NUMERICAL EVALUATION OF THREE INTEGRALS OF THE KIND $\int_0^\infty x^m dx/(1+x^n\sin^2 x)$

RICHARD J. MATHAR

Abstract. Definite integrals along the real line from zero to infinity with functions with denominator $1+x^n\sin^2 x$ suffer from dominant peaks at all x-values that are close to π, which impedes sampling the function with generic discrete numerical methods. We demonstrate the method of integrating along a closed contour around a circular sector in the complex x-plane and collecting the sum of all (infinitely many) residues inside the sector with an adapted series acceleration.

1. Statement of the Problem

Integrals of the type [3, 3.241.4][2, Table 17 (22)]

(1)
$$\int_0^\infty \frac{x^{\mu-1}}{(p+qx^\nu)^{n+1}}dx = \frac{1}{\nu p^{n+1}}\left(\frac{p}{q}\right)^{\mu/\nu} B(1+n-\frac{\mu}{\nu},\frac{\mu}{\nu}),\quad 0<\mu/\nu<n+1, p\neq 0, q\neq 0$$

are manageable by locating the finite number of isolated poles of the integrand in the upper half complex plane, taking a contour integral along a pie-shaped region in the complex plane, and summing the residues to obtain a closed form [1, 14.4.3.2].

This work considers kernels with a squared sine introduced in the denominator,

$$f_{m,n}(x) \equiv \frac{x^m}{1+x^n\sin^2 x},\quad m=0,1,2,\ldots,n>2m+2. \tag{2}$$

The suppression of the nth power nearby zeros of the sine introduces spikes near multiples of π in the kernel, illustrated in Figure 1.

The nature of these spikes is uncovered by computing the poles of $f_{m,n}$, the roots of $1+z^n\sin^2 z$, in the first quadrant of the complex plane. Table 1 shows for $n=6$ the real and imaginary part of the first 12 poles z_0, and the real and imaginary part of the residue at that pole sorted by increasing real value of $\Re z_0$.

The list starts with two poles $\approx 0.349+0.906i$ and $\approx 0.93+0.427i$ further away from the real axis, and continues with an infinite list of poles with small imaginary part near multiples of π, $z_0\approx s\pi$, $s\geq 1$.

The poles are easily computed numerically by a Newton-type of iteration solving $1+z^n\sin^2 z=0$ starting at $z_0\approx s\pi+0.03i$ as a first estimate. Once the pole is located, the residue is $\mathrm{Res}_{z_0} f_{m,n} = \lim_{z\to z_0}(z-z_0)f_{m,n}z)$ [4, 8.8.8], and with l'Hôpital's rule evaluated as

$$\mathrm{Res}_{z=z_0} f_{m,n}(z) = \frac{z_0^m}{nz_0^{n-1}\sin^2 z_0 + 2z_0^n \sin z_0\cos z_0}. \tag{3}$$

Date: September 6, 2024.

2020 *Mathematics Subject Classification.* Primary 26A36.

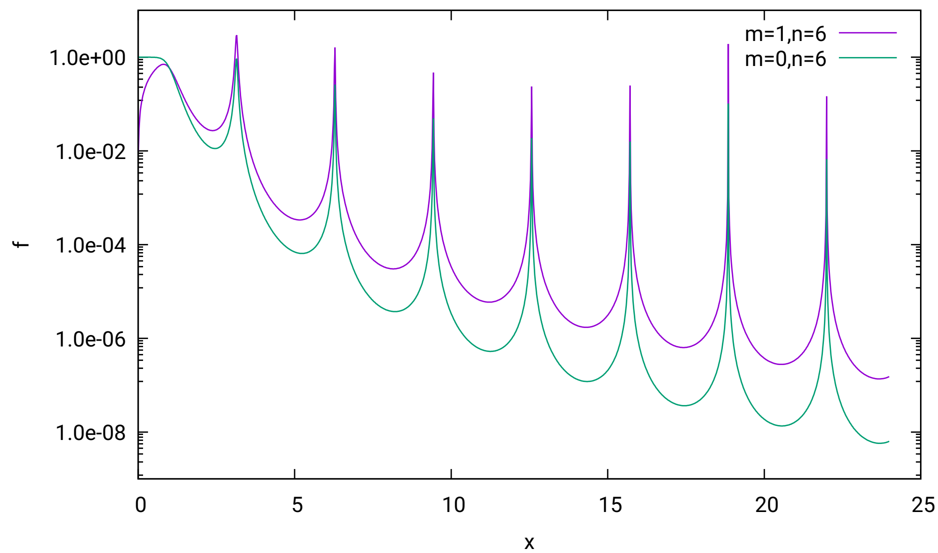

FIGURE 1. Kernels $f_{0,6}(x)$ and $f_{1,6}(x)$ of the integral along the real axis.

s	$\Re z_0$	$\Im z_0$	$\Re\,\mathrm{Res}\, z_0$	$\Im\,\mathrm{Res}\, z_0$
-1	3.490165284929085e-01	9.065484600592319e-01	8.596313253916175e-02	-7.087683959697169e-02
0	9.363994229692140e-01	4.267291693393096e-01	-8.333709883856446e-02	-1.123270619276566e-01
1	3.142582093924117e+00	3.219524885560753e-02	-2.580936674615053e-03	-5.044386363228393e-02
2	6.283193067006738e+00	4.031405989970889e-03	-4.062938232817706e-05	-1.266485734463813e-02
3	9.424778414943871e+00	1.194500703433598e-03	-3.567072499511272e-06	-5.628947375948893e-03
4	1.256637067498422e+01	5.039301920344882e-04	-6.348638856230781e-07	-3.166286403486014e-03
5	1.570796328066296e+01	2.580122715587622e-04	-1.664258169097328e-07	-2.026423585714707e-03
6	1.884955592508700e+01	1.493126587176246e-04	-5.573567362992561e-08	-1.407238642833671e-03
7	2.199114857633466e+01	9.402779701700128e-05	-2.210306629721098e-08	-1.033889623752575e-03
8	2.513274122919198e+01	6.299127814225012e-05	-9.919752758682190e-09	-7.915717454563160e-04
9	2.827433388251581e+01	4.424078795815415e-05	-4.893118286716063e-09	-6.254394045324211e-04
10	3.141592653599726e+01	3.225153442709850e-05	-2.600403679115772e-09	-5.066059179289920e-04

TABLE 1. The first 12 poles of $f_{1,6}(z)$ in the first quadrant and their residues. The pole labeled $s = -1$ is outside the 45° contour and does not contribute to (6).

A standard numerical strategy to evaluate the integral

$$I_{m,n} \equiv \int_0^\infty f_{m,n}(x)dx \tag{4}$$

is to map the x-interval onto $[0,1]$ with the substitution $y = e^{-x}$,

$$I_{m,n} = \int_0^1 \frac{(-\log y)^m}{1+\log^n y \sin^2(-\log y)} \frac{dy}{y}. \tag{5}$$

This generates a cluster of all poles near $y = 0$ and prevents standard discrete methods of integration.

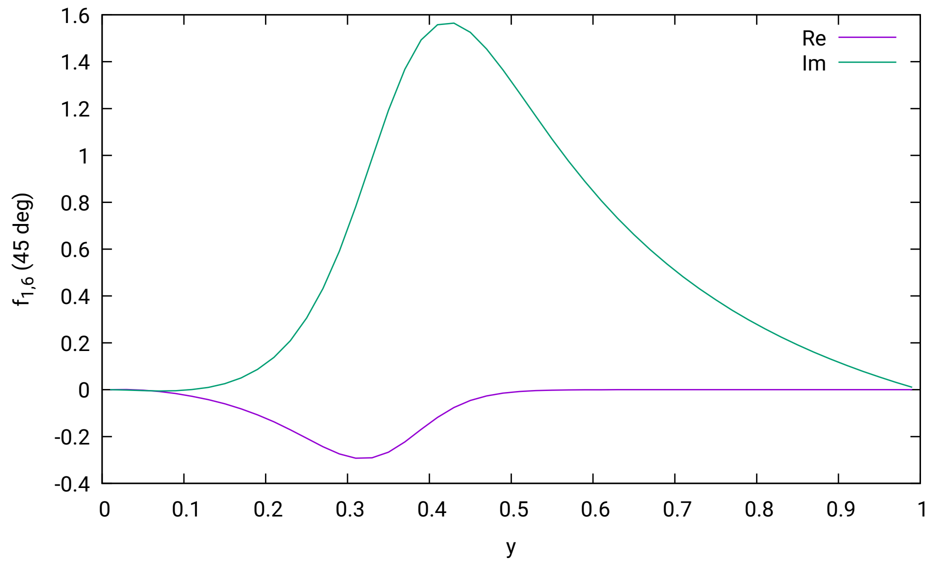

FIGURE 2. Real and imaginary part of $J_{1,6}$-integral kernel along the $e^{i\pi/4}x$ line after the substitution $y = e^{-x}$. This includes the $1/y$ factor induced by $dy = -ydx$ that is hidden in Eq. (7).

2. INTEGRATION ALONG THE DIAGONAL PATH

The numerical strategy employed here is to calculate the contour integral along the real line from 0 to $+\infty$, to take an arc of 45° at radius $R = \infty$—which does not contribute because $f_{m,n}(x)$ decays faster than $\propto 1/R$—, and to return on a straight line $z = e^{i\pi/4}x$ along the diagonal of the complex plane to the origin:

$$I_{m,n} = J_{m,n} + 2\pi i \sum_{\arg z_0 < \pi/4} \mathrm{Res}_{z=z0}\, f_{m,n}(z) \tag{6}$$

where the first residue in Table 1 must be skipped because the pole is outside the 45° sector of the path. Here $J_{m,n}$ denotes the integral at the 45° angle heading outwards from the origin:

$$\begin{aligned} J_{1,6} \equiv \int_0^{e^{i\pi/4}\infty} \frac{z}{1+z^6\sin^2 z} dz &= i\int_0^\infty \frac{x^m}{1-ix^6\sin^2(e^{i\pi/4}x)} dx \\ &= i\int_0^\infty \frac{x}{1-ix^6\sin^2(\frac{x}{\sqrt{2}}+i\frac{x}{\sqrt{2}})} dx \\ &= i\int_0^\infty \frac{x}{1-ix^6[\sin\frac{x}{\sqrt{2}}\cosh\frac{x}{\sqrt{2}}+i\cos\frac{x}{\sqrt{2}}\sinh\frac{x}{\sqrt{2}}]^2} dx. \end{aligned} \tag{7}$$

If this is mapped to the interval $0 \le y \le 1$ with the substitution $y = e^{-x}$, a very smooth integrand emerges, as shown in Figure 2.

Remark 1. *The strategy is to close the contour along a path that stays away from the poles that accumulate near the real line. The choice of the path along a* 45° *angle*

N	$\Re J_{1,6}$	$\Im J_{1,6}$
2001	-0.05835434709635286	0.54012229076889572
4001	-0.05835434695578123	0.54012229123734220
8001	-0.05835434702810185	0.54012229133767059
16001	-0.05835434706029046	0.54012229133830445
32001	-0.05835434706553047	0.54012229133149408
320001	-0.05835434706409912	0.54012229132915830
640001	-0.05835434706410969	0.54012229132919222
1280001	-0.05835434706411958	0.54012229132919540

TABLE 2. Estimates of the $J_{1,6}$-integral along the 45° line with N-point Simpson integration.

from the real line is almost arbitrary; it suits the idea of passing approximately midway through the first two poles of Table 1 which have arguments $\approx 68.9°$ *and* $\approx 24.5°$ *—an idea lent from saddle-point approaches to integration. The strategy of leaving one pole of the first quadrant outside the contour works at least for* n *=5–7. For* $n = 8$ *these noticeable poles have arguments* $90°$, $19.4°$, *and* $55.7°$. *For* $n = 9$ *they have arguments* $82.2°$, $50.7°$ *and* $17.5°$, *so the contour dismisses* two *of these.*

Numerical integration of $J_{1,6}$ with the Simpson rule and N points on the y-axis gives Table 2.

3. Gathering the Residues

Considering the last line in Table 2, a numerical accuracy near 10^{-14} has been achieved in the auxiliary path. According to Eq. (6) one needs to gather approximately 6 times the partial sum of column $\Im \operatorname{Res} z_0$ in Table 1 to establish $I_{1,6}$ to similar precision. Because the convergence of these imaginary parts is rather slow, the following extrapolating methodology is applied to accelerate convergence: The sequence of poles start to be near $s\pi$, $s \geq 1$. The Taylor series of the denominator of $f_{1,6}$ near such small values of $\sin x$ is

$$1 + x^6 \sin^2(x) \approx 1 + (s\pi)^6(x - s\pi)^2 + 6(s\pi)^5(x - s\pi)^3 + O((x - s\pi)^4), \tag{8}$$

Chopping this after the quadratic order and equating with zero, a good approximation of the pole is

$$z_0 \approx s\pi + \frac{i}{(s\pi)^3}, \quad m = 1. \tag{9}$$

To that lowest order Eq. (3) yields an estimate of $-i/[2(s\pi)^2]$ of the residues. The contribution to the integral (6) is $2\pi i$ times that, $1/(s^2\pi)$. The extrapolation uses the actual numerical sum of the residues up to some maximum $\hat{s}$, and approximates the sum of all contributions not yet accounted for via

$$\sum_{s=\hat{s}+1}^{\infty} \frac{1}{s^2\pi} = \frac{\pi}{6} - \sum_{s=1}^{\hat{s}} \frac{1}{s^2\pi} \tag{10}$$

as deduced by the well-known Riemann-$\zeta(2)$ value.

Remark 2. *This acceleration for logarithmic convergence is known* [5].

$\hat{s}$	$\Re z_0$	$\Im z_0$	$\Re I_{1,6}$	$\Im I_{1,6}$	$\Re I_{1,6}^{(ex)}$
1	3.142582	0.032195	0.964365540850	0.000283352971	1.169654430264629
2	6.283193	0.004031	1.043941186435	0.000028071033	1.169652604304037
3	9.424778	0.001194	1.079308905883	0.000005658455	1.169652558619954
4	12.566371	0.000504	1.099203270092	0.000001669488	1.169652554942173
5	15.707963	0.000258	1.111935664992	0.000000623804	1.169652554394706
6	18.849556	0.000149	1.120777606156	0.000000273606	1.169652554276171
7	21.991149	0.000094	1.127273726249	0.000000134729	1.169652554243179
8	25.132741	0.000063	1.132247318210	0.000000072401	1.169652554232187
9	28.274334	0.000044	1.136177069887	0.000000041657	1.169652554227994
10	31.415927	0.000032	1.139360168747	0.000000025318	1.169652554226218
11	34.557519	0.000024	1.141990828962	0.000000016095	1.169652554225399
12	37.699112	0.000019	1.144201314283	0.000000010623	1.169652554224995
13	40.840704	0.000015	1.146084804733	0.000000007238	1.169652554224783
14	43.982297	0.000012	1.147708834765	0.000000005068	1.169652554224667
15	47.123890	0.000010	1.149123545370	0.000000003634	1.169652554224600
16	50.265482	0.000008	1.150366943363	0.000000002660	1.169652554224560
17	53.407075	0.000007	1.151468361654	0.000000001983	1.169652554224536
18	56.548668	0.000006	1.152450799574	0.000000001502	1.169652554224521
19	59.690260	0.000005	1.153332544411	0.000000001155	1.169652554224511
20	62.831853	0.000004	1.154128319127	0.000000000900	1.169652554224504

TABLE 3. Upper limit $\hat{s}$ of index of pole included, real and imaginary part of the pole—copies of Table 1—real and imaginary part of $I_{1,6}$ including residues up to $\hat{s}$, and extrapolated real part of $I_{1,6}$.

The efficiency of that extrapolation scheme is illustrated in Table 3. If Eq. (6) is computed based on the estimate $J_{1,6}$ in the last line of Table 2 and including the residues up to index $\hat{s}$, real and imaginary value of $I_{1,6}$ become column $\Re I_{1,6}$ and $\Im I_{1,6}$ in the table.

Remark 3. $\Im I_{m,n}$ *approaches zero down the table as it should: Because the integral* $I_{m,n}$ *is obviously real-valued, only* $\Re J_{m,n}$ *is of final interest.* $\Im J_{m,n}$ *is essentially a tracer value for the final integral which needs to balance the contribution from the residues to maintain* $\Im I_{m,n} \to 0$ *if that numerical strategy is executed correctly.*

Including the first two dozen residues lets $I_{1,6}$ converge only to approximately 3 digits, but when the estimator (10) for the missing sum is added, the extrapolated $\Re I_{1,6}^{(ex)}$ converges much faster, last column in Table 3, and the number of valid digits is rather limited by precision in $J_{1,6}$ than the truncation of the series of residues.

s	$\Re z_0$	$\Im z_0$	$\Re \operatorname{Res} z_0$	$\Im \operatorname{Res} z_0$
-1	1.981320404079660e-01	9.407429853781349e-01	-2.256306937385727e-02	-1.254471168298916e-01
0	9.053538257189063e-01	4.855989772375105e-01	-1.283073538303410e-01	-8.541246448071564e-02
1	3.144168807783924e+00	5.693497567621882e-02	-2.552255516908106e-03	-2.824091055136655e-02
2	6.283225933409751e+00	1.010487636364353e-02	-4.062115370064213e-05	-5.051988598436316e-03
3	9.424781527808762e+00	3.667081055676098e-03	-3.566995524567449e-06	-1.833526410612220e-03
4	1.256637124922172e+01	1.786384803946352e-03	-6.348609147103144e-07	-8.931910683059836e-04
5	1.570796343437467e+01	1.022586870207360e-03	-1.664255726554569e-07	-5.112932108409547e-04
6	1.884955597727442e+01	6.482569124858596e-04	-5.573564152231397e-08	-3.241284026929050e-04
7	2.199114859723162e+01	4.409407153186883e-04	-2.210306048598415e-08	-2.204703414872214e-04
8	2.513274123863810e+01	3.157914321532129e-04	-9.919751432080494e-09	-1.578957102982059e-04
9	2.827433388720126e+01	2.352442626931989e-04	-4.893117925498037e-09	-1.176221290038466e-04
10	3.141592653849834e+01	1.807695661653257e-04	-2.600403566151274e-09	-9.038478203455015e-05

TABLE 4. The first 12 poles of $f_{0,5}(z)$ in the first quadrant and their residues. The pole labeled $s = -1$ is outside the 45° contour and does not contribute to (6).

N	$\Re J_{0,5}$	$\Im J_{0,5}$
2001	0.58805542896296801	0.82249847495855990
4001	0.58805542910354102	0.82249847542700417
8001	0.58805542903122048	0.82249847552733242
16001	0.58805542899903187	0.82249847552796628
32001	0.58805542899379187	0.82249847552115591
64001	0.58805542899445157	0.82249847551890665
320001	0.58805542899522322	0.82249847551882012
640001	0.58805542899521265	0.82249847551885405
1280001	0.58805542899520276	0.82249847551885722
2560001	0.58805542899520032	0.82249847551885534

TABLE 5. Estimates of the $J_{0,5}$-integral along the 45° line with N-point Simpson integration.

4. THE INTEGRAL $\int_0^\infty dx/(1 + x^5 \sin^2 x)$

The fundamental case for the integral over

$$f_{0,5} = \frac{1}{1 + x^5 \sin^2 x} \tag{11}$$

starts with the poles and residues of Table 4.

For the extrapolation of the sum of residues, the location of the poles is estimated generalized from (8)

$$1 + x^n \sin^2(x) \approx 1 + (s\pi)^n (x - s\pi)^2 + n(s\pi)^{n-1}(x - s\pi)^3 + O((x - s\pi)^4), \tag{12}$$

with pole position generalizing (9),

$$z_0 \approx s\pi + \frac{i}{(s\pi)^{n/2}}. \tag{13}$$

The estimator for the residues at these poles is $-i/[2(s\pi)^{n/2-m}]$. The relevant contribution to $I_{m,n}$ is $2\pi i$ times this, $1/[s^{n/2-m}\pi^{n/2-m-1}]$. Generalized from (10)

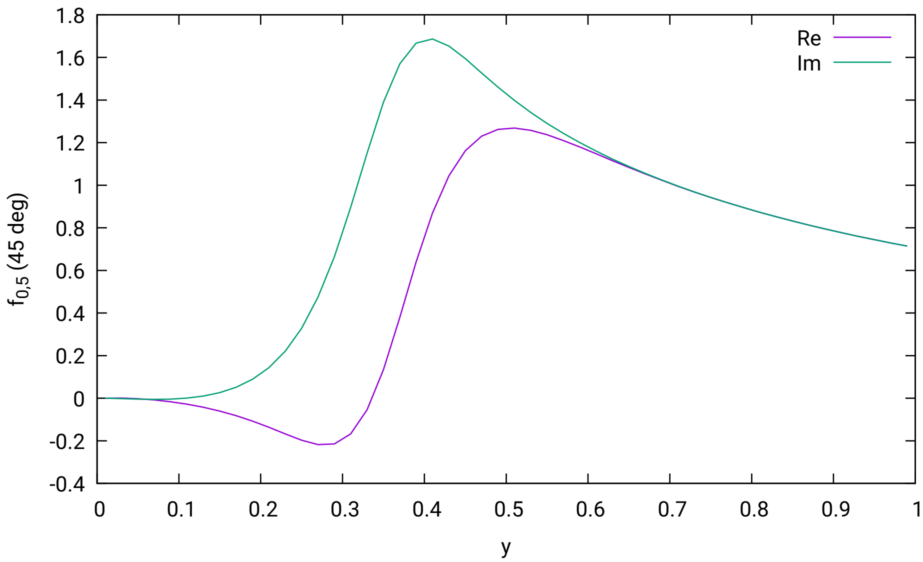

FIGURE 3. Real and imaginary part of $J_{0,5}$-integral kernel along the $e^{i\pi/4}x$ line after the substitution $y = e^{-x}$. This includes the $1/y$ factor induced by $dy = -ydx$ that is hidden in Eq. (7).

the guess of the unaccounted residues is

$$\sum_{s=\hat{s}+1}^{\infty} \frac{1}{s^{n/2-m}\pi^{n/2-m-1}} = \frac{1}{\pi^{n/2-m-1}}\zeta(n/2-m) - \sum_{s=1}^{\hat{s}} \frac{1}{s^{n/2-m}\pi^{n/2-m-1}}. \quad (14)$$

With the extrapolation algorithm we arrive at Table 6 for the estimated integral $I_{0,5}$ including partial sums up to residue $\hat{s}$.

$\hat{s}$	$\Re z_0$	$\Im z_0$	$\Re I_{0,5}$	$\Im I_{0,5}$	$\Re I_{0,5}^{(ex)}$
1	3.144169	0.056935	1.302160645108	0.000283300765	1.363487358880259
2	6.283226	0.010105	1.333903225642	0.000028070529	1.363483121446872
3	9.424782	0.003667	1.345423611845	0.000005658435	1.363482988394584
4	12.566371	0.001786	1.351035696842	0.000001669486	1.363482975825057
5	15.707963	0.001023	1.354248246832	0.000000623804	1.363482973711430
6	18.849556	0.000648	1.356284805650	0.000000273606	1.363482973206733
7	21.991149	0.000441	1.357670061660	0.000000134729	1.363482973054314
8	25.132741	0.000316	1.358662149667	0.000000072401	1.363482972999854
9	28.274334	0.000235	1.359401191300	0.000000041657	1.363482972977774
10	31.415927	0.000181	1.359969095634	0.000000025318	1.363482972967896
11	34.557519	0.000142	1.360416596081	0.000000016095	1.363482972963114
12	37.699112	0.000115	1.360776612305	0.000000010623	1.363482972960644
13	40.840704	0.000094	1.361071337191	0.000000007238	1.363482972959297
14	43.982297	0.000078	1.361316218197	0.000000005068	1.363482972958529
15	47.123890	0.000066	1.361522303510	0.000000003634	1.363482972958072
16	50.265482	0.000056	1.361697681559	0.000000002660	1.363482972957792
17	53.407075	0.000048	1.361848395313	0.000000001983	1.363482972957615
18	56.548668	0.000042	1.361979040654	0.000000001502	1.363482972957500
19	59.690260	0.000036	1.362093168386	0.000000001155	1.363482972957423
20	62.831853	0.000032	1.362193560640	0.000000000900	1.363482972957371

TABLE 6. Upper limit $\hat{s}$ of index of pole included, real and imaginary part of the pole—copies of Table 4—real and imaginary part of $I_{0,5}$ including residues up to $\hat{s}$, and extrapolated real part of $I_{0,5}$.

s	$\Re z_0$	$\Im z_0$	$\Re \operatorname{Res} z_0$	$\Im \operatorname{Res} z_0$
-1	4.655576051906159e-01	8.626662923989105e-01	9.947453817025014e-02	1.395337284404808e-02
0	9.565859545073164e-01	3.791775558797673e-01	-4.697875849336042e-02	-1.192038621198058e-01
1	3.141960944982686e+00	1.818271865522610e-02	-2.592703241408274e-03	-8.965753180765293e-02
2	6.283186748057586e+00	1.608309772566175e-03	-4.063094558581943e-05	-3.174669292549203e-02
3	9.424778016990430e+00	3.890908592263626e-04	-3.567081935119911e-06	-1.728077638642918e-02
4	1.256637061998763e+01	1.421560911913720e-04	-6.348641540365564e-07	-1.122419496141821e-02
5	1.570796326889326e+01	6.509991626228185e-05	-1.664258343907358e-07	-8.031380236520490e-03
6	1.884955592175837e+01	3.439110017037273e-05	-5.573567553344607e-08	-6.109677961976103e-03
7	2.199114857519254e+01	2.005082772041217e-05	-2.210306659138222e-08	-4.848396568382816e-03
8	2.513274122874033e+01	1.256494208283463e-05	-9.919752722216699e-09	-3.968352245536757e-03
9	2.827433388231671e+01	8.320063914220988e-06	-4.893118300874818e-09	-3.325687446635609e-03
10	3.141592653590162e+01	5.754074036484633e-06	-2.600403683095544e-09	-2.839521721700870e-03

TABLE 7. The first 12 poles of $f_{2,7}(z)$ in the first quadrant and their residues. The pole labeled $s=-1$ is outside the 45° contour and does not contribute to (6).

N	$\Re J_{2,7}$	$\Im J_{2,7}$
2001	-0.25206129166402547	0.31175004232140071
4001	-0.25206129152345661	0.31175004278984637
8001	-0.25206129159577740	0.31175004289017470
16001	-0.25206129162796602	0.31175004289080856
32001	-0.25206129163320603	0.31175004288399819
64001	-0.25206129163254633	0.31175004288174894
320001	-0.25206129163177468	0.31175004288166241
640001	-0.25206129163178525	0.31175004288169633
1280001	-0.25206129163179514	0.31175004288169950
2560001	-0.25206129163179759	0.31175004288169764

TABLE 8. Estimates of the $J_{2,7}$-integral along the 45° ray with N-point Simpson integration.

5. THE INTEGRAL $\int_0^\infty x^2dx/(1+x^7\sin^2 x)$

The fundamental case for the integral over

$$f_{2,7} = \frac{x^2}{1+x^7\sin^2 x} \tag{15}$$

starts with the poles and residues of Table 7.

With the extrapolation algorithm we arrive at Table 9 for the estimated integral $I_{2,7}$ including partial sums up to residue $\hat{s}$.

$\hat{s}$	$\Re z_0$	$\Im z_0$	$\Re I_{2,7}$	$\Im I_{2,7}$	$\Re I_{2,7}^{(ex)}$
1	3.141961	0.018183	1.060253549930	0.000283362854	1.969938926427793
2	6.283187	0.001608	1.259723904471	0.000028071094	1.969938140768070
3	9.424778	0.000389	1.368302224759	0.000005658457	1.969938125077511
4	12.566371	0.000142	1.438825921626	0.000001669488	1.969938124000543
5	15.707963	0.000065	1.489288571924	0.000000623804	1.969938123858618
6	18.849556	0.000034	1.527676810726	0.000000273606	1.969938123830757
7	21.991149	0.000020	1.558140184808	0.000000134729	1.969938123823611
8	25.132741	0.000013	1.583074077331	0.000000072401	1.969938123821391
9	28.274334	0.000008	1.603969987832	0.000000041657	1.969938123820595
10	31.415927	0.000006	1.621811228993	0.000000025318	1.969938123820275
11	34.557519	0.000004	1.637275734442	0.000000016095	1.969938123820135
12	37.699112	0.000003	1.650848026439	0.000000010623	1.969938123820069
13	40.840704	0.000002	1.662884798466	0.000000007238	1.969938123820035
14	43.982297	0.000002	1.673655227664	0.000000005068	1.969938123820018
15	47.123890	0.000001	1.683366769269	0.000000003634	1.969938123820008
16	50.265482	0.000001	1.692182231512	0.000000002660	1.969938123820002
17	53.407075	0.000001	1.700231412293	0.000000001983	1.969938123819999
18	56.548668	0.000001	1.707619232301	0.000000001502	1.969938123819997
19	59.690260	0.000001	1.714431546372	0.000000001155	1.969938123819996
20	62.831853	0.000000	1.720739377677	0.000000000900	1.969938123819995

TABLE 9. Upper limit $\hat{s}$ of index of pole included, real and imaginary part of the pole—copies of Table 7—real and imaginary part of $I_{2,7}$ including residues up to $\hat{s}$, and extrapolated real part of $I_{2,7}$.

6. SUMMARY

The integrals evaluated here numerically are

$$\int_0^\infty \frac{1}{1+x^5\sin^2 x}dx = I_{0,5} \approx 1.36348297295737; \quad (16)$$

$$\int_0^\infty \frac{x}{1+x^6\sin^2 x}dx = I_{1,6} \approx 1.1696525542245; \quad (17)$$

$$\int_0^\infty \frac{x^2}{1+x^7\sin^2 x}dx = I_{2,7} \approx 1.96993812381999. \quad (18)$$

Not reported here in any detail, the same numerical recipe yields

$$\int_0^\infty \frac{1}{1+x^6\sin^2 x}dx = I_{0,6} \approx 1.21795716787160; \tag{19}$$

$$\int_0^\infty \frac{1}{1+x^7\sin^2 x}dx = I_{0,7} \approx 1.14173560173977; \tag{20}$$

$$\int_0^\infty \frac{x}{1+x^7\sin^2 x}dx = I_{1,7} \approx 0.854731117093820; \tag{21}$$

$$\int_0^\infty \frac{1}{1+x^4\sin^2 x}dx = I_{0,4} \approx 1.6795575946; \tag{22}$$

$$\int_0^\infty \frac{x}{1+x^5\sin^2 x}dx = I_{1,5} \approx 2.16378494901; \tag{23}$$

$$\int_0^\infty \frac{1}{1+x^3\sin^2 x}dx = I_{0,3} \approx 2.65743947. \tag{24}$$

References

1. I. N. Bronshtein, K. A. Semendyayev, G. Musiol, and H. Muehlig, *Handbook of mathematics*, 5 ed., Springer, 2007. MR 2374226
2. Bierens de Haan, *Nouvelle tables d'intégrales definies*, Hafner Publications, New York, 1957.
3. I. Gradstein and I. Ryshik, *Summen-, Produkt- und Integraltafeln*, 1st ed., Harri Deutsch, Thun, 1981. MR 0671418
4. Paul J. Nahin, *Inside interesting integrals*, 2nd ed., Undergrad. Lect. Not. Phys., Springer, 2020. MR 4171483
5. E. Joachim Weniger, *Nonlinear sequence transformations for the acceleration of convergence and the summation of divergent series*, Comp. Phys. Rep. **10** (1989), no. 5–6, 189–371.

URL: `https://www.mpia-hd.mpg.de/homes/mathar`
Email address: `mathar@mpia.de`

Max-Planck Institute for Astronomy, Königstuhl 17, 69117 Heidelberg, Germany