mathcal{A}^{qnil}$, we see that $\lim\limits_{n\to\infty}\sqrt[n]{\parallel b^n\parallel}=0$; hence, $\lim\limits_{n\to\infty}\sqrt[n]{|c_n|\parallel b^n\parallel}=0$. This implies that $\sum\limits_{i=0}^{\infty}c_ib^i$ absolutely converges.

Thus, the equation $xa+1=xbx$ has a solution $x=\sum\limits_{i=0}^{\infty}c_ia^{-1}b^i$, where $c_0=-1, c_{k+1}=\sum\limits_{i=0}^{k}c_ic_{k-i}$. Moreover, we verify that

$$\begin{array}{c} x=-a^{-1}[1+(\sum\limits_{i=1}^{\infty}c_ib^i)b]\in \mathcal{A}^{-1},\\ bx=-ba^{-1}\sum\limits_{i=0}^{\infty}c_ib^i=-[\sum\limits_{i=0}^{\infty}c_ib^i]b\in \mathcal{A}^{qnil},\\ 1-(bx)a^{-1}=1-bx\in \mathcal{A}^{-1},\\ a-bx=a[1-a^{-1}(bx)]\in \mathcal{A}^{-1}. \end{array}$$

This completes the proof. $\square$

Lemma 2.2. *Let $\mathcal{A}$ be a Banach algebra and $M=\left(\begin{array}{cc} a & b\\ 1 & 0 \end{array}\right)$ with $a\in \mathcal{A}^{-1}, b\in \mathcal{A}^{qnil}$. If $ba=b$, then $M\in M_2(\mathcal{A})^d$.*

Proof. In view of Lemma 2.1, the equation $xa+1=xbx$ has a solution x such that $x, a-bx\in \mathcal{A}^{-1}, bx\in \mathcal{A}^{qnil}$. It is easy to verify that

$$M=\left(\begin{array}{cc} 1 & 0\\ -x & 1 \end{array}\right)\left(\begin{array}{cc} a-bx & b\\ 0 & xb \end{array}\right)\left(\begin{array}{cc} 1 & 0\\ x & 1 \end{array}\right).$$

Since $bx\in \mathcal{A}^{qnil}$, then so is xb by [11, Theorem 2.2]. Thus, $\left(\begin{array}{cc} a-bx & b\\ 0 & xb \end{array}\right)$ has g-Drazin inverse. Therefore M has g-Drazin inverse,a s asserted. $\square$

Lemma 2.3. *Let $\mathcal{A}$ be a Banach algebra and $M=\left(\begin{array}{cc} a & b\\ 1 & 0 \end{array}\right)$ with $a\in \mathcal{A}^{d}, b\in \mathcal{A}^{qnil}$. If $ba=b$, then $M\in M_2(\mathcal{A})^d$.*

Proof. Clearly, we have $M=P+Q$, where

$$P=\left(\begin{array}{cc} a^2a^d & aa^db\\ aa^d & 0 \end{array}\right), Q=\left(\begin{array}{cc} aa^{\pi} & a^{\pi}b\\ a^{\pi} & 0 \end{array}\right).$$

Claim 1. P has g-Drazin inverse.

$$\begin{array}{c} P=\left(\begin{array}{cc} a^2a^d & aa^dbaa^d\\ aa^d & 0 \end{array}\right)\in M_2(aa^d\mathcal{A}aa^d).\\ a^2a^d\in (aa^d\mathcal{A}aa^d)^{-1}, aa^dbaa^d\in (aa^d\mathcal{A}aa^d)^{qnil}. \end{array}$$

By virtue of Lemma 2.2, P has g-Drazin inverse.

Claim 2. Q has g-Drazin inverse.

$$\begin{aligned} Q &= \begin{pmatrix} aa^{\pi} & a^{\pi}b \\ a^{\pi} & 0 \end{pmatrix} = \begin{pmatrix} a^{\pi} & 0 \\ 0 & a^{\pi} \end{pmatrix} \begin{pmatrix} aa^{\pi} & a^{\pi}b \\ a^{\pi} & 0 \end{pmatrix}, \\ \begin{pmatrix} aa^{\pi} & 0 \\ a^{\pi} & 0 \end{pmatrix} &= \begin{pmatrix} aa^{\pi} & a^{\pi}b \\ a^{\pi} & 0 \end{pmatrix} \begin{pmatrix} a^{\pi} & 0 \\ 0 & a^{\pi} \end{pmatrix}. \end{aligned}$$

Then Q has g-Drazin inverse.

Since $PQ = 0$, it follows by [24, Lemma 4.1] that $M = P + Q$ has g-Drazin inverse, as asserted. □

Lemma 2.4. *Let $\mathcal{A}$ be a Banach algebra and $M = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$ with $a, b \in \mathcal{A}^d$. If $ba = b$, then $M \in M_2(\mathcal{A})^d$.*

Proof. Obviously, we have $M = P + Q$, where

$$P = \begin{pmatrix} abb^d & b^2b^d \\ bb^d & 0 \end{pmatrix}, Q = \begin{pmatrix} ab^{\pi} & bb^{\pi} \\ b^{\pi} & 0 \end{pmatrix}.$$

Claim 1. P has g-Drazin inverse.

$$\begin{aligned} \begin{pmatrix} abb^d & b^2b^d \\ bb^d & 0 \end{pmatrix} &= \begin{pmatrix} abb^d & b^2b^d \\ bb^d & 0 \end{pmatrix} \begin{pmatrix} bb^d & 0 \\ 0 & bb^d \end{pmatrix}, \\ \begin{pmatrix} bb^d & b^2b^d \\ bb^d & 0 \end{pmatrix} &= \begin{pmatrix} bb^d & 0 \\ 0 & bb^d \end{pmatrix} \begin{pmatrix} abb^d & b^2b^d \\ bb^d & 0 \end{pmatrix}. \end{aligned}$$

$$\begin{pmatrix} bb^d & b^2b^d \\ 1 & 0 \end{pmatrix}$$

Claim 2. Q has g-Drazin inverse.

It is easy to verify that

$$(bb^{\pi})(ab^{\pi}) = b^{\pi}(ba)b^{\pi} = bb^{\pi}.$$

Since $bb^{\pi} \in \mathcal{A}^{qnil}$.

Since $bb^dab^{\pi} = 0$ and $bb^da = bb^d \in \mathcal{A}^d$, it follows by [20, Lemma 2.2] that $ab^{\pi} \in \mathcal{A}^d$. In light of Lemma 2.3, $\begin{pmatrix} ab^{\pi} & bb^{\pi} \\ 1 & 0 \end{pmatrix}$ has g-Drazin inverse. By using Cline's formula (see [11, Theorem 2.2]), Q has g-Drazin inverse. Obviously, $abb^dab^{\pi} = abb^db^{\pi} = 0$ and $bb^dab^{\pi} = 0$. Since

$$PQ = \begin{pmatrix} abb^d & b^2b^d \\ bb^d & 0 \end{pmatrix} \begin{pmatrix} ab^{\pi} & bb^{\pi} \\ b^{\pi} & 0 \end{pmatrix} = 0,$$

it follows by [24, Lemma 4.1] that $M = P + Q$ has g-Drazin inverse. This completes the proof. □

Theorem 2.5. *Let $\mathcal{A}$ be a Banach algebra and $M = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ with $a, cb \in \mathcal{A}^d$. If $ca = c$, then $M \in M_2(\mathcal{A})^d$.*

Proof. Since $cb \in \mathcal{A}^d$, by using Cline's formula (see [11, Theorem 2.2]), $bc \in \mathcal{A}^d$. In light of Lemma 2.4, $\begin{pmatrix} a & bc \\ 1 & 0 \end{pmatrix}$ has g-Drazin inverse. One easily checks that

$$\begin{array}{rcl} M & = & \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}, \\ \begin{pmatrix} a & bc \\ 1 & 0 \end{pmatrix} & = & \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}. \end{array}$$

By using Cline's formula, we prove that M has g-Drazin inverse. □

Lemma 2.6. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^{\#}$.
(2) $a \in \mathcal{A}^d$ and $\mathcal{A}a = \mathcal{A}a^2$.
(3) $a \in \mathcal{A}^d$ and $a\mathcal{A} = a^2\mathcal{A}$.

Proof. (1) ⇒ (2) This direction is obvious.

(2) ⇒ (1) Write $a = xa^2$ for some $x \in \mathcal{A}$. Then we have

$$\begin{array}{rcl} a - a^2a^d & = & x^na^{n+1} - a^2a^d \\ & = & x^na(a^n - a^{n+1}a^d) + x^na^{n+2}a^d - a^2a^d \\ & = & x^na(a^n - a^{n+1}a^d) + (x^na^{n+1})aa^d - a^2a^d \\ & = & x^na(a^n - a^{n+1}a^d). \end{array}$$

Hence,

$$\| a - a^2a^d \|^{\frac{1}{n}} \leq \| x \| \| a \|^{\frac{1}{n}} \| a^n - a^{n+1}a^d \|^{\frac{1}{n}} .$$

This implies that $\lim\limits_{n\to\infty} \| a-a^2a^d \|^{\frac{1}{n}} = 0$, and so $a = a^2a^d$. Thus $a \in a^2\mathcal{A} \bigcap \mathcal{A}a^2$, as required.

(1) ⇔ (3) This is obvious by the symmetry. □

We are ready to prove:

Theorem 2.7. *Let $\mathcal{A}$ be a Banach algebra and $M = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ with $a, cb \in \mathcal{A}^d$. If $ca = c$, then*

(1) $M \in M_2(\mathcal{A})^{\#}$.
(2) $a, cb \in \mathcal{A}^{\#}$ and $a^{\pi}b(cb)^{\pi} = 0$.

In this case,

$$M^{\#} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where

$$\begin{array}{rcl} \alpha & = & a^{\#} - (a^{\#})^2 b(cb)^{\pi} c - a^{\#} b(cb)^{\#} c - a^{\#} b(cb)^{\pi} c - a^{\pi} b[(cb)^{\#}]^2 c, \\ \beta & = & (a^{\#})^2 b(cb)^{\pi} + a^{\pi} b[(cb)^{\#}]^2 + a^{\#} b(cb)^{\#} + a^{\pi} b(cb)^{\#}, \\ \gamma & = & (cb)^{\pi} + (cb)^{\#} c, \\ \delta & = & -(cb)^{\#}. \end{array}$$

Proof. Since $ca = c$, we verify that

$$M^2 = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} a^2 & ab - bcb \\ 0 & (cb)^2 \end{pmatrix}.$$

In light of Theorem 2.5, M has generalized Drazin inverse. By virtue of Lemma 2.6, M has group inverse if and only if there exists some X such that $M = XM^2$ if and only if there exists some Y such that $M = M^2 Y$. By the preceding conditions, we deduce that

$$M = X \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} a^2 & ab - bcb \\ 0 & (cb)^2 \end{pmatrix} \ (1),$$

$$\begin{array}{rcl} \begin{pmatrix} a^2 & ab - bcb \\ 0 & (cb)^2 \end{pmatrix} Y & = & \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix} \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} M^2 Y \\ & = & \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix} \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} M \ (2). \end{array}$$

(1) $\Rightarrow$ (2) It follows from equations (1) that $a \in \mathcal{A}a^2$. In light of Lemma 2.6, $a \in \mathcal{A}^{\#}$.

Write $Y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$. Then

$$\begin{gathered} a^2 y_1 + (ab - bcb) y_3 = a - bc \ (3), \\ a^2 y_2 + (ab - bcb) y_4 = b \ (4), \\ (cb)^2 y_3 = cbc \ (5), \\ (cb)^2 y_4 = -cb \ (6). \end{gathered}$$

Since $cb \in (cb)^2 \mathcal{A}$, it follows by Lemma 2.6 that $cb \in \mathcal{A}^{\#}$.

Moreover, we have

$$\begin{aligned} a^{\pi}b(cb)^{\pi} &= a^{\pi}[a^2y_2+(ab-bcb)y_4](cb)^{\pi} \\ &= -a^{\pi}bcby_4(cb)^{\pi} \\ &= -a^{\pi}b(cb)^{\#}[(cb)^2y_4](cb)^{\pi} \\ &= a^{\pi}b(cb)^{\#}cb(cb)^{\pi} \\ &= 0, \end{aligned}$$

as required.

(2) $\Rightarrow$ (1) By hypothesis, we have $a, cb \in \mathcal{A}^{\#}$ and $a^{\pi}b(cb)^{\pi} = 0$. Then $ca^{\#} = ca^2a^{\#} = ca = c$. Let $y_1 = a^{\#} - (a^{\#})^2b(cb)^{\pi}c - a^{\#}b(cb)^{\#}c, y_2 = (a^{\#})^2b(cb)^{\pi}c + a^{\#}b(cb)^{\#}, y_3 = (cb)^{\#}c, y_4 = -(cb)^{\#}$. Then we verify that equations $(3) - (6)$ hold. This implies that $M = M^2Y$, and so M has group inverse by Lemma 2.6.

Let

$$Z = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where

$$\begin{aligned} \alpha &= a^{\#} - (a^{\#})^2b(cb)^{\pi}c - a^{\#}b(cb)^{\#}c - a^{\#}b(cb)^{\pi}c - a^{\pi}b[(cb)^{\#}]^2c, \\ \beta &= (a^{\#})^2b(cb)^{\pi} + a^{\pi}b[(cb)^{\#}]^2 + a^{\#}b(cb)^{\#} + a^{\pi}b(cb)^{\#}, \\ \gamma &= (cb)^{\pi}c + (cb)^{\#}c, \\ \delta &= -(cb)^{\#}. \end{aligned}$$

We verify that $ca^{\#} = c$. Then

$$\begin{aligned} a\alpha + b\gamma &= aa^{\#} - a^{\#}b(cb)^{\pi}c + a^{\pi}b(cb)^{\#}c - aa^{\#}b(cb)^{\pi}c + b(cb)^{\pi}c, \\ a\beta + b\delta &= a^{\#}b(cb)^{\pi} - a^{\pi}b(cb)^{\#}, \\ c\alpha &= (cb)^{\pi}c. \\ c\beta &= cb(cb)^{\#}, \end{aligned}$$

Since $a^{\pi}b(cb)^{\pi} = 0$, we deduce that

$$\begin{aligned} a\alpha + b\gamma &= aa^{\#} - a^{\#}b(cb)^{\pi}c + a^{\pi}b(cb)^{\#}c - aa^{\#}b(cb)^{\pi}c + b(cb)^{\pi}c \\ &= a^{\#}a - a^{\#}b(cb)^{\pi}c + a^{\pi}b(cb)^{\#}c \\ &= \alpha a + \beta c. \end{aligned}$$

Likewise, we verify that

$$\begin{aligned} a\beta + b\delta &= \alpha b, \\ c\alpha &= \gamma a + \delta c, \\ c\beta &= \gamma b. \end{aligned}$$

We compute that

$$\begin{aligned} MZ &= \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \\ &= \begin{pmatrix} a\alpha+b\gamma & a\beta+b\delta \\ c\alpha & c\beta \end{pmatrix}, \\ ZM &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \\ &= \begin{pmatrix} \alpha a+\beta c & \alpha b \\ \gamma a+\delta c & \gamma b \end{pmatrix}. \end{aligned}$$

Then $MZ = ZM$. Similarly, we check that $MZM = M$ and $Z = ZMZ$. Therefore $M^{\#} = Z$, as asserted. □

Theorem 2.8. *Let $\mathcal{A}$ be a Banach algebra and $M = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ with $a, cb \in \mathcal{A}^d$. If $ab = b$, then*

(1) $M \in M_2(\mathcal{A})^{\#}$.
(2) $a, cb \in \mathcal{A}^{\#}$ and $(cb)^{\pi}ca^{\pi} = 0$.

In this case,

$$M^{\#} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where

$$\begin{aligned} \alpha &= a^{\#} - b(cb)^{\pi}ca^{\#} - b(cb)^{\pi}c(a^{\#})^2 - b(cb)^{\#}ca^{\#} - b[(cb)^{\#}]^2ca^{\pi}, \\ \beta &= b(cb)^{\#} + b(cb)^{\pi}, \\ \gamma &= (cb)^{\#}ca^{\#} + (cb)^{\#}ca^{\pi} + [(cb)^{\#}]^2ca^{\pi} + (cb)^{\pi}c(a^{\#})^2, \\ \delta &= -(cb)^{\#}. \end{aligned}$$

Proof. This is proved similarly to Theorem 2.7. □

3. GENERALIZED GROUP INVERSES

This aim of this section is to investigate various conditions under which the representation of the generalized group inverse of M is presented. We begin with

Lemma 3.1. *Let $a \in \mathcal{A}^{ⓖ}$ and $b \in \mathcal{A}$. Then the following are equivalent:*

(1) $a^{\pi}b = 0$.
(2) $a^{\tau}b = 0$.
(3) $(1 - a^{ⓖ}a)b = 0$.

Proof. (1) $\Rightarrow$ (3) Since $a^{\pi}b = 0$, we have that $b = aa^d b$. Then $a^{ⓖ}ab = a^{ⓖ}a^2a^d b = aa^d b = b$. Hence $(1 - a^{ⓖ}a)b = 0$, as required.

(3) $\Rightarrow$ (2) Since $(1 - a^{ⓖ}a)b = 0$, we have $b = a^{ⓖ}ab$. Thus, $(1 - aa^{ⓖ})b = (1 - aa^{ⓖ})a^{ⓖ}ab = 0$.

(2) $\Rightarrow$ (1) Since $(1 - aa^{ⓖ})b = 0$, we have $b = aa^{ⓖ}b$. Then

$$aa^d b = a^2a^d a^{ⓖ}b = aa^{ⓖ}b = b.$$

This implies that $a^{\pi}b = 0$, as required. □

Lemma 3.2. *Let $a \in \mathcal{A}^{ⓖ}$ and $b \in \mathcal{A}^{qnil}$. If $a^*b = 0$ and $ba = 0$, then $a + b \in \mathcal{A}^{ⓖ}$. In this case,*

$$(a + b)^{ⓖ} = a^{ⓖ}.$$

Proof. Since $a \in \mathcal{A}^{ⓖ}$, by virtue of Theorem 1.1, there exist $x \in \mathcal{A}^{\#}$ and $y \in \mathcal{A}^{qnil}$ such that $a = x + y, x^*y = 0, yx = 0$. As in the proof of [7, Theorem 2.2], $x = a^2a^{ⓖ}$ and $y = a - a^2a^{ⓖ}$. Then $a = x + (y + b)$. Since $by = b(a - a^2a^{ⓖ}) = 0$, it follows by [23, Lemma 4.1] that $y + b \in \mathcal{A}^{qnil}$. Obviously, $a^*(y + b) = a^*y + a^*b = 0$. In light of Theorem 1.1, $a + b \in \mathcal{A}^{ⓖ}$. In this case,

$$(a + b)^{ⓖ} = x^{\#} = a^{ⓖ},$$

as asserted. □

Theorem 3.3. *Let $\mathcal{A}$ be a Banach algebra and $M = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ with $a \in \mathcal{A}^{ⓖ}, cb \in \mathcal{A}^{\#}$. If*

$$ca = ca^* = c, aa^{\pi}b = 0, b^*aa^{\tau} = 0, a^{\pi}b(cb)^{\pi} = 0,$$

then $M \in M_2(\mathcal{A})^{ⓖ}$. In this case,

$$M^{ⓖ} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where

$$\begin{aligned} \alpha &= a^{ⓖ} - (a^{ⓖ})^2b(cb)^{\pi}c - a^{ⓖ}b(cb)^{\#}c - a^{ⓖ}b(cb)^{\pi}c - a^{\tau}b[(cb)^{\#}]^2c, \\ \beta &= (a^{ⓖ})^2b(cb)^{\pi} + a^{\tau}b[(cb)^{\#}]^2 + a^{ⓖ}b(cb)^{\#} + a^{\tau}b(cb)^{\#}, \\ \gamma &= (cb)^{\pi} + (cb)^{\#}c, \\ \delta &= -(cb)^{\#}. \end{aligned}$$

Proof. Since $ca = ca^* = c$, it follows by [5, Theorem 15.2.12] that $ca^d = c$ and $c = c(a^*)^d = c(a^d)^*$. By virtue of Theorem 1.1, we derive that

$$\begin{array}{rcl} c & = & ca = [c(a^d)^*]a = c[(a^d)^*a] \\ & = & c[(a^d)^*a^2a^{\circledS}] \\ & = & ca^{\circledS}. \end{array}$$

Since $a \in \mathcal{A}^{\circledS}$, we have the generalized group decomposition: $a = x + y$, where $x \in \mathcal{A}^{\#}, y \in \mathcal{A}^{qnil}$ and $x^*y = 0, yx = 0$. As in the proof of [6, Theorem 2.2],

$$\begin{array}{rcl} x & = & a^2a^{\circledS}, \\ y & = & d - d^2d^{\circledS}. \end{array}$$

Write $M = P + Q$, where

$$P = \begin{pmatrix} x & b \\ c & 0 \end{pmatrix}, Q = \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $y \in \mathcal{A}^{qnil}$, Q is quasinilpotent. One easily checks that

$$\begin{array}{rcl} P^*Q & = & \begin{pmatrix} x^* & c^* \\ b^* & 0 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ b^*y & 0 \end{pmatrix} = 0, \\ QP & = & \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x & b \\ c & 0 \end{pmatrix} = \begin{pmatrix} 0 & yb \\ 0 & 0 \end{pmatrix} = 0. \end{array}$$

We verify that

$$\begin{array}{rcl} cx & = & ca^2a^{\circledS} = ca^{\circledS} = c, \\ x^{\pi}b(cb)^{\pi} & = & [1 - xx^{\#}]b(cb)^{\pi} = [1 - (a^2a^{\circledS})a^{\circledS}]b(cb)^{\pi} \\ & = & a^{\tau}b(cb)^{\pi}. \end{array}$$

Since $a^{\pi}b(cb)^{\pi} = 0$, it follows by Lemma 3.1 that $a^{\tau}b(cb)^{\pi} = 0$. This implies that $x^{\pi}b(cb)^{\pi} = 0$. In light of Lemma 3.2 and Theorem 2.7, we derive that

$$M^{\circledS} = P^{\#} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where

$$\begin{array}{rcl} \alpha & = & x^{\#} - (x^{\#})^2b(cb)^{\pi}c - a^{\#}b(cb)^{\#}c - x^{\#}b(cb)^{\pi}c - x^{\pi}b[(cb)^{\#}]^2c, \\ \beta & = & (x^{\#})^2b(cb)^{\pi} + x^{\pi}b[(cb)^{\#}]^2 + x^{\#}b(cb)^{\#} + x^{\pi}b(cb)^{\#}, \\ \gamma & = & (cb)^{\pi}c + (cb)^{\#}c, \\ \delta & = & -(cb)^{\#}. \end{array}$$

Obviously, $x^{\#} = a^{\circledS}$ and $x^{\pi} = 1 - xx^{\#} = 1 - (a^2a^{\circledS})a^{\circledS} = a^{\tau}$. This completes the proof. $\square$

We are ready to prove:

Theorem 3.4. *Let $\mathcal{A}$ be a Banach algebra and $M = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ with $a \in \mathcal{A}^{\circledS}$ and $bc \in \mathcal{A}^{\#}$. If*

$$\begin{gathered} ca = ca^* = c, aa^{\pi}b = 0, b^*aa^{\tau} = 0, \\ a^*b(cb)^{\pi} = 0, b^*b(cb)^{\pi} = 0. \end{gathered}$$

then $M \in M_2(\mathcal{A})^{\circledS}$. In this case,

$$M^{\circledS} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where

$$\begin{array}{rcl} \alpha & = & a^{\#} - a^{\#}b(cb)^dc - a^{\pi}b[(cb)^d]^2c, \\ \beta & = & a^{\pi}b[(cb)^d]^2 + a^{\#}b(cb)^d + a^{\pi}b(cb)^d, \\ \gamma & = & (cb)^{\pi} + (cb)^dc, \\ \delta & = & -(cb)^d. \end{array}$$

Proof. Write $M = P + Q$, where

$$P = \begin{pmatrix} a & b(cb)(cb)^d \\ c & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & b(cb)^{\pi} \\ 0 & 0 \end{pmatrix}.$$

Then $Q^2 = 0$, and so it is nilpotent.

By hypothesis, we check that

$$ca = ca^* = c, aa^{\pi}b(cb)(cb)^d = 0, [b(cb)(cb)^d]^*aa^{\tau} = 0.$$

Moreover, we derive that

$$a^{\pi}b(cb)(cb)^d[(cb)^2(cb)^d]^{\pi} = a^{\pi}b(cb)(cb)^d[1 - (cb)^2((cb)^d)^2] = 0.$$

Since $c[b(cb)(cb)^d] = (cb)^2(cb)^d \in \mathcal{A}^{\#}$, it follows by Theorem 3.4 that P has generalized group inverse and

$$P^{\circledS} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where

$$\begin{array}{rcl} \alpha & = & a^{\#} - a^{\#}b(cb)^dc - a^{\pi}b[(cb)^d]^2c, \\ \beta & = & a^{\pi}b[(cb)^d]^2 + a^{\#}b(cb)^d + a^{\pi}b(cb)^d, \\ \gamma & = & (cb)^{\pi} + (cb)^dc, \\ \delta & = & -(cb)^d. \end{array}$$

Furthermore, we verify that

$$\begin{array}{rcl} P^*Q & = & \begin{pmatrix} a^* & c^* \\ \big((cb)(cb)^d\big)^*b^* & 0 \end{pmatrix}\begin{pmatrix} 0 & b(cb)^\pi \\ 0 & 0 \end{pmatrix} \\ & = & \begin{pmatrix} 0 & a^*b(cb)^\pi \\ 0 & \big((cb)(cb)^d\big)^*b^*b(cb)^\pi \end{pmatrix} = 0, \\ QP & = & \begin{pmatrix} 0 & b(cb)^\pi \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b(cb)(cb)^d \\ c & 0 \end{pmatrix} = \begin{pmatrix} b(cb)^\pi c & 0 \\ 0 & 0 \end{pmatrix} \\ & = & \begin{pmatrix} bc(bc)^\pi & 0 \\ 0 & 0 \end{pmatrix} = 0. \end{array}$$

According to Lemma 3.2, $M = P + Q$ has generalized group inverse and $M^{ⓖ} = P^{ⓖ}$, as required. □

Corollary 3.5. *Let $\mathcal{A}$ be a Banach algebra and $M = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ with $a \in \mathcal{A}^{ⓖ}$.*

(1) If $cb \in \mathcal{A}^{\#}$ and $ca = ca^* = c, a^*b = 0, a^\pi b = 0$, then $M \in M_2(\mathcal{A})^{ⓖ}$. In this case,

$$M^{ⓖ} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where

$$\begin{array}{rcl} \alpha & = & a^{ⓖ} - (a^{ⓖ})^2 b(cb)^\pi c - a^{ⓖ} b(cb)^{\#}c - a^{ⓖ} b(cb)^\pi c, \\ \beta & = & (a^{ⓖ})^2 b(cb)^\pi + a^{ⓖ} b(cb)^{\#}, \\ \gamma & = & (cb)^\pi + (cb)^{\#}c, \\ \delta & = & -(cb)^{\#}. \end{array}$$

(2) If $bc \in \mathcal{A}^{\#}$ and $ca = ca^* = c, a^*b = 0, aa^\pi b = 0, b(cb)^\pi = 0$ then $M \in M_2(\mathcal{A})^{ⓖ}$. In this case,

$$M^{ⓖ} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where

$$\begin{array}{rcl} \alpha & = & a^{\#} - a^{\#}b(cb)^d c - a^\pi b[(cb)^d]^2 c, \\ \beta & = & a^\pi b[(cb)^d]^2 + a^{\#}b(cb)^d + a^\pi b(cb)^d, \\ \gamma & = & (cb)^\pi + (cb)^d c, \\ \delta & = & -(cb)^d. \end{array}$$

Proof. (1) Since $a^\pi b = 0$, by virtue of Lemma 3.1, $a^\tau b = 0$. The result follows by Theorem 3.3.

(2) It is immediate from Theorem 3.4. □

Theorem 3.6. *Let $\mathcal{A}$ be a Banach algebra and $M = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ with $a \in \mathcal{A}^{\circledR}$.*

(1) If $cb \in \mathcal{A}^{\#}$ and $ab = a^*b = b, b^*a^\tau = 0, (cb)^\pi ca^\tau = 0$, then $M \in M_2(\mathcal{A})^{\circledR}$. In this case,

$$M^{\circledR} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where

$$\begin{array}{rcl} \alpha & = & a^{\circledR} - b(cb)^\pi ca^{\circledR} - b(cb)^\pi c(a^{\circledR})^2 - b(cb)^{\#}ca^{\circledR} - b[(cb)^{\#}]^2ca^\tau, \\ \beta & = & b(cb)^{\#} + b(cb)^\pi, \\ \gamma & = & (cb)^{\#}ca^{\circledR} + (cb)^{\#}ca^\tau + [(cb)^{\#}]^2ca^\tau + (cb)^\pi c(a^{\circledR})^2, \\ \delta & = & -(cb)^{\#}. \end{array}$$

(2) If $bc \in \mathcal{A}^{\#}$ and $ab = a^*b = b, b^*a^\tau = 0, (cb)^\pi ca^\tau = 0, a^*(bc)^\pi b = 0, b^*(bc)^\pi b = 0$, then $M \in M_2(\mathcal{A})^{\circledR}$. In this case,

$$M^{\circledR} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where

$$\begin{array}{rcl} \alpha & = & a^{\circledR} - bc(bc)^d b(cb)^d ca^{\circledR} - b[(cb)^d]^2ca^\tau, \\ \beta & = & b(cb)^d, \\ \gamma & = & (cb)^d ca^{\circledR} + (cb)^d ca^\tau + [(cb)^d]^2ca^\tau + (cb)^\pi c(a^{\circledR})^2, \\ \delta & = & -(cb)^d. \end{array}$$

Proof. (1) Since $ab = b$, it follows by [5, Theorem 15.2.12] that $a^d b = b$, and so $aa^\pi b = a(1 - aa^d)b = ab - a(a^d b) = 0$. Construct x, y, P and Q as in Theorem 3.3. Then $M = P + Q, P^*Q = 0, QP = 0$ and Q is quasinilpotent.

By hypothesis, we verify that

$$\begin{array}{rcl} xb & = & a^2(a^{\circledR}b) = a^2b = b, \\ (cb)^\pi cx^\pi & = & (cb)^\pi c[1 - xx^{\#}] = (cb)^\pi ca^\tau = 0. \end{array}$$

Applying Theorem 2.8 to $P = \begin{pmatrix} x & b \\ c & 0 \end{pmatrix}$, P has group inverse. According to Theorem 1.1, we deduce that

$$M^{\circledR} = P^{\#} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where

$$\begin{array}{rcl}\alpha & = & x^{\#}-b(cb)^{\pi}cx^{\#}-b(cb)^{\pi}c(x^{\#})^2-b(cb)^{\#}cx^{\#}-b[(cb)^{\#}]^2cx^{\pi},\\ \beta & = & b(cb)^{\#}+b(cb)^{\pi},\\ \gamma & = & (cb)^{\#}cx^{\#}+(cb)^{\#}cx^{\pi}+[(cb)^{\#}]^2cx^{\pi}+(cb)^{\pi}c(x^{\#})^2,\\ \delta & = & -(cb)^{\#}.\end{array}$$

Since $x^{\#}=a^{ⓖ}$ and $x^{\pi}=1-xx^{\#}=1-(a^2a^{ⓖ})a^{ⓖ}=a^{\tau}$. The proof is true.

(2) Write $M=P+Q$, where

$$P=\begin{pmatrix} a & (bc)(bc)^db\\ c & 0\end{pmatrix}, Q=\begin{pmatrix} 0 & (bc)^{\pi}b\\ 0 & 0\end{pmatrix}.$$

Then $Q^2=0$; hence, it is nilpotent.

By hypothesis, we check that

$$\begin{array}{rcl}a(bc)(bc)^db & = & a^*(bc)(bc)^db=(bc)(bc)^db, aa^{\pi}(bc)(bc)^db=0,\\ \big((bc)(bc)^db\big)^*aa^{\tau} & = & 0, \big(c(bc)(bc)^db\big)^{\pi}ca^{\tau}=(cb)^{\pi}ca^{\tau}=0.\end{array}$$

Since $c[(bc)(bc)^db]=c(bc)b[(cb)^d]^2cb=(cb)^2(cb)^d\in\mathcal{A}^{\#}$, by the argument above, P has generalized group inverse and

$$P^{ⓖ}=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix},$$

where

$$\begin{array}{rcl}\alpha & = & a^{ⓖ}-bc(bc)^db(cb)^dca^{ⓖ}-b[(cb)^d]^2ca^{\tau},\\ \beta & = & b(cb)^d,\\ \gamma & = & (cb)^dca^{ⓖ}+(cb)^dca^{\tau}+[(cb)^d]^2ca^{\tau}+(cb)^{\pi}c(a^{ⓖ})^2,\\ \delta & = & -(cb)^d.\end{array}$$

Furthermore, we verify that

$$\begin{array}{rcl}P^*Q & = & \begin{pmatrix} a^* & c^*\\ \big((bc)(bc)^db\big)^* & 0\end{pmatrix}\begin{pmatrix} 0 & (bc)^{\pi}b\\ 0 & 0\end{pmatrix}\\ & = & \begin{pmatrix} 0 & a^*(bc)^{\pi}b\\ 0 & \big((bc)(bc)^db\big)^*(bc)^{\pi}b\end{pmatrix}=0,\\ QP & = & \begin{pmatrix} 0 & (bc)^{\pi}b\\ 0 & 0\end{pmatrix}\begin{pmatrix} a & (bc)(bc)^db\\ c & 0\end{pmatrix}=\begin{pmatrix} (bc)^{\pi}bc & 0\\ 0 & 0\end{pmatrix}\\ & = & 0.\end{array}$$

According to Lemma 3.2, $M=P+Q$ has generalized group inverse and $M^{ⓖ}=P^{ⓖ}$, as required. □

Corollary 3.7. *Let $\mathcal{A}$ be a Banach algebra and $M = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ with $a \in \mathcal{A}^{ⓖ}, cb \in \mathcal{A}^d$.*

(1) If $ab = a^*b = b, b^*a^\tau = 0, (cb)^\pi c = 0$, then $M \in M_2(\mathcal{A})^{ⓖ}$. In this case,

$$M^{ⓖ} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where

$$\begin{array}{rcl} \alpha & = & a^{ⓖ} - b(cb)^\# ca^{ⓖ} - b[(cb)^\#]^2 ca^\tau, \\ \beta & = & b(cb)^\# + b(cb)^\pi, \\ \gamma & = & (cb)^\# ca^{ⓖ} + (cb)^\# ca^\tau + [(cb)^\#]^2 ca^\tau, \\ \delta & = & -(cb)^\#. \end{array}$$

(2) If $ab = a^*b = b, b^*a^\tau = 0, (cb)^\pi c = 0, (bc)^\pi b = 0$, then $M \in M_2(\mathcal{A})^{ⓖ}$. In this case,

$$M^{ⓖ} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where

$$\begin{array}{rcl} \alpha & = & a^{ⓖ} - bc(bc)^d b(cb)^d ca^{ⓖ} - b[(cb)^d]^2 ca^\tau, \\ \beta & = & b(cb)^d, \\ \gamma & = & (cb)^d ca^{ⓖ} + (cb)^d ca^\tau + [(cb)^d]^2 ca^\tau + (cb)^\pi c(a^{ⓖ})^2, \\ \delta & = & -(cb)^d. \end{array}$$

Proof. It is immediate from Theorem 3.6. $\square$

We come now to investigate the special cases of Theorem 3.3 and Theorem 3.6 and establish explicit representations of generalized group inverse for related anti-triangular block operator matrices.

Theorem 3.8. *Let $\mathcal{A}$ be a Banach algebra and $M = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ with $a \in \mathcal{A}^{ⓖ}, cb \in \mathcal{A}^d$.*

(1) If $cb \in \mathcal{A}^\#, ab = a^*b = b, ca = ca^* = c$, then $M \in M_2(\mathcal{A})^{ⓖ}$. In this case,

$$M^{ⓖ} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where

$$\begin{array}{rcl} \alpha &=& a^{\textcircled{g}} - 2b(cb)^{\pi}c - b(cb)^{\#}c, \\ \beta &=& b(cb)^{\#} + b(cb)^{\pi}, \\ \gamma &=& (cb)^{\#}c + (cb)^{\pi}c, \\ \delta &=& -(cb)^{\#}. \end{array}$$

(2) If $bc \in \mathcal{A}^{\#}, ab = a^*b = b, ca = ca^* = c, a^*b(cb)^{\pi} = 0, b^*b(cb)^{\pi} = 0$, then $M \in M_2(\mathcal{A})^{\textcircled{g}}$. In this case,

$$M^{\textcircled{g}} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where

$$\begin{array}{rcl} \alpha &=& a^{\textcircled{g}} - b(cb)^{d}c, \\ \beta &=& b(cb)^{d}, \\ \gamma &=& (cb)^{d}c + (cb)^{\pi}c, \\ \delta &=& -(cb)^{d}. \end{array}$$

Proof. (1) Since $ca = ca^* = c$, it follows by [5, Theorem 15.2.12] that $ca^d = c$ and $c = c(a^*)^d = c(a^d)^*$. By virtue of Theorem 1.1, we derive that

$$\begin{array}{rcl} c &=& ca = [c(a^d)^*]a = c[(a^d)^*a] \\ &=& c[(a^d)^*a^2a^{\textcircled{g}}] \\ &=& ca^{\textcircled{g}}. \end{array}$$

In view of [5, Theorem 15.2.12], $a^db = b$, and then $aa^{\pi}b = 0$, and so $aa^{\tau}b = 0$. Furthermore, we have

$$b^*aa^{\tau} = (a^*b)^*a^{\tau} = b^* - b^*aa^{\textcircled{g}} = b^* - b^*a^{\textcircled{g}} = 0.$$

Since $a \in \mathcal{A}^{\textcircled{g}}$, we have the generalized group decomposition: $a = x + y$, where $x \in \mathcal{A}^{\#}, y \in \mathcal{A}^{qnil}$ and $x^*y = 0, yx = 0$. As in the proof of [6, Theorem 2.2],

$$\begin{array}{rcl} x &=& a^2a^{\textcircled{g}}, \\ y &=& d - d^2d^{\textcircled{g}}. \end{array}$$

Write $M = P + Q$, where

$$P = \begin{pmatrix} x & b \\ c & 0 \end{pmatrix}, Q = \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $y \in \mathcal{A}^{qnil}$, Q is quasinilpotent. One easily checks that

$$\begin{array}{rcl} P^*Q &=& \begin{pmatrix} x^* & c^* \\ b^* & 0 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ b^*y & 0 \end{pmatrix} = 0, \\ QP &=& \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x & b \\ c & 0 \end{pmatrix} = \begin{pmatrix} 0 & yb \\ 0 & 0 \end{pmatrix} = 0. \end{array}$$

We verify that

$$\begin{array}{rcl} cx & = & ca^2a^{\textcircled{g}} = ca^{\textcircled{g}} = c, \\ xb & = & a^2a^{\textcircled{g}}b = a^2b = b. \end{array}$$

By virtue of [3, Theorem 3], P has group inverse. In light of Theorem 1.1, we have

$$M^{\textcircled{g}} = P^{\#} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where

$$\begin{array}{rcl} \alpha & = & x^{\#} - 2b(cb)^{\pi}c - b(cb)^{\#}c, \\ \beta & = & b(cb)^{\#} + b(cb)^{\pi}, \\ \gamma & = & (cb)^{\#}c + (cb)^{\pi}c, \\ \delta & = & -(cb)^{\#}. \end{array}$$

Since $x^{\#} = a^{\textcircled{g}}$ and $x^{\pi} = 1 - xx^{\#} = 1 - (a^2a^{\textcircled{g}})a^{\textcircled{g}} = a^{\tau}$. We obtain the result.

(2) Write $M = P + Q$, where

$$P = \begin{pmatrix} a & b(cb)(cb)^d \\ c & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & b(cb)^{\pi} \\ 0 & 0 \end{pmatrix}.$$

By hypothesis, we verify that

$$\begin{array}{rcl} cb(cb)(cb)^d & = & (cb)^2(cb)^d \in \mathcal{A}^{\#}, \\ ab(cb)(cb)^d & = & a^*b(cb)(cb)^d = b(cb)(cb)^d, \\ ca & = & ca^* = c. \end{array}$$

By the argument above, P has generalized group inverse and

$$P^{\#} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where

$$\begin{array}{rcl} \alpha & = & a^{\textcircled{g}} - b(cb)^dc, \\ \beta & = & b(cb)^d, \\ \gamma & = & (cb)^dc + (cb)^{\pi}c, \\ \delta & = & -(cb)^d. \end{array}$$

As in the proof of Theorem 3.6, we easily check that

$$\begin{array}{rcl} P^*Q & = & \begin{pmatrix} 0 & a^*b(cb)^{\pi} \\ 0 & \big((cb)(cb)^d\big)^*b^*b(cb)^{\pi} \end{pmatrix} = 0, \\ QP & = & \begin{pmatrix} b(cb)^{\pi}c & 0 \\ 0 & 0 \end{pmatrix} = 0. \end{array}$$

Clearly, Q is nilpotent, and so it is quasinilpotent. In light of Lemma 3.2, $M = P + Q$ has generalized group inverse and $M = P^{\textcircled{g}}$, as required. $\square$

4. WEAK GROUP INVERSES

Lemma 4.1. *Let $a \in \mathcal{A}$. Then $a \in R^{Ⓦ}$ if and only if $a \in R^D \bigcap R^{ⓖ}$. In this case, $a^{Ⓦ} = a^{ⓖ}$.*

Proof. See [6, Lemma 4.5]. $\square$

Theorem 4.2. *Let $\mathcal{A}$ be a Banach algebra and $M = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ with $a \in \mathcal{A}^{Ⓦ}$.*

(1) If $cb \in \mathcal{A}^{\#}$ and $ca = ca^* = c, aa^{\pi}b = 0, b^*aa^{\tau} = 0, a^{\pi}b(cb)^{\pi} = 0$, then $M \in M_2(\mathcal{A})^{Ⓦ}$. In this case,

$$M^{Ⓦ} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where

$$\begin{array}{rcl} \alpha & = & a^{Ⓦ} - (a^{Ⓦ})^2 b(cb)^{\pi}c - a^{Ⓦ}b(cb)^{\#}c - a^{Ⓦ}b(cb)^{\pi}c \\ & - & [1 - aa^{Ⓦ}]b[(cb)^{\#}]^2 c, \\ \beta & = & (a^{Ⓦ})^2 b(cb)^{\pi} + [1 - aa^{Ⓦ}]b[(cb)^{\#}]^2 + a^{Ⓦ}b(cb)^{\#} \\ & + & [1 - aa^{Ⓦ}]b(cb)^{\#}, \\ \gamma & = & (cb)^{\pi} + (cb)^{\#}c, \\ \delta & = & -(cb)^{\#}. \end{array}$$

(2) If $bc \in \mathcal{A}^{\#}$ and

$$ca = ca^* = c, aa^{\pi}b = 0, b^*aa^{\tau} = 0,$$
$$a^*b(cb)^{\pi} = 0, b^*b(cb)^{\pi} = 0,$$

then $M \in M_2(\mathcal{A})^{Ⓦ}$. In this case,

$$M^{Ⓦ} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where

$$\begin{array}{rcl} \alpha & = & a^{\#} - a^{\#}b(cb)^D c - a^{\pi}b[(cb)^D]^2 c, \\ \beta & = & a^{\pi}b[(cb)^D]^2 + a^{\#}b(cb)^D + a^{\pi}b(cb)^D, \\ \gamma & = & (cb)^{\pi} + (cb)^D c, \\ \delta & = & -(cb)^D. \end{array}$$

Proof. In view of Lemma 4.1, $a \in \mathcal{A}^{Ⓦ}$ if and only if $a \in \mathcal{A}^{ⓖ} \bigcap \mathcal{A}^D$ and $a^{Ⓦ} = a^{ⓖ}$. Therefore we obtain the result by Theorem 3.3 and Theorem 3.4. $\square$

Theorem 4.3. *Let $\mathcal{A}$ be a Banach algebra and $M = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ with $a \in \mathcal{A}^{Ⓦ}$.*

(1) If $cb \in \mathcal{A}^{\#}$ and $ab = a^*b = b, b^*a^\tau = 0, (cb)^\pi ca^\tau = 0$, then $M \in M_2(\mathcal{A})^{\textcircled{W}}$. In this case,

$$M^{\textcircled{W}} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where

$$\begin{array}{rcl} \alpha & = & a^{\textcircled{W}} - b(cb)^\pi ca^{\textcircled{W}} - b(cb)^\pi c(a^{\textcircled{W}})^2 - b(cb)^{\#}ca^{\textcircled{W}} - b[(cb)^{\#}]^2ca^\tau, \\ \beta & = & b(cb)^{\#} + b(cb)^\pi, \\ \gamma & = & (cb)^{\#}ca^{\textcircled{W}} + (cb)^{\#}ca^\tau + [(cb)^{\#}]^2ca^\tau + (cb)^\pi c(a^{\textcircled{W}})^2, \\ \delta & = & -(cb)^{\#}. \end{array}$$

(2) If $bc \in \mathcal{A}^{\#}$ and $ab = a^*b = b, b^*a^\tau = 0, (cb)^\pi ca^\tau = 0, a^*(bc)^\pi b = 0, b^*(bc)^\pi b = 0$, then $M \in M_2(\mathcal{A})^{\textcircled{W}}$. In this case,

$$M^{\textcircled{W}} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where

$$\begin{array}{rcl} \alpha & = & a^{\textcircled{W}} - bc(bc)^D b(cb)^D ca^{\textcircled{W}} - b[(cb)^D]^2ca^\tau, \\ \beta & = & b(cb)^d, \\ \gamma & = & (cb)^D ca^{\textcircled{W}} + (cb)^d ca^\tau + [(cb)^D]^2ca^\tau + (cb)^\pi c(a^{\textcircled{W}})^2, \\ \delta & = & -(cb)^D. \end{array}$$

Proof. It follows directly from Lemma 4.1 and Theorem 3.6. □

Theorem 4.4. *Let $\mathcal{A}$ be a Banach algebra and $M = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ with $a \in \mathcal{A}^{\textcircled{W}}, cb \in \mathcal{A}^D$. If $ab = a^*b = b, ca = ca^* = c, b(cb)^\pi = 0$, then $M \in M_2(\mathcal{A})^{\textcircled{W}}$. In this case,*

$$M^{\textcircled{W}} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where

$$\begin{array}{rcl} \alpha & = & a^{\textcircled{W}} - b(cb)^D c, \\ \beta & = & b(cb)^D, \\ \gamma & = & (cb)^D c + (cb)^\pi c, \\ \delta & = & -(cb)^D. \end{array}$$

Proof. Since $b(cb)^\pi = 0$, we see that $b(cb)^\pi c = 0$, and so $bc - b((cb)(cb)^D c = 0$. By using Cline's formula, we have $bc = bcbc[(bc)^D]^2bc = (bc)^2(bc)^D$. Hence, $bc \in \mathcal{A}^{\#}$. This completes the proof by Lemma 4.1 and Theorem 3.8. □

We conclude this paper with an example to illustrate Theorem 4.4.

Example 4.5. *Let* $M = \begin{pmatrix} A & B \\ C & 0_{6\times 6} \end{pmatrix} \in \mathbb{C}^{12\times 12}$, *where*

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{C}^{6\times 6}.$$

Then $A^{ⓦ} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$. *We directly verify that*

$$AB = A^*B = B, CA = CA^* = C, B(CB)^{\pi} = 0.$$

By using the formula in Theorem 4.4, we compute that

$$\begin{aligned} M^{ⓦ} &= \begin{pmatrix} A^{ⓦ} - B & B \\ C & -C \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

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