

# On Expressions for Gravitational Time Dilation

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## Abstract

Schwarzschild's gravitational time dilation expression is derived assuming a ~flat Minkowski spacetime. This time dilation effect is spherically symmetric, occurring in all radial directions for a gravitational source.

A way to derive Schwarzschild's expression is with a model that assumes a mass starting from rest at some distance from a gravitational field source (such as Earth). One can use Newtonian kinetic energy and gravitational potential Energy to create an energy balance. This is then used to derive escape velocity: the mass steadily starts moving through the gravitational potential field, gaining speed until it hits escape velocity upon reaching the gravitational field source.

This document explores the derivation using escape velocity. Moreover, this document modifies the approach by using relativistic mass to derive a slightly different expression for gravitational time dilation. Finally, an alternative interpretation of gravitational time dilation is also explored, using the concept of relativistic mass.

## Gravitational Time Dilation Derivations

The Schwarzschild metric can be derived by assuming an asymptotically flat spacetime (Weinstein, 2023). By extension, Schwarzschild's gravitational time dilation expression can also be derived assuming a ~flat Minkowski spacetime. More specifically, it has been shown in scientific literature that Newtonian escape velocity, the equivalence principle, and special relativity can be used to derive the Schwarzschild's expression for gravitational time dilation (Dikshit, 2021).

The derivation for escape velocity is as follows:

$$KE = \frac{1}{2}mv^2 \text{ [Eqn. 1]}$$

$$U_{grav} = -\frac{GMm}{r} \text{ [Eqn. 2]}$$

$$KE = -U_{grav} \text{ [Eqn. 3]}$$

$$\frac{1}{2}mv^2 = \frac{GMm}{r} \text{ [Eqn. 4]}$$

$$v^2 = \frac{2GM}{r} \text{ [Eqn. 5]}$$

Escape velocity can then be plugged into special relativity's time dilation equation, for the following gravitational time dilation expression:

$$\frac{t_0}{t_f} = \sqrt{1 - \frac{v^2}{c^2}} \quad [\text{Eqn. 6}]$$

$$\frac{t_0}{t_f} = \sqrt{1 - \frac{2GM}{rc^2}} \quad [\text{Eqn. 7}]$$

However, there are mathematical quirks with this expression. Singularities form at  $\frac{2GM}{rc^2} = 1$ , and imaginary values form at  $\frac{2GM}{rc^2} > 1$ .

Admittedly, there are existing solutions to remedy these quirks, via simple coordinate transformations (Weinstein, 2023). However, there may also be an alternate gravitational time dilation expression that can be used. Special relativity shows that, for flat Minkowski spacetime, Newtonian kinetic energy is only an approximation. Thus, a new expression for gravitational time dilation can be found by using the following well-known expression for relativistic kinetic energy (LibreTexts, 2021):

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad [\text{Eqn. 8}]$$

$$KE_{rel} = (\gamma - 1)mc^2 \quad [\text{Eqn. 9}]$$

In short: this document argues that relativistic kinetic energy applies for flat spacetime, so it should not be neglected when deriving gravitational escape velocity. For gravitational potential energy, a relativistic treatment also exists. However, because the mass for escape velocity is modeled to start at rest, the relativistic component of potential energy should be neglected. Newtonian potential energy (equation 2) can be used instead:

$$U_{grav} = -\frac{GMm}{r}$$

From here, a new relativistic escape velocity can be found by building off the energy balance:

$$KE_{rel} = -U_{grav} \quad [\text{Eqn. 10}]$$

$$(\gamma - 1)mc^2 = \frac{GMm}{r} \quad [\text{Eqn. 11}]$$

$$(\gamma - 1) = \frac{GM}{rc^2} \quad [\text{Eqn. 12}]$$

$$\gamma = \frac{GM}{rc^2} + 1 \quad [\text{Eqn. 13}]$$

$$\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{GM}{rc^2} + 1 \quad [\text{Eqn. 14}]$$

With the relativistic escape velocity equation derived, the value can then be plugged in for the standard time dilation equation from special relativity (equation 6):

$$\frac{t_0}{t_f} = \sqrt{1 - \frac{v^2}{c^2}}$$

This becomes:

$$\frac{t_0}{t_f} = \frac{1}{\frac{GM}{rc^2} + 1} \quad [\text{Eqn. 16}]$$

The newly derived expression does not see the formation of singularities or imaginary values when substituted within the Schwarzschild metric. A graph comparing the two gravitational time dilation expressions (equations 7 and 16) was produced where “ $M = \frac{c^2}{G}kg$ ” and the radius “ $r$ ” was varied from 0-250 meters. The gravitational time dilation expressions closely agreed, up until  $r \leq 2m$  which corresponded with  $\frac{2GM}{rc^2} \geq 1$  for the Schwarzschild expression.

$$\frac{t_0}{t_f} = \frac{1}{\frac{GM}{rc^2} + 1} \approx \sqrt{1 - \frac{2GM}{rc^2}} \quad [\text{Eqn. 17}]$$

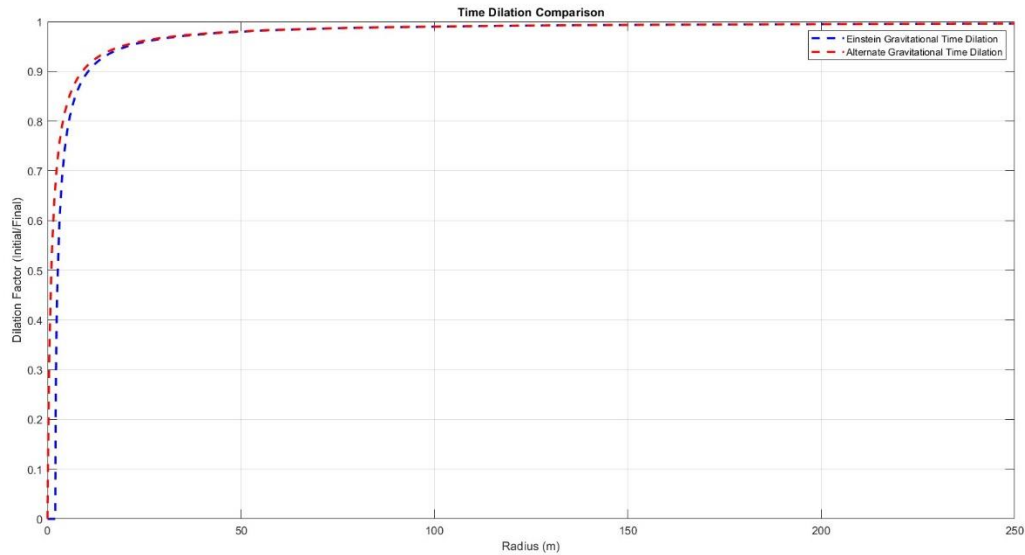


Figure 1 - Time Dilation Comparison

### Alternative Physical Interpretation of the Relativistic Field

While escape velocity and relativistic mass are useful for deriving gravitational time dilation, justification should be provided for why this effect will always apply for the gravitational field.

Suppose that the source of a gravitational field generates its field in all radial directions. Since the field lines cancel relative to the source, the source should be taken as an inertial reference frame in spacetime.

Following this, suppose that every point the gravitational field moves through can also be taken as inertial. The gravitational field itself can be imagined as motion that is generated against an inertial backdrop.

For the spacetime backdrop of the field to be inertial, each point of the backdrop must have a kinetic energy which exactly cancels the energy imparted by the gravitational field lines. Furthermore, relativistic effects must be considered.

This model provides a new justification for use of equation 11:

$$(\gamma - 1)mc^2 = \frac{GMm}{r}$$

A physical interpretation for how this inertial backdrop is maintained might be via massless particles that exist along the field lines. If these particles were made to be light, they would always move at constant speed relative to a center of mass. However, what was to be shown can also be employed: if the source of a field is taken to be an inertial frame, then it inherently corresponds with an inertial backdrop.

To reinforce this idea, an appeal to conservation of energy can also be made. Suppose that a beam of light of energy  $E = mc^2$  starts at the center of a gravitational field source and is sent high into its gravitational field. The gravitational field will be unable to decelerate the light, since light always moves at a constant speed.

Next, suppose that the light's energy is then converted into an inert mass and dropped back down to the source. If the process is taken to be 100% efficient, the light should gain energy as it falls to the source in the gravitational field. This gain in energy would appear to violate the conservation of energy.

However: if gravitational time dilation is employed throughout the field, it can be shown that a beam of light will lose energy as it climbs through a gravitational field (Einstein, 1911). The light beam sees no change in its travel distance or speed, but its energy should experience time dilation (via equation 9):

$$E_{loss} = (\gamma - 1)mc^2$$

Thus, if the light energy is converted into an inert mass and dropped back down to the source of the field, conservation of energy will remain intact.

It is worth re-emphasizing that this approach focuses on a relativistic field, which is generated via relativistic mass. This is used in place of a relativistic spacetime. While the real and observed effects of length contraction and time dilation from special relativity are still in play, they are treated as effects that occur within sources of mass and energy.

Inertial length and time within the field remains unchanged. The spacetime backdrop, external to sources of mass and energy, is taken to always be inertial.

## References

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