# A Fourier derivative collocation method for the solution of the Navier–Stokes problem

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A proposed solution to the millennium problem on the existence and smoothness of the Navier–Stokes equations.

## 1. Introduction

The Navier–Stokes equations are thought to govern the motion of a fluid in $\mathbb{R}^3$, see [1–5]. Let $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) \in \mathbb{R}^3$ be the fluid velocity and let $p = p(\mathbf{x}, t) \in \mathbb{R}$ be the fluid pressure, each dependent on position $\mathbf{x} \in \mathbb{R}^3$ and time $t \geqslant 0$. I take the externally applied force acting on the fluid to be identically zero. The fluid is assumed to be incompressible with constant viscosity $\nu > 0$ and to fill all of $\mathbb{R}^3$. The Navier–Stokes equations can then be written as

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu \nabla^2 \mathbf{u} - \nabla p, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

with initial condition

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^\circ \tag{3}$$

where $\mathbf{u}^\circ = \mathbf{u}^\circ(\mathbf{x}) \in \mathbb{R}^3$. In these equations

$$\nabla = \left( \frac{\partial}{\partial \mathbf{x}_1}, \frac{\partial}{\partial \mathbf{x}_2}, \frac{\partial}{\partial \mathbf{x}_3} \right) \tag{4}$$

is the gradient operator and

$$\nabla^2 = \sum_{i=1}^{3} \frac{\partial^2}{\partial {\mathbf{x}_i}^2} \tag{5}$$

is the Laplacian operator. When $\nu = 0$ equations (1), (2), (3) are called the Euler equations. When $\nabla p = 0$ equations (1), (3) are called the Burgers equations. Solutions of (1), (2), (3) are to be found with

$$\mathbf{u}^\circ(\mathbf{x} + e_i) = \mathbf{u}^\circ(\mathbf{x}) \tag{6}$$

for $1 \leqslant i \leqslant 3$ where $e_i$ is the $i^{\text{th}}$ unit vector in $\mathbb{R}^3$. The initial condition $\mathbf{u}^\circ$ is a given $C^\infty$ divergence-free vector field on $\mathbb{R}^3$. A solution of (1), (2), (3) is then accepted to be physically reasonable [3] if

$$\mathbf{u}(\mathbf{x} + e_i, t) = \mathbf{u}(\mathbf{x}, t), \quad p(\mathbf{x} + e_i, t) = p(\mathbf{x}, t) \tag{7}$$

on $\mathbb{R}^3 \times [0, \infty)$ for $1 \leqslant i \leqslant 3$ and

$$\mathbf{u}, p \in C^\infty(\mathbb{R}^3 \times [0, \infty)). \tag{8}$$

## 2. Solution of the Navier–Stokes problem

**Theorem**. Take $\nu > 0$. Let $\mathbf{u}^\circ$ be any smooth, divergence-free vector field satisfying (6). Then there exist smooth functions $\mathbf{u}$, $p$ on $\mathbb{R}^3 \times [0, \infty)$ that satisfy (1), (2), (3), (7), (8).

**Proof**. A Fourier derivative collocation method is as follows. Let $\mathbf{u}$, $p$ be given by

$$\mathbf{u} = \sum_{\mathbf{L}=-\infty}^{\infty} \mathbf{u}_{\mathbf{L}} \mathrm{e}^{\mathrm{i}k\mathbf{L}\cdot\mathbf{x}}, \tag{9}$$

$$p = \sum_{\mathbf{L}=-\infty}^{\infty} p_{\mathbf{L}} \mathrm{e}^{\mathrm{i}k\mathbf{L}\cdot\mathbf{x}} \tag{10}$$

respectively. Here $\mathbf{u}_{\mathbf{L}} = \mathbf{u}_{\mathbf{L}}(t) \in \mathbb{C}^3$, $p_{\mathbf{L}} = p_{\mathbf{L}}(t) \in \mathbb{C}$, $\mathrm{i} = \sqrt{-1}$, $k = 2\pi$, and $\sum_{\mathbf{L}=-\infty}^{\infty}$ denotes the sum over all $\mathbf{L} \in \mathbb{Z}^3$. The initial condition $\mathbf{u}^\circ$ is a Fourier series [2] of which is convergent for all $\mathbf{x} \in \mathbb{R}^3$. Equations (1), (2) can be written as

$$\frac{\partial \mathbf{u}_i}{\partial t} + \sum_{j=1}^{3} \mathbf{u}_j \frac{\partial \mathbf{u}_i}{\partial \mathbf{x}_j} = \nu \sum_{j=1}^{3} \frac{\partial^2 \mathbf{u}_i}{\partial {\mathbf{x}_j}^2} - \frac{\partial p}{\partial \mathbf{x}_i} \quad \text{for } i = 1, 2, 3, \tag{11}$$

and

$$\sum_{j=1}^{3} \frac{\partial \mathbf{u}_j}{\partial \mathbf{x}_j} = 0 \tag{12}$$

respectively. In this method we have for a quantity $q$ that

$$\left\lceil \frac{\partial q}{\partial \mathbf{x}_j} \right\rceil = \left[G_j\right] \lceil q \rceil \tag{13}$$

valid at $\mathbf{x} = \mathbf{x}^*{}_n$ for $n = 1, 2, \ldots, N$. Here $\left[G_j\right]$ is a known constant $N \times N$ matrix with $\left[G_j\right]_{m,n} = G_{j,m,n}$ and $\lceil r \rceil$ means to vectorise $r$ where the components are equal to $r|_{\mathbf{x}=\mathbf{x}^*{}_n}$, $n = 1, 2, \ldots, N$. We denote $q|_{\mathbf{x}=\mathbf{x}^*{}_n} = \lceil q \rceil_n = q_{,n}$. Then

$$\left\lceil \frac{\partial \mathbf{u}_i}{\partial \mathbf{x}_j} \right\rceil_n = \sum_{\alpha=1}^{N} G_{j,n,\alpha} u_{i,\alpha}, \quad \left\lceil \frac{\partial p}{\partial \mathbf{x}_i} \right\rceil_n = \sum_{\alpha=1}^{N} G_{i,n,\alpha} p_{,\alpha}, \tag{14}$$

$$\left\lceil \frac{\partial^2 \mathbf{u}_i}{\partial {\mathbf{x}_j}^2} \right\rceil_n = \sum_{\alpha=1}^{N} G_{j,n,\alpha} \left\lceil \frac{\partial \mathbf{u}_i}{\partial \mathbf{x}_j} \right\rceil_\alpha = \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N} G_{j,n,\alpha} G_{j,\alpha,\beta} u_{i,\beta}, \tag{15}$$

and

$$\left\lceil \frac{\partial \mathbf{u}_i}{\partial t} \right\rceil_n = \frac{\partial}{\partial t} \lceil \mathbf{u}_i \rceil_n = \frac{\partial}{\partial t} u_{i,n}. \tag{16}$$

Equations (11), (12) at $\mathbf{x} = \mathbf{x}^*_n$ imply

$$\frac{\partial}{\partial t}\lceil \mathbf{u}_i \rceil_n + \sum_{j=1}^{3} \left\lceil \mathbf{u}_j \right\rceil_n \left\lceil \frac{\partial \mathbf{u}_i}{\partial \mathbf{x}_j} \right\rceil_n = \nu \sum_{j=1}^{3} \left\lceil \frac{\partial^2 \mathbf{u}_i}{\partial {\mathbf{x}_j}^2} \right\rceil_n - \left\lceil \frac{\partial p}{\partial \mathbf{x}_i} \right\rceil_n \tag{17}$$

and

$$\sum_{j=1}^{3} \left\lceil \frac{\partial \mathbf{u}_j}{\partial \mathbf{x}_j} \right\rceil_n = 0 \tag{18}$$

respectively. Equations (17), (18) imply

$$\frac{\partial}{\partial t} u_{i,n} + \sum_{j=1}^{3} \sum_{\alpha=1}^{N} u_{j,n} G_{j,n,\alpha} u_{i,\alpha} = \nu \sum_{j=1}^{3} \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N} G_{j,n,\alpha} G_{j,\alpha,\beta} u_{i,\beta} - \sum_{\alpha=1}^{N} G_{i,n,\alpha} p_{,\alpha} \tag{19}$$

and

$$\sum_{j=1}^{3} \sum_{\alpha=1}^{N} G_{j,n,\alpha} u_{j,\alpha} = 0 \tag{20}$$

respectively. Let $U$ be a matrix where $U_{i,n} = u_{i,n}$ and let $P$ be a matrix where $P_{\alpha,n} = p_{,\alpha}$. Then equations (19), (20) imply

$$\frac{\partial U}{\partial t} + U(A(n)U) = \nu U B(n) - A(n)^T P \tag{21}$$

and

$$\mathrm{trace}(UA(n)) = 0 \tag{22}$$

respectively. Herein $A(n)$ and $B(n)$ are matrices where

$$A(n)_{\alpha,j} = G_{j,n,\alpha} \tag{23}$$

and

$$B(n)_{\beta,n} = \sum_{j=1}^{3} \sum_{\alpha=1}^{N} G_{j,n,\alpha} G_{j,\alpha,\beta}. \tag{24}$$

The $i, n$ component of (21) recovers (19) since

$$\begin{aligned} [U(A(n)U)]_{i,n} &= \sum_{l=1}^{N} U_{i,l} [A(n)U]_{l,n} = \sum_{l=1}^{N} U_{i,l} \left[ \sum_{m=1}^{3} A(n)_{l,m} U_{m,n} \right] \\ &= \sum_{j=1}^{3} \sum_{\alpha=1}^{N} U_{i,\alpha} A(n)_{\alpha,j} U_{j,n} = \sum_{j=1}^{3} \sum_{\alpha=1}^{N} U_{i,\alpha} G_{j,n,\alpha} U_{j,n}, \end{aligned} \tag{25}$$

$$[UB(n)]_{i,n} = \sum_{l=1}^{N} U_{i,l} B(n)_{l,n} = \sum_{\beta=1}^{N} U_{i,\beta} B(n)_{\beta,n} = \sum_{j=1}^{3} \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N} U_{i,\beta} G_{j,n,\alpha} G_{j,\alpha,\beta}, \quad (26)$$

and

$$\begin{aligned} \left[A(n)^T P\right]_{i,n} &= \sum_{l=1}^{N} A(n)^T_{i,l} P_{l,n} = \sum_{l=1}^{N} A(n)_{l,i} P_{l,n} = \sum_{l=1}^{N} G_{i,n,l} P_{l,n} \\ &= \sum_{\alpha=1}^{N} G_{i,n,\alpha} P_{\alpha,n} = \sum_{\alpha=1}^{N} G_{i,n,\alpha} p_{,\alpha}. \end{aligned} \quad (27)$$

Equation (22) recovers (20) since

$$\begin{aligned} \text{trace}(UA(n)) &= \sum_{j=1}^{3} [UA(n)]_{j,j} = \sum_{j=1}^{3} \sum_{l=1}^{N} U_{j,l} A(n)_{l,j} \\ &= \sum_{j=1}^{3} \sum_{l=1}^{N} U_{j,l} G_{j,n,l} = \sum_{j=1}^{3} \sum_{\alpha=1}^{N} U_{j,\alpha} G_{j,n,\alpha}. \end{aligned} \quad (28)$$

Let $Q(n)$ be a matrix such that $(A(n)^T P)Q(n) = 0$. We have

$$\begin{aligned} \left[\left(A(n)^T P\right) Q(n)\right]_{i,j} &= \left[A(n)^T \left(PQ(n)\right)\right]_{i,j} = \sum_{l=1}^{3} A(n)^T_{i,l} (PQ(n))_{l,j} \\ &= \sum_{l=1}^{3} \sum_{m=1}^{N} A(n)^T_{i,l} P_{l,m} Q(n)_{m,j} = 0. \end{aligned} \quad (29)$$

Here nonzero $Q(n)$ is possible to construct because $p$ at one arbitrary $\mathbf{x}$ point is arbitrary and can be set to zero without loss of generality. Then (21) implies

$$\left(\frac{\partial U}{\partial t}\right) Q(n) + (U(A(n)U))Q(n) = (\nu UB(n))Q(n). \quad (30)$$

Equation (30) is the same as we would get for the Burgers equations. Now we consider a matrix Riccati equation problem.

$$\frac{\partial X}{\partial t} = aX + bY, \quad (31)$$

$$\frac{\partial Y}{\partial t} = cX + dY, \quad (32)$$

with

$$X = (U\lambda)Y. \quad (33)$$

Then we get

$$\left(\frac{\partial U}{\partial t}\lambda\right) Y + (U\lambda)\frac{\partial Y}{\partial t} = a(U\lambda)Y + bY \tag{34}$$

which implies

$$\left(\frac{\partial U}{\partial t}\lambda\right) Y + (U\lambda)[c(U\lambda)Y + dY] = a(U\lambda)Y + bY \tag{35}$$

implying

$$\frac{\partial U}{\partial t}\lambda + (U\lambda)c(U\lambda) + (U\lambda)d = a(U\lambda) + b. \tag{36}$$

We then let $a = b = 0$, $\lambda = Q(n)$, $c = Q(n)^{-1}A(n)$, $d = -\nu Q(n)^{-1}B(n)Q(n)$ to recover (30). Then (31) implies

$$X = X|_{t=0}. \tag{37}$$

Equation (32) implies

$$\frac{\partial Y}{\partial t} = cX|_{t=0} + dY \tag{38}$$

and so

$$\frac{\partial}{\partial t}\left(\mathrm{e}^{-dt}Y\right) = \mathrm{e}^{-dt}cX|_{t=0} \tag{39}$$

which integrating with respect to $t$ yields

$$\mathrm{e}^{-dt}Y = \int_0^t \mathrm{e}^{-d\tau}cX|_{t=0}\,d\tau + Y|_{t=0} \tag{40}$$

which implies

$$\mathrm{e}^{-dt}Y = \left[\mathrm{e}^{-d\tau}(-d)^{-1}cX|_{t=0}\right]_0^t + Y|_{t=0} \tag{41}$$

to obtain

$$Y = \left(\mathrm{e}^{-dt}\right)^{-1}\left\{\left(\mathrm{e}^{-dt} - I\right)(-d)^{-1}cX|_{t=0} + Y|_{t=0}\right\}. \tag{42}$$

Equation (33) then implies

$$\begin{aligned} U\lambda &= X|_{t=0}Y^{-1} \\ &= (U|_{t=0}\lambda)Y|_{t=0}\left[\left\{\left(\mathrm{e}^{-dt} - I\right)(-d)^{-1}c(U|_{t=0}\lambda Y|_{t=0}) + Y_{t=0}\right\}^{-1}\mathrm{e}^{-dt}\right] \\ &= (U|_{t=0}\lambda)\left[\left\{\left(\mathrm{e}^{-dt} - I\right)(-d)^{-1}c(U|_{t=0}\lambda) + I\right\}^{-1}\mathrm{e}^{-dt}\right]. \end{aligned} \tag{43}$$

No blowup is possible since the Burgers equations are regular. □
For the Euler equations we have

$$U\lambda = (U|_{t=0}\lambda)\{c(U|_{t=0}\lambda)t + I\}^{-1} \tag{44}$$

and blowup is possible since for odd $N$ the equation

$$\det\left(c(U|_{t=0}\lambda)t + I\right) = 0 \tag{45}$$

can have a solution $t$ where $0 < t < \infty$.

## References


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