An Oscillatory Integral

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ABSTRACT: Some remarks on an oscillatory integral.

I. Introduction

Recall that

$$\pi = -\int_0^\infty \frac{\ln\left((\tan x)^2\right)}{x^2 + (\ln t)^2}\, dx \tag{1}$$

where

$$t = \frac{1}{3}\left(1 + \left(19 - 3\sqrt{33}\right)^{1/3} + \left(19 + 3\sqrt{33}\right)^{1/3}\right) \tag{2}$$

and

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots\right) \tag{3}$$

Remark: t is the tribonacci constant.

In this note we give some formulas related to (1)

II. Some Remarks

Entry 1.

$$\pi = \sum_{n=0}^\infty \int_0^{\pi/2} \left(\frac{-\ln\left((\tan x)^2\right)}{(x + n\pi)^2 + (\ln t)^2} + \frac{-\ln\left((\cot x)^2\right)}{\left(x + \frac{2n+1}{2}\pi\right)^2 + (\ln t)^2}\right) dx \tag{4}$$

$$\pi = \sum_{n=0}^\infty \int_0^{\pi/2} \left(\frac{\ln\left((\cot x)^2\right)}{(x + n\pi)^2 + (\ln t)^2} + \frac{\ln\left((\tan x)^2\right)}{\left(x + \frac{2n+1}{2}\pi\right)^2 + (\ln t)^2}\right) dx \tag{5}$$

$$\pi = \sum_{n=0}^{\infty} \int_0^{\pi/2} \left(\frac{1}{\left(x + \frac{2n+1}{2}\pi\right)^2 + (\ln t)^2} - \frac{1}{(x + n\pi)^2 + (\ln t)^2} \right) \ln\left((\tan x)^2\right) dx \tag{6}$$

$$\pi = 2 \sum_{n=0}^{\infty} \int_0^{\pi/2} \left(\frac{1}{\left(x + \frac{2n+1}{2}\pi\right)^2 + (\ln t)^2} - \frac{1}{(x + n\pi)^2 + (\ln t)^2} \right) \ln(\tan x)\, dx \tag{7}$$

$$\pi = 2 \sum_{n=0}^{\infty} \int_0^{\infty} \left(\frac{1}{\left(\frac{2n+1}{2}\pi + \tan^{-1} x\right)^2 + (\ln t)^2} - \frac{1}{\left(n\pi + \tan^{-1} x\right)^2 + (\ln t)^2} \right) \frac{\ln(x)}{1 + x^2}\, dx \tag{8}$$

$$\pi = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \left(\frac{1}{\left(\frac{2n+1}{2}\pi + \tan^{-1}(e^x)\right)^2 + (\ln t)^2} - \frac{1}{\left(n\pi + \tan^{-1}(e^x)\right)^2 + (\ln t)^2} \right) \frac{x}{\cosh(x)}\, dx \tag{9}$$

$$\pi = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \left(\frac{1}{\left(n\pi + \tan^{-1}(e^{-x})\right)^2 + (\ln t)^2} - \frac{1}{\left(\frac{2n+1}{2}\pi + \tan^{-1}(e^{-x})\right)^2 + (\ln t)^2} \right) \frac{x}{\cosh(x)}\, dx \tag{10}$$

$$\pi = 2 \sum_{n=0}^{\infty} (-1)^n \int_0^{\pi/2} \frac{-\ln(\tan x)}{\left(\frac{n}{2}\pi + x\right)^2 + (\ln t)^2}\, dx \tag{11}$$

$$\pi = -2 \int_0^{\pi/2} \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{\left(\frac{n\pi}{2} + x\right)^2 + (\ln t)^2} \right) \ln(\tan x)\, dx \tag{12}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\left(\frac{n\pi}{2} + x\right)^2 + (\ln t)^2} = \frac{i}{2\pi \ln(t)} \left(\psi\left(\frac{x - i\ln(t)}{\pi}\right) - \psi\left(\frac{1}{2} + \frac{x - i\ln(t)}{\pi}\right) - \psi\left(\frac{x + i\ln(t)}{\pi}\right) + \psi\left(\frac{1}{2} + \frac{x + i\ln(t)}{\pi}\right) \right) \tag{13}$$

where $i = \sqrt{-1}$ and $\psi(z)$ is the Psi function.

Entry 2.

For $\lambda = 0.2590933275\ldots$, we have

$$\pi = -\int_0^{\lambda} \frac{\ln\left((\tan x)^2\right)}{x^2 + (\ln t)^2}\, dx \tag{14}$$

$$0 = -\int_{\lambda}^{\infty} \frac{\ln\left((\tan x)^2\right)}{x^2 + (\ln t)^2}\, dx \tag{15}$$

Entry 3.

$$\lambda_1 = 1/4\,,\ \ \lambda_{n+1} = \lambda_n \cot\left(\frac{1}{4} f(\lambda_n)\right),\ \ n = 1, 2, 3, \ \ldots \Longrightarrow \lambda_n \longrightarrow \lambda \tag{16}$$

where

$$f(y) = -\int_0^{y} \frac{\ln\left((\tan x)^2\right)}{x^2 + (\ln t)^2}\, dx \ \ , \ y > 0 \tag{17}$$

Entry 4.

$$\pi = 2\int_0^{\lambda} \frac{-\ln x}{x^2 + (\ln t)^2}\, dx - \frac{1}{(\ln t)^2} \sum_{n=1}^{\infty} \frac{2^{2n+1}\left(2^{2n-1}-1\right)\lambda^{2n+1} B_n}{n\,(2n+1)!}\, {}_2F_1\left(1, n+\frac{1}{2}, n+\frac{3}{2}, -\left(\frac{\lambda}{\ln t}\right)^2\right) \tag{18}$$

where $B_n = \left\{\frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{1}{30}, \frac{5}{66} \ldots\right\}$ are the Bernoulli numbers and ${}_2F_1$ is the Gauss hypergeometric functions.

Entry 5.

$$\pi = -\frac{2}{\ln t}\int_0^{\tan^{-1}(\lambda/\ln t)} \ln(\tan(\ln t \tan x))\, dx \tag{19}$$

$$0 = \int_{\tan^{-1}(\lambda/\ln t)}^{\pi/2} \ln(\tan(\ln t \tan x))\, dx \tag{20}$$

$$\pi = -\frac{2\ln(\tan\lambda)}{\ln t}\tan^{-1}\left(\frac{\lambda}{\ln t}\right) + \int_u^{\infty} \tan^{-1}\left(\frac{\tan^{-1}\left(t^{-x/2}\right)}{\ln t}\right) dx \tag{21}$$

where

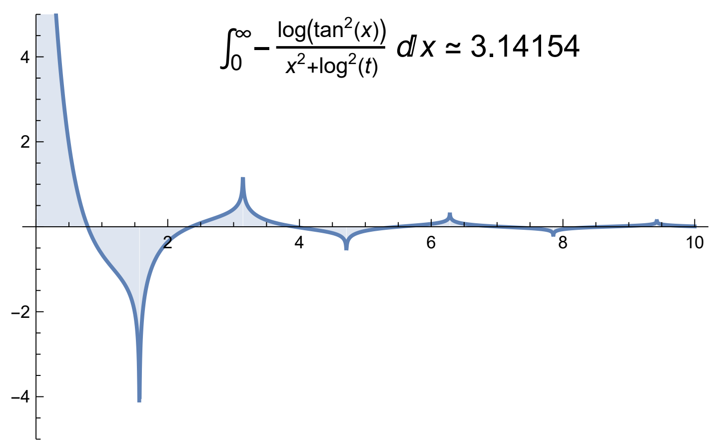
$$\mathrm{u} = -\frac{2\ln(\tan\lambda)}{\ln t} \tag{22}$$

Entry 6.

$$\pi = -\frac{2\ln(\tan\lambda)}{\ln t}\tan^{-1}\left(\frac{\lambda}{\ln t}\right) + \frac{4}{\ln t}\int_0^{\lambda} \frac{1}{\sin(2x)}\tan^{-1}\left(\frac{x}{\ln t}\right) dx \tag{23}$$

$$\pi = \int_{-\ln(\tan\lambda)}^{\infty} \frac{x\,\mathrm{sech}(x)}{\left(\tan^{-1}\left(e^{-x}\right)\right)^2 + (\ln t)^2}\, dx \tag{24}$$

Entry 7. Mathematica⟶NIntegrate⟶Method→LocalAdaptive



III. References

1. Abramowitz, M. and Stegun, I. A.: Handbook of Mathematical Functions. Dover, New York, 1972.

2. Arndt, J., and Haenel, C.: π unleashed. Springer-Verlag, 2001.

3. Biley, D. H., Borwein, J. M., Borwein, P. B., and Pluffe, S.: The quest for Pi. Math. Intelligencer, 19, 50-57, 1997.

4. Davies, K., Strayer, M., and White, G.: Complex-plane methods for evaluating highly oscillatory integrals in nuclear physics. I. J. Phys. G: Nucl. Phys., 14(7), 1988.

5. Deaño, A., and Huybrechs, D.: Complex Gaussian quadrature of oscillatory integrals, Numer. Math. 112, no. 2, 2009.

6. Domínguez, V., Graham, I. G., and Kim, T.: Filon-Clenshaw-Curtis rules for highly oscillatory integrals with algebraic singularities and stationary points, SIAM J. Numer. Anal. 51, no. 3, 2013.

7. Filon, L. N.: On a quadrature formula for trigonometric integrals, Proc. Roy. Soc. Edinburgh 49, 1928.

8. Gradshteyn, I. S., and Ryzhik, I. M.: Table of Integrals, Series, and Products, Academic Press, New York, 1994.

9. Huybrechs, D., and Vandewalle, S.: On evaluation of highly oscillatory integrals by analytic continuation, SIAM J. Numer. Anal. 44, no. 3, 2006.

10. Iserles, A.: On the numerical quadrature of highly-oscillatory integrals I: Fouries transforms. IMA J. Num. Anal., 24, 2004.

11. Levin, D.: Fast integration of rapidly oscillatory functions. J. Comput. Appl. Math., 67(1), 1996.

12. Luke, Y. L.: On the computation of oscillatory integrals, Proc. Cambridge Philos. Soc. 50, 1954.

13. Stein, E. M.: Oscillatory integrals related to Radon-like transforms. Proceedings of the Conference in Honor of Jean-Pierre Kahane (Orsay,1993). J. Fourier Anal. Appl., 1995, Special Issue, 535-551.

14. Xiang, S.: Efficient Filon-type methods for $\int_a^b f(x)\, e^{i\, w\, g(x)}\, dx$, Numer. Math. 105, no. 4, 2007.

15. Zeilberger, D.: Theorems for a price: tomorrow's semi-rigorous mathematical culture. Notices Amer. Math. Soc., 40, 1993. Repr. in Math. Intelligencer, 16, 1994.

16. OEIS: A058265 , Tribonacci constant.