Proof of the conjectures of LEGENDRE, ANDRICA, OPPERMANN and their generalizations.

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Abstract :

This article deals with the conjectures of Legendre, Andrica, Oppermann, Brocard and their generalizations. For this purpose, we study the gaps $\Delta\, Lg_k[p_k\,; p_{k+1}] = Lg_k(p_{k+1}) - Lg_k(p_k)$ between two consecutive primes p_k , p_{k+1} for any integer $k \geq 2$, and the variations $\Delta\, Lg_k[N^2;(N+1)^2]$ of the primes-test- functions Lg_k defined on intervals

$J(N) =](N-1)^2;N^2+N^{1.05}[$, $(N \in \mathbb{N} + 2)$ by : $\forall\, x \in \mathbb{R}$, $Lg_k(x) = R_k\sqrt{x}\,\ln(x)$, with $p_k = \text{Sup}(p \in \mathcal{P} : p < N^2)$ and $R_k = \sqrt{\frac{p_k}{p_{k+1}}}$. Thus, for each integer $N > 1048585$, we verify: $\Delta\, Lg_k[p_k\,; p_{k+1}] < 1$; $\lim \Delta Lg_k\,[p_k\,; p_{k+1}] = 0$ and

$\Delta Lg_k\,[N^2;(N+1)^2] > 2\,R_k\,\frac{(2N+1)}{(N+1)}(\ln(N)+1) > 4\ln(N)$. Thank you to an argument by recursion and by the absurd, we show that $\Delta\pi\,[N^2;(N+1)^2] > 1$, (π is the prime counting function). The Legendre conjecture is thus confirmed. Using a similar approach, the conjectures of Andrica, Oppermann and their generalizations can be demonstrated by fitting test intervals and primes-test-functions of the form : ($Lq_k : x \rightarrow Rq_k x^{(1-\frac{1}{q})}\ln(x)$, $(q \in \mathbb{N} + 3)$; $Rq_k = \sqrt[q]{\frac{p_k}{p_{k+1}}}$). Moreover, the gaps between two consecutive primes are estimated by :

$(\forall\, q \in \mathbb{N} + 2)$, $p_{k+1} - p_k = o(\sqrt[q]{p_k})$; $\sqrt{p_{k+1}} - \sqrt{p_k} < \frac{15}{p_k^{0.4}}$ and $\text{Card}(\,]N^2;(N+1)^2[\, \cap \mathcal{P}\,) > \frac{N}{ln(N)}$.

Keywords:

1 Introduction

The number theory (Hardy, Wright [14], Landau [16], Tchebychev [23]) looks in particular at the distribution of primes and the gaps between two consecutive primes. Numerous results on the subject have been established by Axler [3], [4] Dusart [11],[12] , Salvy [21], Bombieri [7], Chen [8], Cramer [10], Erdos [13],Iwaniec, Pintz , Harman, Baker [5],[6],[15], Ramaré , Saouter [19] and Zhang [24].

In 1772, Adrien-Marie Legendre [17] formulated the following conjecture :

" $\forall$ $N \in \mathbb{N}^*$, the interval of integers $[N^2;(N+1)^2]$ contains at least one prime ".

(This can be limited to the open interval $I(N) =]N^2;(N+1)^2[$; N^2 and $(N+1)^2$ are naturally composite numbers). This conjecture is one of the four questions about primes put forward by Landau [13] at the international Conference of Mathematicians in 1912. Furtheremore, Iwaniec , Pintz [15] have shown that for any integer $n \in \mathbb{N} + 3$, there is always a prime $p \in [n - n^{23/42};n]$. Baker and Harman [5],[6] concluded that for a sufficiently large integer n , there is a prime in the interval $[n;n+o(n^{0.525})]$. The Legendre conjecture provides a better square-root majorization of the gap between two consecutive primes p_k and p_{k+1} of the form :

(1.1) $$\forall\, k \in \mathbb{N}^*, \quad p_{k+1} - p_k < 4\sqrt{p_k} + 3$$

The results obtained by Iwaniec, Pintz, Harman, and Baker [5],[6], [15] allow validation of this conjecture up to a value of $N = 1048585$ by solving the following inequation :

(1.2) $$N^{1.05} < 2N + 1, \quad (N \in \mathbb{N}).$$

T. Oliveira e Silva [20] verified this conjecture on computer up to $N = 4.10^{18}$.

a) For any real $x > 6000$, Axler [3], [4] Dusart [12],[13] and Legendre [17] have shown the following framework :

(1.3) $$\frac{x}{\ln(x)-1} < \pi(x) < \frac{x}{\ln(x) - 1.10806}$$

(1.4) For any integer $k > 15985$ $$\ln(k) + \ln(\ln(k)) - 1 < \frac{p_k}{k} < \ln(k) + \ln(\ln(k)) - 0.9427$$

b)

In line with this advances, we study on one side positive gaps of functions

$Lg_k : x \rightarrow Lg_k(x) = R_k \sqrt{x}\ln(x)$ with $p_k = \mathrm{Sup}(p \in \mathcal{P} : p < N^2)$ and $R_k = \sqrt{\frac{p_k}{p_{k+1}}}$, between two consecutive primes p_k and p_{k+1} of the type :

(1.5) $\Delta Lg_k[p_k ; p_{k+1}] = Lg_k(p_{k+1}) - Lg_k(p_k)$ verifying : $Lg_k(p_{k+1}) - Lg_k(p_k) < 1$.

and positive variations of functions between expressions of the form :

(1.6) $\Delta Lg_k[N^2;(N+1)^2] = Lg_k((N+1)^2) - Lg_k(N^2)$ verifying :

(1.7) $Lg_k((N+1)^2) - Lg_k(N^2) < Lg_k(p_{k+1}) - Lg_k(p_k)$ and $\lim\ [Lg_k((N+1)^2) - Lg_k(N^2)] = +\infty$.

allowing us to prove the conjectures of Legendre, Andrica, Oppermann and Brocard by applying reasoning by recurrence and by the absurd .Their generalizations can be proved by adjusting examination intervals and primes-test-functions of the form :

(1.8) $(Lq_k : x \rightarrow Rq_k x^{1-\frac{1}{q}} \ln(x), (q \in \mathbb{N}+3) ; Rq_k = \sqrt[q]{\frac{p_k}{p_{k+1}}})$.

We also obtain the following framings of the number of primes in the interval of integers

$I(N) =]N^2;(N+1)^2[$: these estimates are given by : $\forall N \in \mathbb{N}^*$,

(1.9) $0 < N < 10^4$, $0.78(N+0.5)/(\ln(N)-0.5) < \mathrm{Card}(I(N) \cap \mathcal{P}) < 1.03(N+0.5)/(\ln(N)-0.55)$

(1.10) $10^4 < N < 10^{15}$, $< \mathrm{Card}(I(N) \cap \mathcal{P}) <$

(1.11) $N > 10^6$, $\mathrm{Card}(I(N) \cap \mathcal{P}) > \frac{N}{\ln(N)}$

Subsequently, estimates of the gaps between two consecutive primes of the form :

$\forall k \in \mathbb{N}^*$; $\forall q \in \mathbb{N} + 2$

(1.12) $p_{k+1} - p_k \leq 2(\sqrt[q]{p_k - 1} - 1)$

(1.13) $p_{k+1} - p_k = o(\sqrt[q]{p_k})$

(1.14) $\sqrt{p_{k+1}} - \sqrt{p_k} < \frac{15}{p_k^{0.4}}$

2 Definitions and notations

(2.1) $[x]$ designate the entire part of the real x.

(2.2) $\mathcal{P}$ designate the infinite set of positive primes , (called primes).

$$(p_1 = 2 ; p_2 = 3 ; p_3 = 5 ; p_4 = 7 ; p_5 = 11 \ldots\ldots).$$

(2.3) π is the prime counting function.

(2.4) $\forall\, k \in \mathbb{N} + 2$, $J(N)$ designate the real interval $](N-1)^2; N^2 + N^{1.05}]$.

(2.5) $N_0 = 1048586$ (Legendre-Iwaniec constant [12], [15]).

(2.6) $\ln(x)$ is the neperian logarithm of the real number x, $(x > 0)$.

(2.7) $\ln_2 (x) = \ln(\ln(x))$ designate the iterated neperian logarithm of order two defined for $x > e$.

(2.8) $\forall\, N \in \mathbb{N} + 2$, the primes p_N^- and p_N^+ are defined by :

$$p_N^- = \text{Sup}(p \in \mathcal{P} : p < N^2) \quad \text{and} \quad p_N^+ = \text{Inf}(p \in \mathcal{P} : p > N^2).$$

(2.9) $\forall\, k \in \mathbb{N} + 2$, Lg_k is a primes-test-function defined by :

$\forall\, x \in J(N)$, $Lg_k(x) = R_k\sqrt{x}\,\ln(x)$; $p_k = \text{Sup}(p \in \mathcal{P} : p < N^2)$ and $R_k = \sqrt{\frac{p_k}{p_{k+1}}}$.

(2.10) $\forall\, k \in \mathbb{N} + 2$, and $\forall\, k \in \mathbb{N} + 3$, Lq_k is a primes-test-function defined by :

$\forall\, x \in J(N)$, $Lq_k : x \mapsto Rq_k\, x^{1-\frac{1}{q}} \ln(x)$; $p_k = \text{Sup}(p \in \mathcal{P} : p < N^2)$ and $Rq_k = \sqrt[q]{\frac{p_k}{p_{k+1}}}$.

(2.11) Li denotes the integral logarithmic deviation function defined for any real $x \geq 2$ by :

$$\text{Li}(x) = \int_2^x \frac{1}{ln(t)}\, dt$$

(2.12) $\#_{\mathcal{P}}\, (I) = \text{Card}(I \cap \mathcal{P})$ denotes the number of primes contained in the set I.

3 Lemma

Legendre's conjecture is verified for any integer $N < 1048586$.

Proof of Lemma 3.

In this case, $(p_N^-)^{0.525} < 2N+1$, so from Iwaniec's results, there is at least one prime

$q \in]p_N^- ; p_N^- + (p_N^-)^{0.525}]$, so $q \in I(N)$.

4 Background

Recall the limited development and framing of Cipolla [9] , Arias [2], Axler [3],[4] , Massias, Robin [18] , Salvy [21].

(4.1) $$\frac{p_k}{k} = \ln(k) + Cip(k;m) + O\left(\left(\frac{ln_2(k)}{ln(k)}\right)^{m+1}\right)$$ with :

$Cip(k;m) = \sum_{q=1}^{m} \frac{P_q(ln_2(k))}{ln(k)^q}$ where, P_q are polynomials such that

$d^o P_q = q$ and verifying the following recursive relation Salvy [21], Axler [3],[4] Dusart [12],[13]

(4.2) $$P_q = q P_{q-1} - P'_{q-1} + \frac{1}{q}\sum_{j=1}^{q-1} j P_{q-j-1}[(j-1)P_{j-1} - P_j - P'_{j-1}]$$

(4.3) Framings and limited development following Axler [3],[4] and Dusart [12],[13].

a) $\frac{p_k}{k} = \ln(k) + \ln_2(k) - 1 + \frac{\ln_2(k)-2}{\ln(k)} + O\left(\frac{1}{\ln(k)^2}\right)$

b) $\ln(k) + \ln_2(k) - 1 < \frac{p_k}{k} < \ln(k) + \ln_2(k) - 0.9247$

c) $k>46254381$, $\ln(k) + \ln_2(k) - 1 + \frac{ln_2(k)-2}{ln(k)} - \frac{ln_2^2(k)-6ln_2(k)+10.667}{2ln^2(k)} < \frac{p_k}{k} <$

$$ln(k) + ln_2(k) - 1 + \frac{ln_2(k)-2}{ln(k)} - \frac{ln_2^2(k)-6ln_2(k)+11.508}{2ln^2(k)}$$

5 Lemma

(5.1) $\forall\, k \in \mathbb{N}+2,\quad Lg_k(p_{k+1}) - Lg_k(p_k) = \sqrt{p_k}\,\{\ln(\frac{p_{k+1}}{p_k}) + \ln(p_k)[1 - R_k]\}\quad ,\ (R_k = \sqrt{\frac{p_k}{p_{k+1}}}\ ).$

Proof of Lemma 5.

$$Lg_k(p_{k+1}) - Lg_k(p_k) = R_k(\sqrt{p_{k+1}}\ln(p_{k+1}) - \sqrt{p_k}\ln(p_k)) = \sqrt{p_k}\ln(p_{k+1}) - \frac{p_k}{\sqrt{p_{k+1}}}\ln(p_k) =$$

$$= \sqrt{p_k}\,ln(p_{k+1}) - \sqrt{p_k}\,ln(p_k) + \sqrt{p_k}.ln(p_k) - \frac{p_k}{\sqrt{p_{k+1}}}\,ln(p_k) =$$

$$= \sqrt{p_k}\,(ln(p_{k+1}) - ln(p_k)) + \sqrt{p_k}.ln(p_k)[1 - R_k]$$

Then,

$$Lg_k(p_{k+1}) - Lg_k(p_k) = \sqrt{p_k}\,\{\ln(\frac{p_{k+1}}{p_k}) + \ln(p_k)\,[1 - R_k]\,\}$$

6 Lemma

(6.1) $$\frac{p_{k+1}}{p_k} = (\frac{k+1}{k})(\frac{ln(k+1)+ln_2(k+1)-1+O(\frac{1}{ln(k)})}{ln(k)+ln_2(k)-1+O(\frac{1}{ln(k)})}) = 1 + \frac{ln(k)+ln_2(k)+\frac{1}{ln(k)}}{(ln(k)+ln_2(k)-1)\,k} + O(\frac{1}{k^2})$$

(6.2) $$\frac{p_k}{p_{k+1}} = 1 - \frac{ln(k)+ln_2(k)+\frac{1}{ln(k)}}{(ln(k)+ln_2(k)-1)\,k} + O(\frac{1}{k^2})$$

(6.3) $$R_k = \sqrt{\frac{p_k}{p_{k+1}}} = 1 - \frac{ln(k)^2 + ln_2(k)\,ln(k) + 1}{2ln(k)\,(ln(k)+ ln_2(k) - 1)\,k} + O(\frac{1}{k^2})$$

Proof of Lemma 6.

$\frac{p_k}{k}$ = ln(k) + $\ln_2(k)$ - 1 + O($\frac{1}{ln(k)}$), hence formulas (6.1) , (6.2) and (6.3) by quotient and radical of limited development.

7 Theorem

(7.1) $$Lg_k(p_{k+1}) - Lg_k(p_k) = \sqrt{p_k}\ \{ \frac{1}{k} + \frac{1}{k\ \ln(k)} + \frac{ln(p_k)}{2k} \} + O(\frac{1}{k^{1.5}})$$

(7.2) $$Lg_k(p_{k+1}) - Lg_k(p_k) = (\frac{\frac{1}{ln(k)} + ln(k) + ln_2(k)}{\sqrt{ln(k)+ln_2(k)-1}} + \frac{(ln(k)^2+ln_2(k)ln(k)+1)(ln(ln(k)+ln_2(k))-1)-ln(k))}{2ln(k)\sqrt{ln(k)+ln_2(k)-1}})\sqrt{\frac{1}{k}} + O(\frac{1}{k^{1.5}})$$

(7.3) $$Lg_k(p_{k+1}) - Lg_k(p_k) \sim 0.5\ \ln_2(k) \sqrt{\frac{ln(k)}{k}}, \quad (k \to +\infty)$$

Proof of Theorem 7.

With Lemma 6, and the rules of limited development theorem 7 is validated.

8 Corollary

(8.1) $\forall\ k \in \mathbb{N}^*$, $Lg_k(p_{k+1}) - Lg_k(p_k) < 1$

(8.2) $\lim \Delta Lg_k\ [p_k\ ; p_{k+1}] = 0$

Proof of Corollary 8.

(8.1) : Validation using Maplesoft Maple for the first terms, then majorizations of the limited developments of (7.2).

(8.2) : Terms in in the limited development of (7.23) which allows us to conclude.

9 Theorem

(9.1) $\forall\, k \in \mathbb{N}^*$, $$Lg_k(p_{k+1}) - Lg_k(p_k) < 0.5\,\frac{\ln(p_k)^{1.5}}{\sqrt{p_k}}$$

Proof of Theorem 9.

10 Lemma

(10.1) $$Lg_k((N+1)^2) - Lg_k(N^2) > 2R_k\,\frac{(2N+1)}{(N+1)}(\ln(N+1)+1)$$

$$> 4\ln(N)\text{ , for } N \text{ large enough}$$

(10.2) $$\lim\ \Delta\, Lg_k[N^2;(N+1)^2] = +\text{oo}$$

Proof of Lemma 10.

For any non-zero integer k, the function Lg_k is derivable on $]p_k;p_{k+1}[$ and $Lg'_k(x) = \frac{ln(x)+2}{\sqrt{x}}$; so, using the inequality of finite increments applied to function Lg_k and the real interval $I(N) =]N^2;(N+1)^2[$, we deduce that :

$$\mathrm{Lg}_k((N+1)^2) - \mathrm{Lg}_k(N^2) > R_k\ \mathrm{Lg}'_k((N+1)^2)$$

$$> R_k\ (2N+1)(\ln((N+1)^2)+2)/\sqrt{(N+1)^2}$$

$$> 2R_k\,\frac{(2N+1)}{(N+1)}\,(ln(N+1)+1)$$

$> 4ln(N)$

11 Theorem *(Legendre Conjecture)*

(11.1) **$\forall\ N \in \mathbb{N}^*$, the interval of integers $I(N) =]N^2;(N+1)^2[$ contains at least one prime.**

$$(\#_{\mathcal{P}}\ (I(N)) \geq 1)$$

Proof of theorem 11.

We argue by recursion and by the absurd :

For every integer $N>1$. Let $P_L(N)$ be the property :

(11.2) ***$P_L(N)$: " $I(N-1)$ contains at least one prime ".***

a) According to Lemma 4, for every integer $i = 2$ to $N_0 - 1$, $P_L(i)$ is true .

b) Let us show that for every integer N, the property $P_L(N)$ is hereditary i.e.

$$(P_L(N) \Rightarrow P_L(N+1)).$$

We suppose that $P_L(N)$ is true; therefore $I(N-1)$ contains at least one prime. Let $p = p_N^-$.

Assuming that the corresponding interval $I(N)$ verifies :

$I(N) \cap \mathcal{P} = \emptyset$, then we put in evidence a contradiction . It follows . :

$$(N + 1)^2 - N^2 < 2N+1 < q - p$$

But ,

(11.3) $\mathrm{Lg_k}(q) - \mathrm{Lg_k}(p) < 1$ according to Corollary 8.

So there exists $m \in \mathbb{N}^*$ such that $p = p_m$ and $q = p_{m+1}$, it follows that :

Now, according to the inequality of finite increases, applied to

the function $\mathrm{Lg_k}$, C^∞ on $[p;q]$, it follows :

(11.4) $\mathrm{Lg_k}((N+1)^2) - \mathrm{Lg_k}(N^2) > R_k((N+1)^2 - N^2)\,\mathrm{Inf'}(\mathrm{Lg}'_k(x)\,;\, x \in \mathrm{J}(N))$

$$> R_k\,(2N+1)\mathrm{Lg}'_k(N+1)$$

$$> 2R_k\frac{(2N+1)}{(N+1)}(\ln(N)+1)$$

$$> 4\ln(N)$$

$$\gg 1 \quad (\lim\ \ln(N) = +\infty)$$

Then finally ,

$$\mathrm{Lg_k}((N+1)^2) - \mathrm{Lg_k}(N^2) \gg 1 > \mathrm{Lg_k}(q) - \mathrm{Lg_k}(p)$$

We end up with an obvious contradiction because : $p < N^2 < (N+1)^2 < q$ and then ,

$\mathrm{Lg_k}((N+1)^2) - \mathrm{Lg_k}(N^2) < \mathrm{Lg_k}(q) - \mathrm{Lg_k}(p)$ because $\mathrm{Lg_k}$ is a C^∞ strictly increasing function defined on $\mathrm{J}(N)$.

So the hypothesis is false and $\mathrm{I(N)} \cap \mathcal{P} \neq \emptyset$. Therefore, the property $\mathrm{P_L}(N)$ is hereditary and since for every integer i = 2 to $\mathrm{N_0}$ - 1 , $\mathrm{P_L}(\mathrm{i})$ and $\mathrm{P_L}(N_0)$ are true, we deduce by recurrence that the property $P_L(N)$ is true for any integer $N \in \mathbb{N} + 2$. There is no N such that $I(N) \cap \mathcal{P} = \emptyset$,

so for any integer $N \in \mathbb{N} + 2$, $\mathrm{I}(N)$ contains at least one prime. Thus , the Legendre conjecture is validated.

12 Corollary

(12.1) $\forall k \in \mathbb{N}^*,\ \ p_{k+1} - p_k \leq [4.\sqrt{p_k - 1} + 2]$

Proof of Corollary 12.

From the results of Theorem 11 on Legendre's conjecture, then

for any integer $k \in \mathbb{N}^*$, there exists an integer N such that the following inequalities are verified.

(12.2) $(N-1)^2 + 1 \leq p_k \leq N^2 - 1 \leq N^2 + 1 \leq p_{k+1} \leq (N+1)^2 - 1$

So, $p_{k+1} - p_k \leq (N+1)^2 - (N-1)^2 - 2 = 4N - 2 = 4(N-1) + 2$

(12.3) $p_{k+1} - p_k \leq 4.\sqrt{p_k - 1} + 2$

(12.4) $p_{k+1} - p_k \leq [4.\sqrt{p_k - 1} + 2]$

13 Theorem

$\forall N \in \mathbb{N}, (N > \mathbf{10^3})$, the interval $I(N) =]N^2; (N+1)^2[$ contains at least

$\boldsymbol{K^-}(N) = 0.78(N + 0.5) / (\ln(N) - 0.5)$ and at most, $\boldsymbol{K^+}(N) = 1.03(N + 0.5)/ (\ln(N) - 0.55)$ *primes .*

$\#_{\mathcal{P}}(I(N))$ verifies :

(13.1) $800 < N < 10000$ $0.3N / \ln(N)^{0.5} < \#_{\mathcal{P}}(I(N)) < 1.03(N + 0.5)/ (\ln(N) - 0.55)$

(13.2) $10000 < N < 100000$ $0.287N / \ln(N)^{0.5} < \#_{\mathcal{P}}(I(N)) <$

(13.3) $10^5 < N < 10^6$ $0.287N / \ln(N)^{0.53} < \#_{\mathcal{P}}(I(N)) <$

(13.4) $N > 10^6$ $\#_{\mathcal{P}}(I(N)) > 0.3N / [\ln(N)]^{1.2}$

***Examples:**

Proof of Theorem 13.

By recurrence , for any integer $N > 20$, let $P(N)$ be the following property :

$P(N)$: " The interval $I(N-1)$ contains at least $0.78(N + 0.5) / (\ln(N) - 0.5)$ and

at most $1.03(N + 0.5)/ (\ln(N) - 0.55)$ primes ".

P(20) is verified ; in fact, there are 7 prime numbers between 400 and 441 et $K^-(20) =$
$0.78\text{x}20.5/(\ln(20)-0.5) \approx 6.41 < 7$ et $K^+(20) = 1.03\text{x}20.5/(\ln(20)-0.55) \approx 7.11 > 7$.

b) Let us show that the property *P(N)* is hereditary : $(P(N) \Rightarrow P(N+1))$

Assume P(N) is true : then, $\pi((N+1)^2) - \pi(N^2) > K^-(N)$

14 Properties of the conjectures of Andrica and Oppermann.

The same type of reasoning as in theorem 11 is developed adapting the intervals of investigation and the primes-test-functions to prove the conjectures of Andrica, Oppermann and Brocard [1].

From this we deduce the framings :

(14.1) $$\sqrt{p_{k+1}} - \sqrt{p_k} < 1$$

(14.2) $$p_{k+1} - p_k \leq 2\sqrt{p_k}$$

(14.3) $\forall\ q \in \mathbb{N}+2$ $$p_{k+1} - p_k = o(\sqrt[q]{p_k})$$

(14.4) The sequence $W_n = \sqrt{p_{n+1}} - \sqrt{p_n}$ is positive , of zero limit and verifies :

$$W_k = \sqrt{p_{k+1}} - \sqrt{p_k} < \frac{15}{p_k^{0.4}}$$

15 Theorem *(Andrica conjecture).*

For any integer $k \in \mathbb{N}^*$ and prime p_k , the following inequality holds :

(15.1) $$\sqrt{p_{k+1}} - \sqrt{p_k} < 1$$

(15.2) $$\sqrt{p_{k+1}} - \sqrt{p_k} \leq 1 - \frac{1}{\sqrt{p_k}} + \frac{0,875}{p_k} - \frac{1,6875}{p_k\sqrt{p_k}} + \frac{307}{128p_k^2}$$

Proof of theorem 15.

Using the same method as for Legendre's conjecture (Theorem 11), but working on intervals of the type :

$I_1(N) =]N^2;N^2 + N[$, and $I_2(N) =]N^2+N;(N+1)^2[$, it follows :

$\forall\ N \in \mathbb{N}^*$, each interval $I_1(N) =]N^2;N^2 + N[$ *and* $I_2(N) =]N^2+N;(N+1)^2[$ contains at least one prime.

It provides a square-root majorization of the gap between two consecutive primes p_k and p_{k+1} of the form :

(15.3) $$\forall k \in \mathbb{N}^*,\ p_{k+1} - p_k \leq 2.\sqrt{p_k}$$

In fact,

$$(N-1)^2 + (N-1) + 1 \leq p_k \leq N^2 - 1 \leq N^2 + 1 \leq p_{k+1} \leq N^2 + N - 1$$

We deduce :

$$p_{k+1} - p_k \leq N^2 + N - (N-1)^2 - (N-1) - 2 = 2N - 2 = 2(N-1) :$$

So,

(15.4) $$\forall k \in \mathbb{N}^*, \qquad p_{k+1} - p_k \leq [2\sqrt{p_k - \sqrt{p_k + 1}}\,]$$

It follows :

(15.5) $$\forall k \in \mathbb{N}^*,\ \sqrt{p_{k+1}} - \sqrt{p_k} = \frac{p_{k+1} - p_k}{\sqrt{p_{k+1}} + \sqrt{p_k}}$$

(15.6) $$\forall k \in \mathbb{N}^*,\ \sqrt{p_{k+1}} - \sqrt{p_k} \leq \frac{2\sqrt{p_k}}{\sqrt{p_{k+1}} + \sqrt{p_k}}$$

However, $\forall k \in \mathbb{N}^*,\ p_k + 2 \leq p_{k+1}$ then,

(15.7) $$\forall k \in \mathbb{N}^*, \qquad \sqrt{p_{k+1}} - \sqrt{p_k} \leq \frac{2\sqrt{p_k}}{\sqrt{p_{k+1}} + \sqrt{p_k+2}} < 1$$

By using *(with Maple)* the Taylor expansion in the vicinity of $x = +\infty$ of the function f defined on *[2;+∞[* by :

$$f(x) = \sqrt{x + 2\sqrt{x - \sqrt{x+1}}} - \sqrt{x}$$

Relation (15.4) is verified, then :

(15.8) $$\sqrt{p_{k+1}} - \sqrt{p_k} \leq \sqrt{p_k + 2\sqrt{p_k - \sqrt{p_k + 1}}} - \sqrt{p_k} \leq 1 - \frac{1}{\sqrt{p_k}} + \frac{0.875}{p_k} - \frac{1.6875}{p_k^{1.5}} + \frac{307}{128p_k^2}$$

16 Corollary

(16.1) $\forall k \in \mathbb{N}^*, \quad p_{k+1} - p_k \leq [2\sqrt{p_k - \sqrt{p_k + 1}}\,]$

Proof of Corollary 16

It is given at the beginning of the proof of theorem 15 with formula (15.4) .

17 Theorem

The sequence (W_n) defined by : $\forall\, n \in \mathbb{N}^*$, $W_n = \sqrt{p_{n+1}} - \sqrt{p_n}$ is positive increased by $M = \sqrt{11} - \sqrt{7}$, and verifies :

(17.1) $\lim\, (W_n) = 0$

Proof of theorem 17.

(W_n) is a positive sequence because the sequence of primes and the square root function are strictly increasing on $]0;+\infty[$; its maximum is : $M = \sqrt{11} - \sqrt{7} \approx 0.670873$.

Using the results of theorem 14, then ,

for any real $\varepsilon > 0$, there exists an integer n_ε such that :

$$\forall\, n \in \mathbb{N}^* ; (n > n_\varepsilon) , \ \sqrt{p_{n+1}} - \sqrt{p_n} = \frac{p_{n+1} - p_n}{\sqrt{p_{n+1}} + \sqrt{p_n}} \leq \frac{\varepsilon\sqrt{p_n}}{\sqrt{p_{n+1}} + \sqrt{p_n}}$$

$$\sqrt{p_{n+1}} - \sqrt{p_n} \leq \frac{\varepsilon\sqrt{p_n}}{\sqrt{p_n+2} + \sqrt{p_n}}$$

$$\leq \frac{\varepsilon}{\sqrt{1+\frac{2}{p_n}}+1}$$

$$\leq 0.5\ \varepsilon$$

thus,

$$\lim (W_n) = \lim(\sqrt{p_{n+1}} - \sqrt{p_n}) = 0 .$$

18 Corollary

(18.1) $\forall n \in \mathbb{N}^*,$ $W_n = \sqrt{p_{n+1}} - \sqrt{p_n} < \frac{15}{p_n^{0.4}}$

Proof of Corollary 18.

We reason by the absurd, assuming that there are two consecutive primes p_k and p_{k+1} such that :

$\sqrt{p_{k+1}} - \sqrt{p_k} > \frac{15}{p_k^{0.4}}$; therefore , $\sqrt{p_{k+1}} > \sqrt{p_k} + \frac{15}{p_k^{0.4}}$; *then,*

by squaring ,

$p_{k+1} > (\sqrt{p_k} + \frac{15}{p_k^{0.4}})^2 = p_k + 30p_k^{0.1} + \frac{225}{p_k^{0.8}} \Rightarrow p_{k+1} - p_k > 30p_k^{0.1} + \frac{225}{p_k^{0.8}}$; but,

according to theorem 17 (17.2) and by using Maple for $k > 10^4$, $p_{k+1} - p_k < 25\ \sqrt[20]{p_k}$.

let f and g be two functions defined on $[2 ; +\infty[$ by :

$f(x) = 30x^{0.1} + \frac{225}{x^{0.8}}$ and $g(x) = 25x^{0.05}$. Since $f(x) - g(x) > 21$, a contradiction is revealed because it is impossible to have $p_{k+1} - p_k > f(p_k) > g(p_k) + 21$ *and* $g(p_k) > p_{k+1} - p_k$; so inequality (16.1) holds for all $k \in \mathbb{N}^*$.

<u>*** Remark :**</u> It can be shown that : $\sqrt{p_{n+1}} - \sqrt{p_n} < \frac{K1}{p_n^c}$ $(0 < c < 0.5)$ *(K1 > 0 very large constant)* using the same principle.

19 Corollary *(Oppermann conjecture)*

(19.1) The intervals I_{p1} = $]N^2;N(N+1)[$ and I_{p2} = $]N(N+1);(N+1)^2[$ contains at least one prime.

Proof of Corollary 19.

According to Corollary 16 , we use the same method with disjoint intervals partitions of I(N) and of length approximately equal to N.

Remark : The proof of Brocard's conjecture follows directly from those of Andrica and Oppermann , (see Paz[ ],[8],[28]).

20 Lemma

(20.1) $\forall\, k \in \mathbb{N}+2,\quad \mathrm{Lq_k}(p_{k+1}) - \mathrm{Lq_k}(p_k) = \sqrt[q]{p_k}\ \{ \ln(\frac{p_{k+1}}{p_k}) + \ln(p_k)\ [1 - Rq_k] \}\quad ;\ (Rq_k = \sqrt[q]{\frac{p_k}{p_{k+1}}}).$

(20.2) $Rq_k = \sqrt[q]{\frac{p_k}{p_{k+1}}} = 1 - \frac{\ln(k)^2 + \ln_2(k)\ln(k) + 1}{q\, ln(k)\ (ln(k) + ln_2(k) - 1)\ k} + O(\frac{1}{k^2})$

(20.3) $\mathrm{Lq_k}(p_{k+1}) - \mathrm{Lq_k}(p_k) = \sqrt[q]{p_k}\ \{ \frac{1}{k} + \frac{1}{k\, ln(k)} + \frac{\ln(p_k)}{q\, k} \} + O(\frac{1}{k^{(2-\frac{1}{q})}})$

(20.4) $\mathrm{Lq_k}(p_{k+1}) - \mathrm{Lq_k}(p_k) = (\frac{\frac{1}{ln(k)} + ln(k) + ln_2(k)}{\sqrt{ln(k) + ln_2(k) - 1}} +$

$\frac{(ln(k)^2 + ln_2(k) ln(k) + 1)(ln(ln(k) + ln_2(k) - 1) - ln(k))}{q\, ln(k)\sqrt{ln(k) + ln_2(k) - 1}}) \frac{1}{k^{(1-\frac{1}{q})}} + O(\frac{1}{k^{(2-\frac{1}{q})}})$

(20.5) $\mathrm{Lq_k}(p_{k+1}) - \mathrm{Lq_k}(p_k) \sim \ln_2(k) \frac{\sqrt{ln(k)}}{q\, k^{(1-\frac{1}{q})}}$

Proof of Lemma 20.

using operating rules for limited developments (controlled by Maple software)

21 Corollary

(21.1) $\forall\, q \in \mathbb{N}+3\,, \exists\, K_q \in \mathbb{N}^* \,/\, \forall\, k \in \mathbb{N}^*\,,\ \ k > K_q \qquad \mathrm{Lq_k}(p_{k+1}) - \mathrm{Lq_k}(p_k) < 1$

(21.2) $\lim \Delta \mathrm{Lq_k}\,[p_k\,;p_{k+1}] = 0$

Proof of Corollary 21.

We examine the first terms using Maple and solve an inequality via limited developments.

22 Theorem

(22.1) $\forall\, k \in \mathbb{N}^*,\qquad \mathrm{Lq_k}(p_{k+1}) - \mathrm{Lq_k}(p_k) < \mathrm{C}\ \dfrac{ln(p_k)^{(1-\frac{1}{q})}}{q\ k^{(1-\frac{1}{q})}}$

Proof of Theorem 22.

23 **Lemma**

(23.1) for $N > N_q$ $\quad Lq_k((N+1)^2)$ - $Lq_k(N^2) > 2Rq_k \frac{(2N+1)}{(N+1)}(ln(N+1)+1)$

$> 4\ln(N)$

(23.2) $\lim \Delta Lq_k[N^2;(N+1)^2] = +oo$

Proof of Theorem 23.

24 Theorem

(24.1)

For any integer $q \geq 2$, there are two integers N_q and K_q such that :

For every integers $N > N_q$, $k > K_q$ and $p < N^{q-1}(N-1)$ the interval

$I_q(N,p) =]N^q + pN ; N^q + (p+1)N[$ *contains at least one prime .*

(24.2) For any integer $k>Kq$ $\quad p_{k+1} - p_k \leq 2(\sqrt[q]{p_k - 1} - 1)$

(24.3) $\forall q \in \mathbb{N} + 3 \quad p_{k+1} - p_k = o(\sqrt[q]{p_k}), (k \to +\infty)$

Proof of theorem 24.

For any integer $q \geq 3$, let us show that there are two integers N_q and K_q such that :

For every integers $N > N_q$, $k > K_q$ *and* $p < N^q - N^{q-1}$ the interval

$I_q(N,p) =]N^q + pN ; N^q + (p+1)N[$ contains at least a prime p_k. The same method of demonstration is used as for Legendre's conjecture in theorem 11 .

We argue by recursion and by the absurd :

For every integer $N > q^{2.1}$, let *Pr(N)* be the property :

a) $f(q^{2.1q} + 2q^{2.1}) - f(q^{2.1q} + q^{2.1}) > q^{2.1}$

b) (14.1) *Pr(N) : " I_q (N-1,p) contains at least one prime ".*

$N^q + pN + 1 \leq p_k \leq N^q + (p+1)N - 1 \leq p_{k+1} \leq N^q + (p+2)N - 1$

$p_{k+1} - p_k \leq N^q + (p+2)N - 1 - N^q - pN - 1 = 2N - 2 \leq 2\sqrt[q]{p_k - 1} - 2$

25 Perspectives and generalizations *(of same type)*

(25.1) $\forall n \in \mathbb{N}^*$, $\sqrt[q]{p_{n+1}} - \sqrt[q]{p_n} < \frac{A_q}{p_n^c}$, $(0 < c < \frac{1}{q})$ $(A_q > 0$ *constant)*

$Primes_q^-(N) = Cm_q N / q\ ln(N)$ *and* $Prismes_q^+(N) = CM_q.N / q\ ln(N)$

We perform the same kind of reasoning by adjusting the intervals of investigation and the minor functions of the variations of the function π to prove the conjecture of Cramer([8]).

In the proof of the Legendre conjecture (Theorem 11 same argumentation) we replace *I(N)* by

$I_C(N,q) = [N + q\,ln^2(N) ; N + (q+1)\,ln^2(N)]$, and Lg_k par $hc_k(x) = c_1 x / (ln(x) - c_2 tanh(x - p_k)) + c_3$

Then, we try to adjust the coefficients *c1,c2,c3* (functions of p_k and p_{k+1}) so that hc_k verifies :

$\forall x \in]p_k ; p_{k+1}[, k < hc_k(x) < k+1$

26 Conclusion

These results are essentially due to the general (global) form of the prime counting function π and therefore, to the variations of its lower convex envelope f (positive concave function), which must satisfy a framework of the form :

$$(26.1) \qquad \forall a,b \in \mathbb{R}, \ (1<a<b) \qquad k_1 \frac{b-a}{ln(b)^{1+\varepsilon}} < f(b) - f(a) < k_2 \frac{b-a}{ln(a)^{1+\varepsilon}}.$$

With this type of method, Cramer and Firoozbakht's conjectures can probably be improved.

Ou test identique avec d'autres fonctions (plus proches des variations de l'enveloppe convexe inf de π)

du type $f_k \sim \frac{c(p_k)\, f}{c(p_{k+1})}$, $f = Li$, $x/(ln(x)-A)$,..........to prove the **<u>CRAMER</u>** conjecture.

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