Skill in Backgammon: Cubeless vs Cubeful

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Abstract. Does the doubling cube make backgammon more skillful? And is the answer the same in both money and match play? This article presents GNUbg rollouts between unequally skilled players which show that use of the doubling cube does not favor the better player in either case.

Keywords: Backgammon, Cubeful, Cubeless, Portes, Skill, ELO

Contents

1. Introduction

Luck is arguably the most common thing that backgammon players complain about. The doubling cube, a rather recent invention compared to the long history of backgammon, introduces a new element of skill and has therefore been touted as a way to reduce luck. But is this actually true and if so, is it true in both money and match play? To answer this question we used GNUbg rollouts between unequally skilled players which show that use of the doubling cube does not favor the better player in either case. We also examine the implications of these data on the ELO system.

2. Money Play

We will compare 4 types of money games:

- cubeful games with the Jacoby rule in effect (cubeful games),
- cubeless games with gammons and backgammons (cubeless games),
- cubeless games with backgammons counting as gammons (Portes games) and
- cubeless games without backgammons or gammons (DMP games).

The reason for examining DMP games is because we'd have to examine them anyway as part of match play. In order to compare the skill in these formats we need 2 things for each one:

- the equity E of the better player and
- the expected value V of a game (assuming optimal play from both sides).

By equity we mean the expected difference in points per game (PPG) and by expected value of a game we mean the average PPG of the winner. The reason we need that second number is because players will bet less money per point in a game where more points are at stake, which means that each point would be worth less. Therefore, the equity of the better player has to be normalized (adjusted by the expected value of the game) before comparing different formats.

2.1. Expected Value of Money Games

We already have a very good estimate for the expected value of cubeless money games. Tom Keith has rolled out every opening roll 46,656 times using GNUbg 2ply and reported a **gammon rate of 27.62%** and a **backgammon rate of 1.22%** [1]. For cubeful games we rolled out **38,880 games with 1ply Normal**. From these data we get the following results:

$$V_{DMP} = 1\ ppg$$
$$V_{Portes} = 1+0.2762 = 1.2762\ ppg$$
$$V_{cubeless} = 1+0.2762+0.0122 = 1.2884\ ppg$$
$$V_{cubeful} = 2.443\ ppg$$

2.2. Equity of the Better Player

Here we rolled out **77,760 games** for each format, with GNUbg playing **one side at 1ply Normal and the other at 0ply**. Of course, half the games were played with the better player going first. These games were rolled out without variance reduction (VR). Because of the way it works, VR would actually skew the results instead of making them more accurate. VR works by making use of the equity difference between 2 consecutive plies. This is often interpreted as canceling out the estimated luck, but it's equivalent to think about it as using subsequent evaluations to estimate the error in previous ones. However, that error is precisely what we want to measure, not adjust for it! Below are the results. Check Table 6 for the outcome probabilities of cubeless games.

$$E_{DMP} = 0.0286\ ppg$$
$$E_{Portes} = 0.0318\ ppg$$
$$E_{cubeless} = 0.0316\ ppg$$
$$E_{cubeful} = 0.0552\ ppg$$

2.3. Comparing Formats

If we assume that the amount of money that players are willing to risk in a single game is constant, it follows that the amount they're willing to bet per point must be inversely proportional to the points at stake. Therefore, we can normalize the equity of the better player by dividing it with the expected value of the game. These normalized equities can be used as a measure of skill.

Table 1. Comparison of Various Money Game Formats

Format	E(ppg)	V(ppg)	E/V
DMP	0.0286	1.0000	0.0286
Portes	0.0318	1.2762	0.0249
Cubeless	0.0316	1.2884	0.0245
Cubeful	0.0552	2.4430	0.0226

As you can see, cubeless backgammon turns out to be more favorable to the better player than cubeful, with DMP specifically coming out on top. This was truly a surprise. Maybe DMP strategy leads to longer games with more decisions compared to Portes. As for cubeful backgammon, perhaps a lot of opportunity for skill is wasted when the better player is forced to make theoretically correct passes in positions where the skill difference would actually allow them to take.

3. Match Play

We will compare 3 types of matches:

- cubeful matches with the Crawford rule in effect (cubeful matches),
- cubeless matches with backgammons counting as gammons (Portes matches) and
- cubeless matches without backgammons or gammons (DMP matches).

We're examining the Portes format because Portes games outperformed cubeless games in money play and the DMP format not only because it outperformed all other formats in money play, but also due to its simplicity which makes it easy to study mathematically. In order to compare the skill in these formats we again need 2 things for each one:

- the probability of the better player winning an N-point match and
- the relative duration of an N-point match compared to DMP (assuming equal players).

The reason we need the expected duration is because the proper way to measure skill is per unit of time. Hence, we must compare matches of equivalent length, either in terms of duration or skill.

3.1. Defining Skill in Match Play

One obvious way to define the skill S of a match relative to a DMP game is as the ratio of the corresponding expectations of the better player:

$$S=\frac{2P-1}{2W-1} \tag{1}$$

where P, W are the probabilities of the better player winning the match or a DMP game respectively. The problem with this definition is that the larger the skill difference of the players is, the smaller the corresponding skill values will be. While it's obviously true that better players gain less from longer matches, what we're interested in is the opportunity for skill inherent in a match, which ideally should be independent of the skill difference. Another way of defining skill would be using the ELO system, according to which the probability of the better player winning a match is

$$P=\frac{1}{1+10^{-\frac{|\Delta R|}{C}}}$$

where R is a player's rating and C is a constant (usually 2000 in backgammon) that determines the width of the distribution. The longer the match length is, the higher the absolute value of the ELO difference gets. Of course, this doesn't mean that good players play better at longer matches, just that longer matches contain more skill. The solution is to multiply the ELO difference at DMP by a factor depending on the match length. That factor can be defined as the skill of that particular match length.

The ELO formula then becomes

$$P(N)=\frac{1}{1+10^{-\frac{|\Delta R|}{C}S(N)}}$$

where N is the length of the match and S is a skill function. Unfortunately, things aren’t always that simple. While in cubeful matches the skill appears to be independent of the ELO difference [2], the skill values of DMP matches increase with the absolute value of the ELO difference. The ELO formula for the better player in cases where the skill function isn't constant with respect to the skill difference can be generalized as follows:

$$P(N,W)=\frac{1}{1+10^{-\frac{|\Delta R|}{C}S(N,W)}}$$

where W is the probability of the better player winning a single DMP game. Solving this formula for S(N,W) we get the generalized skill function:

$$S(N,W)=-\frac{C}{|\Delta R|}\cdot\log\left(\frac{1}{P(N,W)}-1\right)$$

If we define DMP games to contain 1 unit of skill, we have:

$$S(1,W)=1 \Leftrightarrow -\frac{C}{|\Delta R|}=\frac{1}{\log\left(\frac{1}{P(1,W)}-1\right)}=\frac{1}{\log\left(\frac{1}{W}-1\right)} \Rightarrow$$

$$S(N,W)=\frac{\log\left(\frac{1}{P(N,W)}-1\right)}{\log\left(\frac{1}{W}-1\right)} \tag{2}$$

Since we’re interested in the opportunity for skill, that is to say the minimum amount of skill inherent in different length matches, we can define the skill of an N-point match to be the limit of S(N,W) as $W\rightarrow 1/2$. Note that because the linear approximation of $\log(1/x-1)$ at $x=1/2$ is $-4(x-1/2)$, this definition is equivalent to a limit definition using equation (1), which would represent the maximum gain of playing a longer match over a DMP game. If we now find the win probabilities of unequal players that are closely matched, we can plug them in equation (2) and calculate the corresponding skill values at different match lengths.

3.2. Skill in Cubeful Matches

In order to find the skill in an N-point cubeful match, all we have to do is look at the function used to adjust for different length matches. That function is $S(N)=\sqrt{N}$ [3]. Easy, right? Wrong! The ELO formula has a huge problem that many players had noticed long before I did. Specifically, better players win less often than predicted by the formula. The problem is not with the system itself which is well researched, but rather with the skill function used in backgammon, which was chosen on general grounds rather than actual evidence [2]. In order to find the true values of this function, we need a Match Equity Table (MET) for unequal players. There are 2 ways to construct a MET:

- theoretically, using math [4] and
- empirically, through bot self-play.

Tom Keith went the theoretical route when he computed a MET using a constant **win rate of 51%** for the better player and a constant **gammon rate of 25%** for both players [5]. Joseph Heled (one of the authors of GNUbg) went the other route [2,6]. Despite the unrealistic assumption of perfectly efficient doubling required to compute a MET mathematically – which explains the 4-point match anomaly – both approaches are essentially in complete agreement and not even close to the square root hypothesis.

Table 2. Skill in Cubeful Backgammon Matches up to 15 points

Length	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\sqrt{Length}$	1.00	1.41	1.73	2.00	2.24	2.45	2.65	2.83	3.00	3.16	3.32	3.46	3.61	3.74	3.87
Keith	1.00	1.00	1.24	1.23	1.44	1.50	1.62	1.67	1.77	1.84	1.92	–	–	–	–
Heled	1.00	1.00	1.24	1.26	1.45	1.50	1.63	1.67	1.78	1.83	1.92	1.97	2.05	2.10	2.17

3.3. Duration of Cubeful Matches

Along with the Match Equity method, Tom Keith proposed another method for measuring skill at different match lengths, the “Rolls” method [5]. According to this method, the skill in an N-point match is defined as the square root of the ratio of rolls in contact positions of an N-point match to contact position rolls at DMP. The problem with this method is the assumption that skill is a function of the number of decisions only, when in fact it’s also the difficulty of the decisions that matters. Otherwise, all backgammon variants with the same average number of decisions per game would contain exactly the same amount of skill. However, this method without the square root is actually a good lower estimate for the expected duration of a match – it underestimates the duration because it doesn't take cube decisions into account properly.

A better way of measuring the expected duration would be to roll out the entire match and construct a duration table the same way we construct an equity table. Joseph Heled did exactly that using the total number of decisions at each score to estimate the average duration of a game. In his video [7], Heled presents 2 duration tables, one for a 15-point match and another one for a 13-point match. He does so because he uses the match length as his unit of measurement – a score of -N/-N corresponds to 100% of an N-point match. These tables can be converted so that they show the expected number of games remaining (using DMP as our unit of measurement) instead of the percentage of the match remaining, which would be more convenient for our purposes. Also, it helps that we have 2 tables to work with, because we can take their average after we convert them and thus gain back some of the accuracy lost from rounding percentages to integer values.

Table 3. Duration of Cubeful Matches up to 15 points

Length	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
“Rolls”	1	–	1.78	–	2.66	–	3.59	–	4.70	–	5.50	–	–	–	–
Heled	1	1	1.92	2.15	2.92	3.27	3.96	4.39	5.04	5.46	6.04	6.54	7.20	7.67	8.31

3.4. Duration of Cubeless Matches

Fortunately, we don't need to roll out cubeless matches to find out their expected duration. We can calculate the expected duration D[M,N] from any score -M/-N recursively using the following formula:

$$D[M,N]=T+(1-G)\frac{D[M-1,N]+D[M,N-1]}{2}+G\frac{D[M-2,N]+D[M,N-2]}{2} \tag{3}$$

where T is the average duration of a game and G is the gammon rate at the particular score. Since the gammon rate isn't very score-sensitive, we can use the same as in money games for every score further than 1-away. For 1-away scores, we can extract a **gammon rate of 29.05%** from Kazaross' XG2 MET [8] at **-1/-2C**. As for the average duration, unfortunately neither XG nor GNUbg provide the average number of decisions or at least the average number of rolls at the end of a rollout. However, they do report the duration of the rollout which could be an even better measure of time as it also takes into account the relative difficulty of decisions – easy decisions are not sent to higher plies for further analysis. Using **XG 2ply**, we divided the time it takes to roll out 1,080 cubeless games with the time it takes to roll out 1,080 DMP games to get the average duration of a cubeless game (**T=0.867**). These rollouts were performed without VR because even the small amount of time it adds to the results has nothing to do with the actual time it takes to complete a game. The following table shows the expected duration we obtain when we apply formula (3) in Portes matches.

Table 4. Duration of Portes Matches up to 8 points

Length	1	2	3	4	5	6	7	8
Duration	1	1.75	2.85	3.94	5.08	6.24	7.42	8.61

If we set **G=0** and **T=1** in formula (3), we get the expected duration of DMP matches. Note that because we make no assumptions about gammon rates, these results are perfectly accurate.

Table 5. Duration of DMP Matches up to 5 points

Length	1	2	3	4	5
Duration	1	2.5	4.12	5.81	7.54

3.5. Skill in Portes Matches

Similarly to how we calculated the duration of cubeless matches, we also don't need to roll out an entire MET for unequal players in order to compute the skill in Portes matches. Even better, Joseph Heled seems to have beat us to the punch yet again. This time though, he got it slightly wrong. In his video about cubeless gammon rates [9] he uses a constant win rate for the better player and a constant gammon rate for each player at every score. Unfortunately, that's not the proper way to compute a MET because the win and gammon rates vary depending on the score. This effect is not very significant when both players are further than 1-away, but it can't be ignored at scores where the gammon value is very different than for money. As such, we decided to roll out all 1-away scores in a 2-point match using the same settings as for money games.

Table 6. Outcome Probabilities (of 1ply vs 0ply)

Score	L BG	L Gammon	Win	W Gammon	W BG
-1-1			0.5143		
-1-2		0.1319	0.5173		
-2-1			0.5136	0.1449	
Unlimited	0.0062	0.1302	0.5100	0.1419	0.0060

The recursive formula used to calculate the Match Winning Chances (MWC) of player A from any score -M/-N in a Portes match is the following:

$$P_A[M,N]=(w_A-g_A)\cdot P_A[M-1,N]+g_A\cdot P_A[M-2,N]+(w_B-g_B)\cdot P_A[M,N-1]+g_B\cdot P_A[M,N-2] \quad (4)$$

where w,g are the outcome probabilities. Using the outcome probabilities of the -1/-2 scores for farther 1-away scores and the money game probabilities for all remaining scores, we constructed a MET and then plugged the MWC in equation (2) to get the corresponding skill values.

Table 7. Skill in Portes Matches up to 8 points

Length	1	2	3	4	5	6	7	8
MWC	0.5143	0.5191	0.523	0.5259	0.5285	0.5308	0.533	0.5351
Zoidis	1.00	1.33	1.61	1.81	1.99	2.16	2.31	2.45
Heled	1.00	1.35	1.66	1.93	2.17	2.39	2.60	2.80

Unsurprisingly, the naive construction of a MET using the money game probabilities for every score exaggerates the skill. Nevertheless, Portes matches dominate cubeful matches, that is to say Portes matches are shorter in duration AND contain more skill than cubeful matches of approximately equivalent length. Specifically, if we compare Tables 3 & 4, we see that **an N-point Portes match takes about as much time as a cubeful match of length 2N-1**. However, since Portes matches of length N>4 take a bit longer, we can compare them to 2N-point cubeful matches which are closer to matches of length 2N-1 than 2N+1 both in terms of duration and skill.

Table 8. Skill Comparison between Portes & Cubeful Matches

Portes Matches		Cubeful Matches	
Length (Duration)	Skill	Skill	Length (Duration)
2 (1.75)	1.33	1.24	3 (1.92)
3 (2.85)	1.61	1.45	5 (2.92)
4 (3.94)	1.81	1.63	7 (3.96)
5 (5.08)	1.99	1.83	10 (5.46)
6 (6.24)	2.16	1.97	12 (6.54)
7 (7.42)	2.31	2.10	14 (7.67)

These results shouldn't be surprising. When the stakes of the current game are increased, both players get closer to the end of the match, which means less opportunity for skill. Also, consider what happens when the match goes beyond the Crawford game. The trailer – who statistically is more likely to be the worse player – has free access to the most powerful weapon in backgammon.

3.6. Skill in DMP Matches

If we set $g_A = g_B = 0$ and $w_A = 0.5143$ in equation (4), we get the MWC of the better player in DMP matches, which can then plug them in equation (2) to obtain the corresponding skill values. Like the duration of DMP matches, these results are perfectly accurate too.

Table 9. Skill in DMP Matches up to 5 points

Length	1	2	3	4	5
MWC	0.5143	0.5214	0.5268	0.5313	0.5352
Skill	1.00	1.50	1.88	2.19	2.46

As you can see in the following 2 tables, DMP matches dominate cubeful matches and Portes matches, but the latter only at longer matches. This can be seen most clearly if we compare matches of approximately equivalent skill and observe the difference in duration.

Table 10A. Skill Comparison between DMP & Cubeful Matches

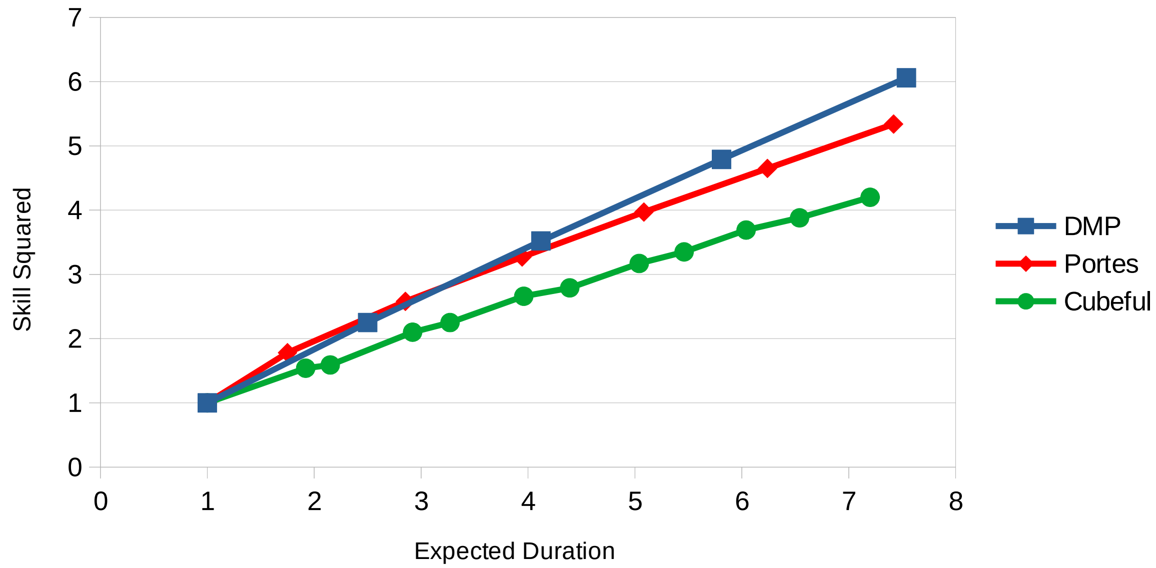
DMP Matches		Cubeful Matches	
Length (Duration)	Skill	Skill	Length (Duration)
2 (2.50)	1.50	1.50	6 (3.27)
3 (4.12)	1.88	1.83	10 (5.46)
4 (5.81)	2.19	2.17	15 (8.31)

Table 10B. Skill Comparison between DMP & Portes Matches

DMP Matches		Portes Matches	
Length (Duration)	Skill	Skill	Length (Duration)
4 (5.81)	2.19	2.16	6 (6.24)
5 (7.54)	2.46	2.45	8 (8.61)

For shorter matches, the skill values of equivalent lengths are much closer and so we must plot them against the duration to see what's happening. Essentially, we want a graph of the function $S \circ D^{-1}$ for the various formats, where S and D are the skill and expected duration functions with respect to the match length. Since the square of this function is close to being a straight line – for reasons that will become apparent in the next chapter – we actually plotted the squares of the skill values to get a more accurate approximation. For comparison, we also plotted the same data from cubeful matches.

Figure 1.



As you can see, the 2-point DMP match contains less skill per unit of time compared to both the 2 and 3-point Portes matches. From then on, DMP matches dominate Portes matches, but the curves are reasonably close – up to the usual match lengths at least. Cubeful matches are clearly at the bottom.

3.7. Skill and Duration Formulas

It would be useful to have formulas (even approximate) for the skill functions of the various formats to use in the ELO system. For DMP matches in particular, we can actually find explicit formulas for both the skill and the expected duration which can be shown to be related. Check the Appendix for derivations of the following 2 formulas.

$$S(N)=\binom{2N}{N}\frac{2N}{2^{2N}} \tag{5}$$

$$D(N)=2N-S(N) \tag{6}$$

This relation might seem unexpected at first, but it actually makes perfect sense. Assuming luck was evenly distributed, the difference in points at the end of a match represents how much better a player the winner is and can thus be used as a measure of skill. Now simply observe that, if the difference in points at the end of an N-point match is S, the duration of the match would be

$$D=N+(N-S)=2N-S$$

Similar relations between skill and expected duration exist for all types of backgammon matches. Specifically, the sum of the duration and skill functions seems to always be a linear function of the match length. Since by definition D(1)=S(1)=1 the line is of the following form:

$$D(N)+S(N)=B\cdot N-B+2$$

B is approximately equal to **1.3 in Portes matches** and **0.6 in cubeful matches**. Furthermore, the skill values can be fitted with a square root function of the following form:

$$S(N)=\sqrt{A\cdot N-A+1}$$

Since by definition S(1)=1 this formula also has only one degree of freedom, namely A which is approximately equal to **0.72 in Portes matches** and **0.26 in cubeful matches**. For reference, we also fitted the skill values of DMP matches and got $A\approx 1.27$ which of course agrees with the coefficient of N we get when we apply Stirling’s approximation:

$$S(N)\sim\sqrt{\frac{4N}{\pi}}$$

This comeback of the square root is not a coincidence. As we noted above, skill can be represented by the difference in points at the end of a match. In a DMP match, that difference corresponds to the distance from the origin in a random walk. And since the average distance form the origin is proportional to the square root of the number of steps, it shouldn't be surprising that the square root makes an appearance. After all, this is precisely the reason why it was chosen in the first place. Approximate formulas for all formats examined are shown in the following table.

Table 11. Useful Approximations for the ELO System

Format	$S(N)$	$D(N)+S(N)$
DMP	$\sqrt{1.27N-0.27}$	$2N$
Portes	$\sqrt{0.72N+0.28}$	$1.3N+0.7$
Cubeful	$\sqrt{0.26N+0.74}$	$0.6N+1.4$

4. Conclusion

We examined various formats in an effort to find out which one favors the better player most. Cubeful backgammon did worse in every setting. It turns out that the most skillful form of play is DMP. Now I know that even if people are convinced of the validity of this article, they might still object to the idea of parting with their doubling cubes, not to mention the idea of playing without gammons. Fortunately for those people, the cube will always have a place in chouettes. But if you are one of the many people who complain about luck in tournaments, now you know what the numbers say.

5. References

1. Keith, Tom. "Rollouts of Opening Moves" *Backgammon Galore*, 2006, https://www.bkgm.com/openings/rollouts.html

2. "The Backgammon Elo system: theory, practice, data - and why it's (sort of) broken" *YouTube*, uploaded by Joseph Heled, 23 May 2023, https://www.youtube.com/watch?v=Lv75r6xH82Y

3. Keith, Tom. "Introduction to Ratings" *Backgammon Galore*, 2006, https://www.bkgm.com/faq/Ratings.html

4. Keith, Tom. "How to Compute a Match Equity Table" *Backgammon Galore*, 1995, https://www.bkgm.com/articles/met.html

5. Keith, Tom. "Different length matches" *Backgammon Galore*, May 1998, https://www.bkgm.com/rgb/rgb.cgi?view+523

6. "PRestigious PR (or: the curious case of Brad Garner)" *YouTube*, uploaded by Joseph Heled, 25 October 2023, https://www.youtube.com/watch?v=pgG8bQgvhI4

7. "Time to Think - Time to GO (or: what the data says about Mochy's time management)" *YouTube*, uploaded by Joseph Heled, 9 August 2024, https://www.youtube.com/watch?v=z2Jp7GXe5PI

8. Keith, Tom. "The Kazaross-XG2 Match Equity Table" *Backgammon Galore*, 2013, https://www.bkgm.com/articles/Keith/KazarossXG2MET/index.html

9. "Gammon rates in cube-less games. or: the unexpected return of the square root." *YouTube*, uploaded by Joseph Heled, 3 July 2023, https://www.youtube.com/watch?v=LVQ2teWqhFs

A.1. Derivation of the Skill Formula

$$S(N)=\lim_{W\to 1/2} S(N,W)=\lim_{W\to 1/2}\frac{\log\left(\frac{1}{P(N,W)}-1\right)}{\log\left(\frac{1}{W}-1\right)}$$

Since an N-point cubeless DMP match is equivalent to a best of 2N-1 match, we can use the binomial distribution to calculate the probability of the better player winning:

$$P(N,W)=\sum_{K=N}^{2N-1}\binom{2N-1}{K}W^K(1-W)^{2N-1-K}$$

Using L' Hopital's rule we get

$$S(N)=\lim_{W\to 1/2}\frac{(1-W)\cdot W}{[1-P(N,W)]\cdot P(N,W)}\sum_{K=N}^{2N-1}\binom{2N-1}{K}\cdot W^{K-1}\cdot(1-W)^{2N-2-K}\cdot[K-(2N-1)\cdot W]$$

Since $P(N,1/2)=1/2$ we have

$\sum_{K=N}^{2N-1}\binom{2N-1}{K}=P(N,1/2)\cdot 2^{2N-1}=2^{2N-2}$ and our limit becomes

$$S(N)=2^{3-2N}\cdot\sum_{K=N}^{2N-1}\binom{2N-1}{K}\cdot K-\frac{2N-1}{2^{2N-2}}\sum_{K=N}^{2N-1}\binom{2N-1}{K}\ \Leftrightarrow$$

$$S(N)=(2N-1)\cdot 2^{3-2N}\cdot\sum_{K=N}^{2N-1}\binom{2N-2}{K-1}-(2N-1)$$

Let $A=\sum_{K=N}^{2N-1}\binom{2N-2}{K-1}=\sum_{K=N-1}^{2N-2}\binom{2N-2}{K}=\sum_{K=0}^{N-1}\binom{2N-2}{K}\ \Rightarrow$

$$2A=\sum_{K=0}^{N-1}\binom{2N-2}{K}+\sum_{K=N-1}^{2N-2}\binom{2N-2}{K}=\binom{2N-2}{N-1}+\sum_{K=0}^{2N-2}\binom{2N-2}{K}=\binom{2N}{N}\frac{N}{2\cdot(2N-1)}+2^{2N-2}\ \Rightarrow$$

$$S(N)=\binom{2N}{N}\frac{2N}{2^{2N}}\quad\blacksquare$$

A.2. Derivation of the Duration Formula

Since an N-point cubeless DMP match will end after a minimum of N games and a maximum of 2N-1 games, the expected duration can be expressed as:

$$D(N)=\sum_{K=N}^{2N-1} K\cdot P(K)$$

where P(K) is the probability of the match lasting exactly K games. In order for the match to end after K games, the winner of game K must win N-1 of the first K-1 games. There are C(K-1,N-1) combinations in which this happens. Assuming both players are equally likely to win a single game, the probability of each combination is 2^{1-K} and thus we have:

$$D(N)=\sum_{K=N}^{2N-1} K\cdot\binom{K-1}{N-1}\cdot 2^{1-K}=N\sum_{K=N}^{2N-1}\binom{K}{N}\cdot 2^{1-K} \Leftrightarrow$$

$$D(N)=N\sum_{K=0}^{N-1}\binom{K+N}{N}2^{1-K-N}=N\cdot 2^{1-2N}\sum_{K=0}^{N-1}\binom{N+K}{N}\cdot 2^{N-K} \Leftrightarrow$$

$$D(N)=N\cdot 2^{1-2N}\left\{\sum_{K=0}^{N}\binom{N+K}{N}\cdot 2^{N-K}-\binom{2N}{N}\right\}$$

To simplify our calculations, let's consider a best of 2N+1 match, equivalent to an (N+1)-point match. Let K be the score of the loser of the match. The game that decides the match is preceded by exactly N games won by the winner and K games won by the loser. These N+K games can occur in any order. Now imagine that the maximum of 2N+1 games are played even if the winner's already decided. Thus, there remain N-K games which can go either way. By symmetry – assuming both players are equally likely to win – the total number of sequences of 2N+1 games is equal to twice the total number of sequences that decide the outcome in the (N+K+1)-st game. Therefore, we have:

$$2\sum_{K=0}^{N}\binom{N+K}{K}\cdot 2^{N-K}=2^{2N+1} \Leftrightarrow \sum_{K=0}^{N}\binom{N+K}{N}\cdot 2^{N-K}=2^{2N} \Rightarrow$$

$$D(N)=2N-S(N) \quad \blacksquare$$