Proof of the Collatz conjecture based on directed graph

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Abstract

The Collatz conjecture considers recursively sequences of positive integers where n is succeeded by $\frac{n}{2}$, if n is even or $\frac{3n+1}{2}$, if n is odd. The conjecture states that for all starting values n the sequence eventually reaches a trivial cycle 1, 2, 1, 2…….The Collatz sequences can be represented as a directed graph. If the Collatz conjecture is false, then either there is a nontrivial cycle, or one sequence goes to infinity. In this paper, we construct a Collatz directed graph by connecting infinite number of basic directed graphs. Each basic directed graph relates to each natural number. We prove that the Collatz directed graph covers all positive integers and there is only a trivial cycle and no sequence goes to infinity.

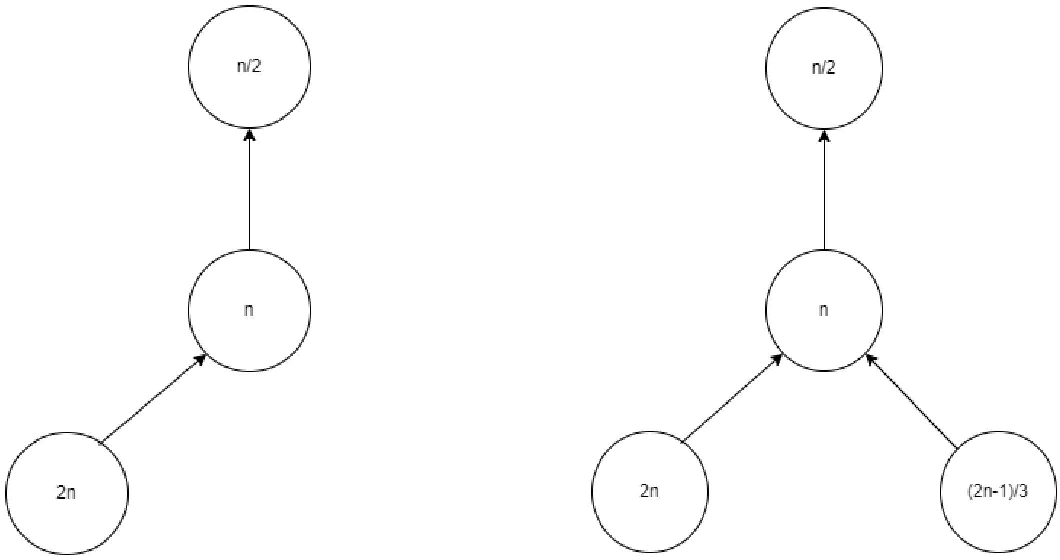
1. Introduction

The Collatz conjecture considers recursively sequences of positive integers where n is succeeded by $\frac{n}{2}$, if n is even, or $\frac{3n+1}{2}$, if n is odd. The conjecture states that for all starting values n the sequence eventually reaches the trivial cycle 1, 2, 1, 2…… The Collatz sequences can be represented as a directed graph. If the Collatz conjecture is false, then either there is a nontrivial cycle, or one sequence goes to infinity [1-2].

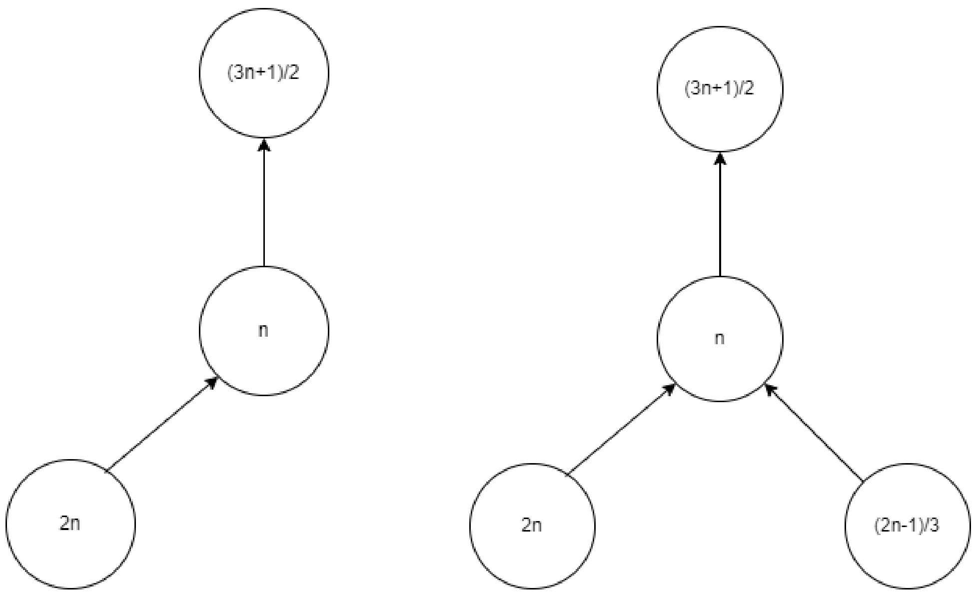
2. A basic directed graph

The basic directed graph is constructed for each natural number as follows:

Let n be a positive integer node. Its parent node is $\frac{n}{2}$, if n is even or $\frac{3n+1}{2}$, if n is odd. Its left child is 2n. Its right child is $\frac{2n-1}{3}$, if $n \equiv 2 \bmod 3$, or no right child , if $n \neq 2 \bmod 3$. Thus there are four types of basis directed graph as shown in Figure 1.



(a) n is even and not equal to 2 mod 3 (b) n is even and equals to 2 mod 3



(c) n is odd and not equal to 2 mod 3 (d) n is odd and equals to 2 mod 3

Figure 1, Four types of basic directed graphs

Examples of basic directed graphs shown in Figure 2.

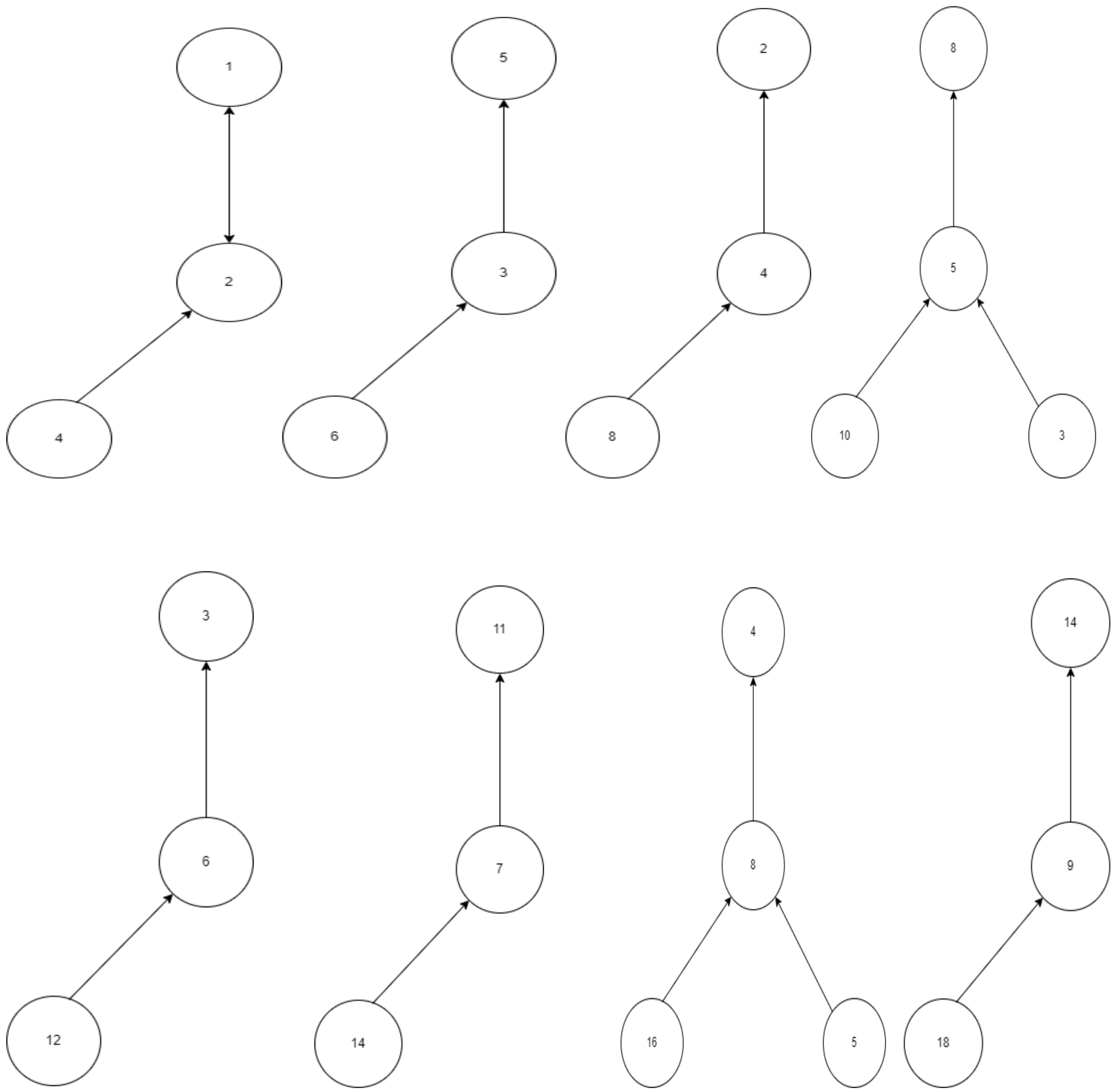


Figure 2. Basic directed graphs of 2, 3, 4, 5, 6, 7, 8, and 9

2. **How to connect two basic directed graphs**

A simple rule to connect two basic directed graph is that these two basic directed graphs must have a common edge as shown in Figure 3.

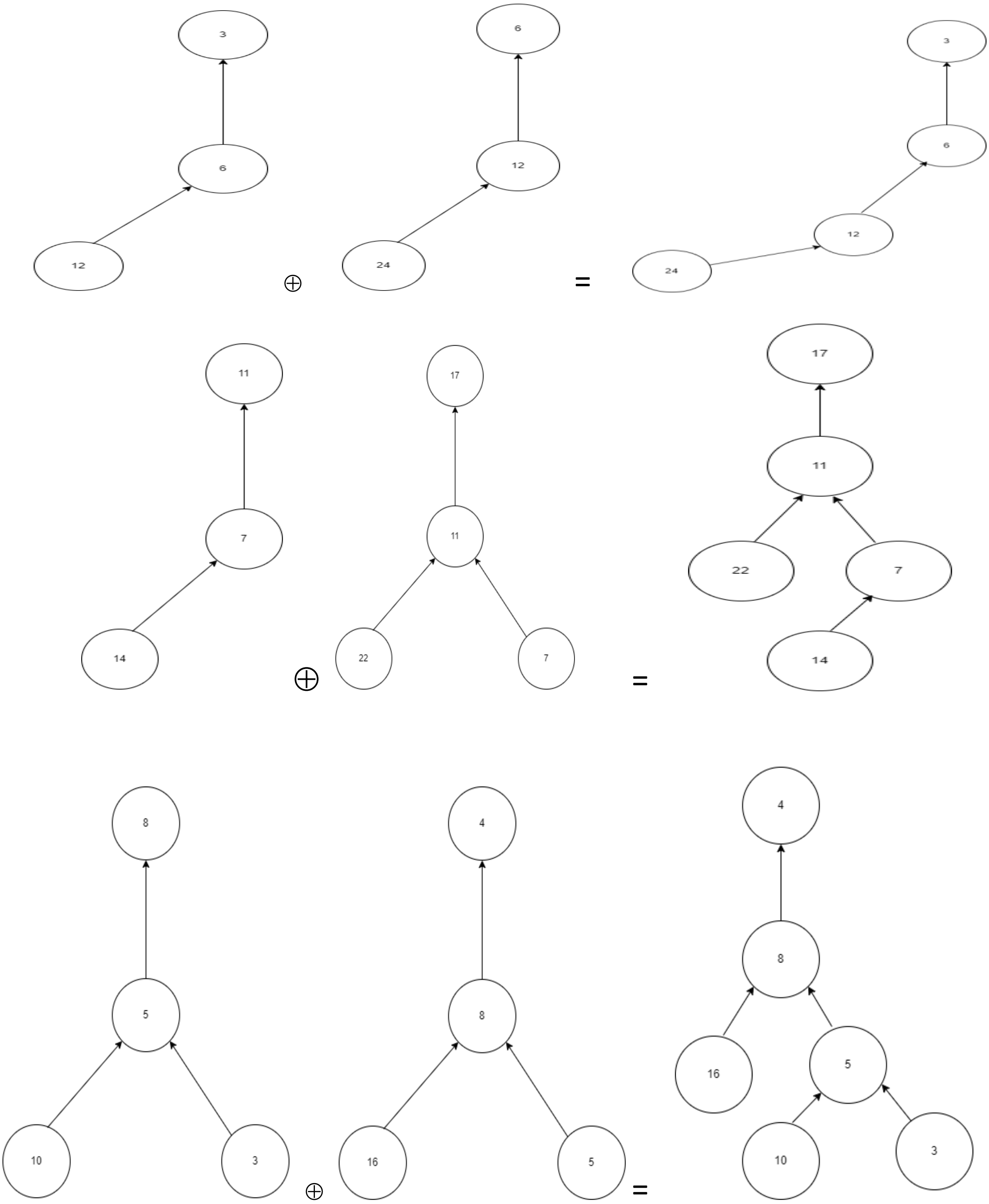


Figure 3. Various cases of basic graphs connection

3. **A Collatz directed graph**

Let start with a basic graph of node 2, connecting basic graphs according to connection rules, the complete Collatz directed graph is shown in Figure 4. This graph is arranged in levels 0 to ∞. There is only node 1 in level 0. For $i \geq 3$, number of nodes in level i is less than number of nodes in level i+1. There is no nontrivial cycle or divergence sequence in this graph. But in order to prove the Collatz conjecture, this graph must cover all positive integers.

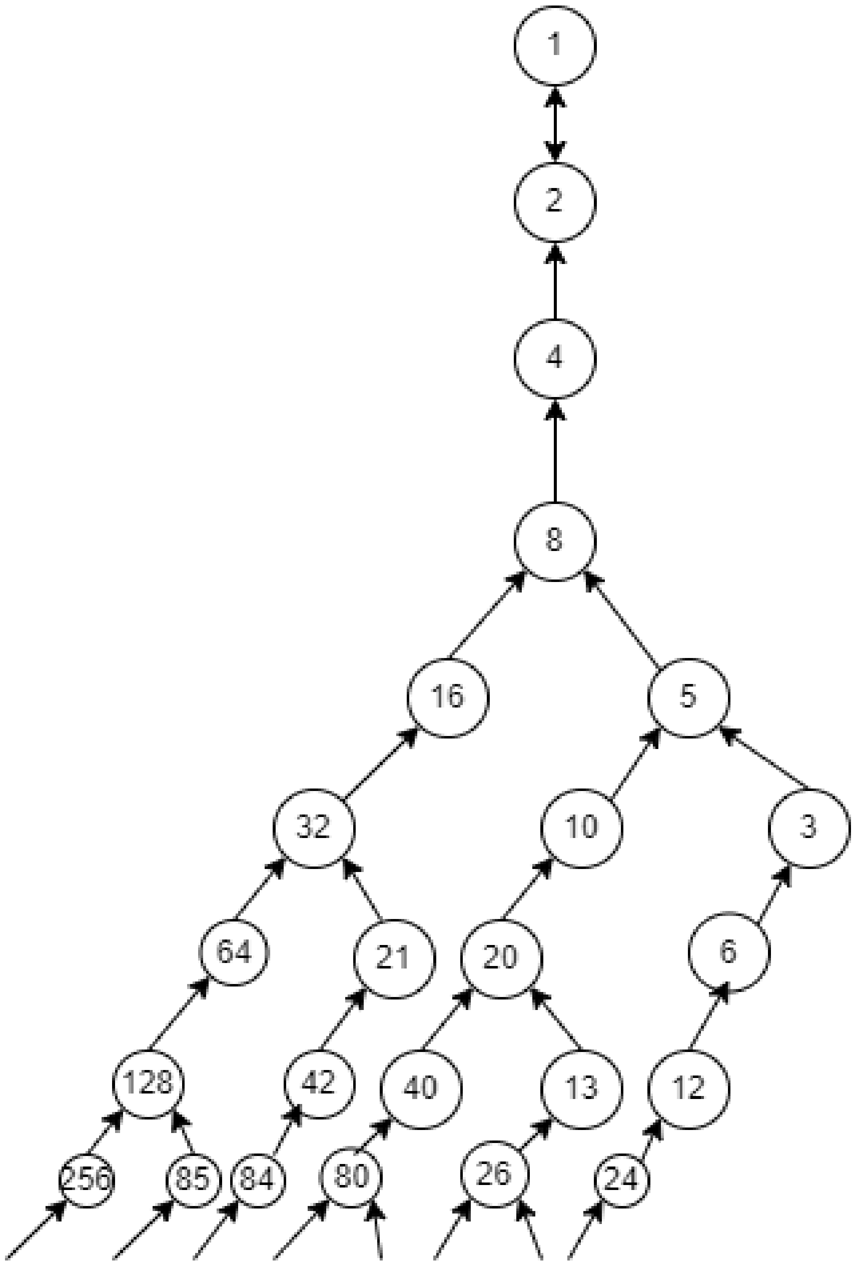


Figure 4. A complete Collatz directed graph

We can show that there is only one Colltaz directed graph for all positive integers. Let assume there is another Collatz directed graph that is completely different from the complete Collatz directed graph shown in Figure 4. For level $i \geq 3$ of this new Collatz directed graph, number of nodes in level i is less than number of nodes in level i+1. Only node 1 which is in the complete Collatz directed graph is in level 0 of this new Collatz directed graph. This result contradicts the previous assumption.

4. Conclusion

A complete Collatz directed graph covers all positive integers**.** By starting at any node in this complete Collatz directed graph, there is a unique path from that node to a node 1.

References

[1] R . Terras, (1976). " A stopping time problem on the positive integers". Acta Arithmetica, 30(3), 241-252.

[2] J. C. Lagarias. The 3x+1 Problem: An Overview. https://arxiv.org/abs/2111.02635, 2021.