Vector Product Approach for Optimization Resolution (VPAOR)

Mamadou NDAO

August 30, 2024

Abstract

This document presents a new method for solving constrained optimization problems, an alternative to the Lagrange multipliers. We introduce the Vector Product Approach for Optimization Resolution (VPAOR), which uses properties of vector products to simplify optimization problems. Our results demonstrate that this approach is an alternative to traditional methods, offering an effective solution for various problems.

1 Introduction

Constrained optimization problems are essential in various fields such as logistics, economics, and engineering. Traditionally, Lagrange multipliers are used to solve these problems by transforming constraints into penalties within the objective function. However, this method has limitations, particularly when dealing with constraints and is not easily solvable by individuals with relatively lower mathematical knowledge. Thus, this new method could be beneficial to a broader audience.

1.1 Objectives of the Contribution

In this document, we introduce the Vector Product Approach for Optimization Resolution (VPAOR). This method relies on concepts from vector products and the implicit function theorem to reformulate constrained optimization problems. We aim to show that this approach offers greater flexibility and simplifies calculations compared to traditional methods while maintaining efficiency and accuracy.

1.2 Contributions and Structure of the Document

The document is structured as follows:

- **Section 2** : Review of traditional methods for constrained optimization, including Lagrange multipliers.

1.2.1 Introduction

The method of Lagrange multipliers is a technique used to find the optimal points of a function under constraints. This method allows converting a constrained optimization problem into an unconstrained problem by introducing a Lagrange multiplier.

1.2.2 Problem Formulation

Let $f(x_1, x_2, \ldots, x_n)$ be an objective function that we wish to optimize (maximize or minimize) under the constraint $g(x_1, x_2, \ldots, x_n) = 0$. We introduce a Lagrange multiplier λ and define the Lagrange function $\mathcal{L}$ as follows:

$$\mathcal{L}(x_1, x_2, \ldots, x_n, \lambda) = f(x_1, x_2, \ldots, x_n) + \lambda \cdot g(x_1, x_2, \ldots, x_n).$$

1.2.3 Optimality Conditions

To find the optimal points, we solve the following system of equations:

$$\nabla \mathcal{L} = 0,$$

where $\nabla \mathcal{L}$ is the gradient of $\mathcal{L}$ with respect to the variables $x_1, x_2, \ldots, x_n$ and λ. This translates to the following equations:

$$\frac{\partial \mathcal{L}}{\partial x_i} = 0 \quad \text{for } i = 1, 2, \ldots, n, \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = g(x_1, x_2, \ldots, x_n) = 0. \tag{2}$$

1.2.4 Example

Consider the problem of maximizing $f(x, y) = xy$ under the constraint $g(x, y) = x^2 + y^2 - 1 = 0$. The Lagrange function is:

$$\mathcal{L}(x, y, \lambda) = xy + \lambda(x^2 + y^2 - 1).$$

The optimality conditions are:

$$\frac{\partial \mathcal{L}}{\partial x} = y + \lambda \cdot 2x = 0, \tag{3}$$

$$\frac{\partial \mathcal{L}}{\partial y} = x + \lambda \cdot 2y = 0, \tag{4}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = x^2 + y^2 - 1 = 0. \tag{5}$$

- **Section 3** : Detailed presentation of the Vector Product Approach for Optimization Resolution (VPAOR).

1.2.5 Implicit Function Theorem

Implicit Function Theorem

Let f be a differentiable function on an open set $\Lambda \subset \mathbb{R}^2$ such that $f(a,b) = 0$ with $\frac{\partial f}{\partial y}(a,b) \neq 0$. Then there exists an open set $U \subset \mathbb{R}$ containing a, an open set $V \subset \mathbb{R}$ containing b, and a function $\varphi : U \to V$ such that

$$f(x, \varphi(x)) = 0,\ \forall x \in U,\ \text{and } \varphi(a) = b$$

$$\frac{\partial f}{\partial y}(x, \varphi(x)) \neq 0,\ \text{for all } x \in U,\ \text{and } y = \varphi(x),$$

Therefore,

$$\begin{gathered} \max\ f(x,y)_{\mathbb{R}^n \times \mathbb{R}} \text{ subject to } g(x,y) = 0 \\ \frac{\partial f}{\partial x}(x, \varphi(x)) + \omega^{\cdot}(x)\frac{\partial g}{\partial y}(x, \varphi(x)) = 0 \\ \frac{\partial g}{\partial x}(x, \varphi(x)) + \omega^{\cdot}(x)\frac{\partial g}{\partial y}(x, \varphi(x)) = 0 \\ \langle \nabla f(x, \varphi(x)) \rangle \geq 0 = \langle \nabla g(x, \varphi(x)) \rangle / \langle \frac{1}{\varphi(x)} \rangle \end{gathered}$$

Thus we can conclude that

$$\nabla f(n, \varphi(n)) \text{ is collinear to } \nabla g(n, \varphi(n))$$

- **Conclusion and Future Perspectives**: VPAOR represents an alternative to the Lagrange multiplier theorem. We are moving towards researching solutions to complex problems and also problems with non-differentiable functions.

2 Methodology

2.1 Development of the Vector Approach

The Vector Approach for Optimization Resolution (VPAOR) relies on concepts from vector geometry and the theory of implicit functions. Given the collinearity of functions as indicated by the implicit function theorem stated above, we use the zero vector product for collinear vectors to arrive at the following formula by NDAO.

2.2 Demonstration and Tests

2.2.1 Functions with 2 Variables

$$\langle \nabla f(x, \varphi(x)) \rangle \text{ and } \frac{1}{\varphi(x)} \rangle = 0 = \langle \nabla g(x, \varphi(x)) \rangle / \langle \frac{1}{\varphi(x)} \rangle$$

The φ functions are derived.

Therefore, we can conclude that

$$\nabla f(x, \varphi(x)) \text{ is collinear to } \nabla g(x, \varphi(x))$$

$$\Rightarrow \det\left(\nabla f(x, \varphi(x)), \nabla g(x, \varphi(x))\right) = 0$$

$$\begin{vmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{vmatrix} = 0$$

Thus, for finding x and y, we have the following relation:

NDAO's Formula

$$\frac{\partial g}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial g}{\partial y}\frac{\partial f}{\partial x}$$

Example

Consider the problem of maximizing $f(x, y) = xy$ under the constraint $g(x, y) = x^2 + y^2 - 1 = 0$. (Example 1.2.4) According to NDAO's formula we have:

$$\frac{\partial g}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial g}{\partial y}\frac{\partial f}{\partial x}$$

For this problem, we get:

$$2y^2 = 2x^2 \Rightarrow x^2 = y^2$$

This is the first relation between y and x without using the Lagrange multiplier. Just replace it in $g(x, y) = 0$ to find the values of x and y.

2.2.2 Generalization

3 Variables

$$\nabla f(x, y, \varphi(x, y)) \text{ is collinear to } \nabla g(x, y, \varphi(x, y))$$

Using the zero vector product between 2 collinear vectors, we arrive at the following relation by projecting the 2 vectors onto a coordinate system:

Generalized NDAO's Formula

$$\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} = \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_i}$$

3 Conclusion

In conclusion, we can assert that VPAOR will allow a broader audience to understand the significance of this optimization problem by relying on a single relation.