Skill in Backgammon: Cubeless vs Cubeful

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Abstract. Does the doubling cube make backgammon more skillful? And is the answer the same in both money and match play? This article presents GNUbg rollouts between unequally skilled players which show that use of the doubling cube favors the better player only in match play.

Keywords: Backgammon, Cubeful, Cubeless, Portes, Skill, ELO

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1. Introduction

Luck is arguably the most common thing that backgammon players complain about. The doubling cube, a rather recent invention compared to the long history of backgammon, introduces a new element of skill and has therefore been touted as a way to reduce luck. But is this actually true and if so, is it true in both money and match play? To answer this question we used GNUbg rollouts between unequally skilled players which show that use of the doubling cube does indeed favor the better player but only in match play. We also examine the implications of these data on the ELO system.

2. Money Play

We will compare 4 types of money games:

- cubeful games with the Jacoby rule in effect (cubeful games),
- cubeless games with gammons and backgammons (cubeless games),
- cubeless games with backgammons counting as gammons (Portes games) and
- cubeless games without backgammons or gammons (DMP games).

In order to compare the skill in these formats we need 2 things for each one:

- the equity E of the better player and
- the expected value V of a game (assuming optimal play from both sides).

By equity we mean the expected difference in points per game (PPG) and by expected value of a game we mean the average PPG of the winner. The reason we need that second number is because players will bet less money per point in a game where more points are at stake, which means that each point would be worth less. Therefore, the equity of the better player has to be adjusted by the expected value of the game before comparing different formats.

2.1. Expected Value of Money Games

We already have a very good estimate for the expected value of cubeless money games. Tom Keith has rolled out every opening roll 46,656 times using GNUbg 2ply and reported a **gammon rate of 27.62%** and a **backgammon rate of 1.22%** [1]. For cubeful games I rolled out **174,960 games** with GNUbg at **1ply Normal**. From these data we get the following results:

$$V_{DMP} = 1\,ppg$$
$$V_{Portes} = 1+0.2762 = 1.2762\,ppg$$
$$V_{cubeless} = 1+0.2762+0.0122 = 1.2884\,ppg$$
$$V_{cubeful} = 2.4494\,ppg \ [SE=0.0041\,ppg]$$

2.2. Equity of the Better Player

Here I rolled out **349,920 games** for each format, with GNUbg playing **one side at 1ply Normal and the other at 0ply**. Of course, half the games were played with the better player going first. These games were rolled out without variance reduction (VR). Because of the way it works, VR would actually skew the results instead of making them more accurate. VR works by making use of the equity difference between 2 consecutive plies. This is often interpreted as canceling out the estimated luck, but it's equivalent to think about it as using subsequent evaluations to estimate the error in previous ones. However, that error is precisely what we want to measure, not adjust for it! Below are the results. Check Table 5 for the outcome probabilities of cubeless games.

$$E_{DMP} = 0.0284\, ppg \;\; [SE = 0.0017\, ppg]$$
$$E_{Portes} = 0.0303\, ppg \;\; [SE = 0.0023\, ppg]$$
$$E_{cubeless} = 0.0307\, ppg \;\; [SE = 0.0023\, ppg]$$
$$E_{cubeful} = 0.0592\, ppg \;\; [SE = 0.0049\, ppg]$$

2.3. Comparing Formats

If we assume that the amount of money that players are willing to risk in a single game is constant, it follows that the amount they're willing to bet per point must be inversely proportional to the points at stake. Therefore, we can normalize the equity of the better player by dividing it with the expected value of the game. These normalized equities can be used as a measure of skill.

Table 1. Comparison of Various Money Game Formats

Format	E(ppg)	V(ppg)	E/V
DMP	0.0284	1.0000	0.0284
Portes	0.0303	1.2762	0.0237
Cubeless	0.0307	1.2884	0.0238
Cubeful	0.0592	2.4494	0.0242

As you can see, almost all formats are virtually indistinguishable from each other in terms of skill with the exception of DMP which turns out to be the most favorable format for the better player. One possible explanation is that DMP strategy leads to longer games with more difficult decisions. As for cubeful play, the opportunity for skill wasted when the game ends with a pass could be cancelling out the added skill from cube decisions.

3. Match Play

We will compare 3 types of matches:

- cubeful matches with the Crawford rule in effect (cubeful matches),
- cubeless matches with backgammons counting as gammons (Portes matches) and
- cubeless matches without backgammons or gammons (DMP matches).

Because the results from cubeless matches with backgammons counting where nearly identical to the results from Portes matches, I chose to present only the latter which were slightly better. In order to compare the skill in these formats we again need 2 things for each one:

- the probability of the better player winning an N-point match and
- the relative duration of an N-point match compared to DMP (assuming equal players).

The reason we need the expected duration is because the only way to compare different formats is to compare matches of equivalent length, either in terms of duration or skill.

3.1. Defining Skill in Match Play

An obvious way to define the skill S of a match relative to a DMP game is as the ratio of the corresponding expectations of the better player:

$$S=\frac{2P-1}{2W-1} \tag{1}$$

where P, W are the probabilities of the better player winning the match or a DMP game respectively. The problem with this definition is that the larger the skill difference of the players is, the smaller the corresponding skill values will be. While it’s obviously true that the advantage better players have at longer matches grows slower the better they are, what we’re interested in is the opportunity for skill inherent in a match, which ideally should be independent of the skill difference.

Another way of defining skill would be using the ELO system, according to which the probability of the better player winning a match is

$$P=\frac{1}{1+10^{-\frac{|\Delta R|}{C}}}$$

where R is a player’s rating and C is a constant (usually 2000 in backgammon) that determines the width of the distribution. The longer the match length is, the larger the absolute value of the ELO difference gets. Since using different ratings for different match lengths would be impractical, the ELO difference at DMP is multiplied by a factor depending on the match length. This factor can be defined as the skill of that particular match length.

The ELO formula then becomes

$$P(N)=\frac{1}{1+10^{-\frac{|\Delta R|}{C}S(N)}}$$

where N is the length of the match and S is a skill function. Unfortunately, things aren't that simple. This time the skill values increase with the absolute value of the ELO difference. This happens because, as we will see in Chapter 3.5, the skill is related to the expected duration of a match and therefore better players can extract more skill by winning having played fewer games on average. The ELO formula for the better player in cases where the skill function isn't constant with respect to the skill difference can be generalized as follows:

$$P(N,W)=\frac{1}{1+10^{-\frac{|\Delta R|}{C}S(N,W)}}$$

where W is the probability of the better player winning a single DMP game.
Solving this formula for S(N,W) we get the generalized skill function:

$$S(N,W)=-\frac{C}{|\Delta R|}\cdot\log\left(\frac{1}{P(N,W)}-1\right)$$

If we define DMP games to contain 1 unit of skill, we have:

$$S(1,W)=1 \iff -\frac{C}{|\Delta R|}=\frac{1}{\log\left(\frac{1}{P(1,W)}-1\right)}=\frac{1}{\log\left(\frac{1}{W}-1\right)} \Rightarrow$$

$$S(N,W)=\frac{\log\left(\frac{1}{P(N,W)}-1\right)}{\log\left(\frac{1}{W}-1\right)} \qquad (2)$$

Since we're interested in the opportunity for skill inherent in a match, that is to say the minimum opportunity for skill it gives to the better player regardless of their skill difference, we can define the skill of an N-point match to be the limit of S(N,W) as $W \to 1/2$. Note that because the linear approximation of $\log(1/x-1)$ at $x=1/2$ is $-4(x-1/2)$, this definition is equivalent to a limit definition using equation (1), which would represent the maximum gain of playing a longer match over a DMP game. If we now find the win probabilities of unequal players that are closely matched, we can plug them in equation (2) and calculate the corresponding skill values at different match lengths.

3.2. Expected Duration of Matches

The expected duration D[M,N] of cubeless matches can be calculated recursively from any away score -M/-N using the following formula:

$$D[M,N]=T+(1-G)\frac{D[M-1,N]+D[M,N-1]}{2}+G\frac{D[M-2,N]+D[M,N-2]}{2} \tag{3}$$

where T is the average duration of a game and G is the gammon rate at the particular score. Since the gammon rate isn't very score-sensitive, we can use the same as in money games for every score farther than 1-away. For 1-away scores, we can extract a **gammon rate of 29.05%** from Kazaross' XG2 MET [2] at **-1/-2C**. For DMP matches, we simply set **G=0**.

As for the average duration, unfortunately neither XG nor GNUbg provide the average number of decisions at the end of a rollout. However, they do report the duration of the rollout which could be an even better measure of time as it also takes into account the relative difficulty of decisions – easy decisions are not sent to higher plies for further analysis. Being our unit of measurement, we obviously have **T=1** for DMP games. For cubeless games, we used **XG 2ply** to get **T=0.867** by dividing the time it takes to roll out 1,080 cubeless games with the time it takes to roll out 1,080 DMP games. These rollouts were performed without VR because even the small amount of time it adds to the results has nothing to do with the actual time it takes to complete a game. The following 2 tables show the expected duration of Portes and DMP matches. Note that because we make no assumptions about DMP matches, their results are perfectly accurate.

Table 2. Duration of Portes Matches up to 7 points

Length	1	2	3	4	5	6	7
Duration	1	1.75	2.85	3.94	5.08	6.24	7.42

Table 3. Duration of DMP Matches up to 5 points

Length	1	2	3	4	5
Duration	1	2.5	4.12	5.81	7.54

Similarly to how we construct an equity table, we can use the total number of decisions to estimate the average duration of a game at each score and construct a duration table. In his video about time management [3], Joseph Heled presents 2 cubeful duration tables based on data from his research on the ELO system, one for the 15-point match and another one for the 13-point match. The reason he presents 2 tables is because he uses the match length as his unit of measurement – a score of -N/-N corresponds to 100% of an N-point match.

These tables can be converted so that they show the expected number of games remaining (measured in DMP games) instead. Also, we can take their average after we convert them and thus gain back some of the accuracy lost from rounding percentages to integer values.

Table 4. Duration of Cubeful Matches up to 13 points

Length	1	2	3	4	5	6	7	8	9	10	11	12	13
Duration	1	1	1.92	2.15	2.92	3.27	3.96	4.39	5.04	5.46	6.04	6.54	7.2

If we compare Table 4 with Tables 2 & 3, we see 2 patterns emerge:

- An N-point Portes match takes about as much time as a cubeful match of length 2N-1.
- An N-point DMP match takes at least as much time as a cubeful match of length 3N-2.

3.3. Skill in Cubeless Matches

In his video about cubeless gammon rates [4], Joseph Heled uses a constant win rate for the better player and a constant gammon rate for each player to calculate the Match Winning Chances (MWC) of the better player at different scores and thus obtain the skill values at different match lengths. Unfortunately, that's not the proper way to compute a MET because the win and gammon rates vary depending on the score. This effect is not very significant when both players are farther than 1-away, but it can't be ignored at scores where the gammon value is very different than for money. As such, I decided to roll out all 1-away scores in a 2-point match using the same settings as for money games.

Table 5. Outcome Probabilities (of 1ply vs 0ply)

Score	L BG	L Gammon	Win	W Gammon	W BG
-1-1			0.5142		
-1-2		0.1308	0.5148		
-2-1			0.5131	0.1467	
Unlimited	0.0059	0.1317	0.5100	0.1420	0.0063

The MWC of player A from any away score -M/-N in a cubeless match can be calculated recursively using the following formula:

$$P_A[M,N]=(w_A-g_A)\cdot P_A[M-1,N]+g_A\cdot P_A[M-2,N]+(w_B-g_B)\cdot P_A[M,N-1]+g_B\cdot P_A[M,N-2] \quad (4)$$

where w,g are the outcome probabilities. For Portes matches, we can use the outcome probabilities of the -1/-2 scores for farther 1-away scores and the money game probabilities for all remaining scores, whereas for DMP matches, we simply set $g_A = g_B = 0$. The resulting skill values are shown in the following 2 tables along with Heled's results for comparison. Unsurprisingly, the naive construction of a MET using the money game probabilities for every score exaggerates the skill. As for DMP matches, their skill values are perfectly accurate like their duration.

Table 6. Skill in Portes Matches up to 7 points

Length	1	2	3	4	5	6	7
Heled	1.00	1.35	1.66	1.93	2.17	2.39	2.60
Zoidis	1.00	1.31	1.57	1.76	1.93	2.09	2.24

Table 7. Skill in DMP Matches up to 5 points

Length	1	2	3	4	5
Skill	1.00	1.50	1.88	2.19	2.46

3.4. Skill in Cubeful Matches

In order to find the skill in an N-point cubeful match, all we have to do is look at the function used to adjust for different length matches. That function is $S(N)=\sqrt{N}$ [5]. Easy, right? Wrong! The ELO formula has a huge problem that many players had noticed long before I did. Specifically, better players win less often than predicted by the formula. The problem is not with the system itself which is well researched, but rather with the skill function used in backgammon, which was chosen on general grounds rather than actual evidence [6]. In order to find the true values of this function, we need a Match Equity Table (MET) for unequal players. Tom Keith has computed such a MET mathematically using a constant **win rate of 51%** for the better player and a constant **gammon rate of 25%** for both players [7]. Joseph Heled used bot self-play instead [6,8]. As you can see in the following table, both approaches are essentially in complete agreement and not even close to the square root hypothesis.

Table 8A. Skill in Cubeful Backgammon Matches up to 13 points

Length	1	2	3	4	5	6	7	8	9	10	11	12	13
Keith	1.00	1.00	1.24	1.23	1.44	1.50	1.62	1.67	1.77	1.84	1.92	–	–
Heled	1.00	1.00	1.24	1.26	1.45	1.50	1.63	1.67	1.78	1.83	1.92	1.97	2.05

However, along with the match equity method, Tom Keith proposed another method for measuring skill which doesn't agree with the previous ones, the "Rolls" method [7]. According to this method, the skill in an N-point match is defined as the square root of the ratio of rolls in contact positions of an N-point match to contact position rolls at DMP. Initially, I dismissed this method as merely a better guess than the square root of the match length until I noticed that the skill values of odd match lengths are very close to Heled's exaggerated skill values for Portes matches. Naturally, I had to investigate further, so I rolled out the entire 13-point match (the resulting MET can be found in the Appendix) using the same settings as for money games. This time though, I performed only **38,880 trials per score**, which should be enough for the big picture. Remarkably, the results are almost in complete agreement with the "Rolls" method, which by the way uses data from matches between humans. So which one of us is right? Heled's data seem to have the advantage of theoretical validation. Or do they? It turns out that Keith's assumption of the gammon rate being the same for both players might be incorrect. According to Heled's own research [4], not only does the better player have a higher gammon rate, but also the gammon rates are related to then win rate through the following equation:

$$W_A = \frac{0.5 - G_B}{1 - G_B - G_A} \tag{5}$$

where G is the relative gammon rate, not the total gammons. This equation fits very well with our own data from money games. If we keep the 51% win rate for player A and the 25% gammon rate for player B, we can solve for the **gammon rate of the better player** which is approximately **26%**. Using the method described by Tom Keith [9] with these values and a **backgammon rate of 1%** for both players, I computed a 13-point MET and extracted the skill values from the MWC. All these results are shown in the following table along with the "Rolls" method and the square root hypothesis for comparison.

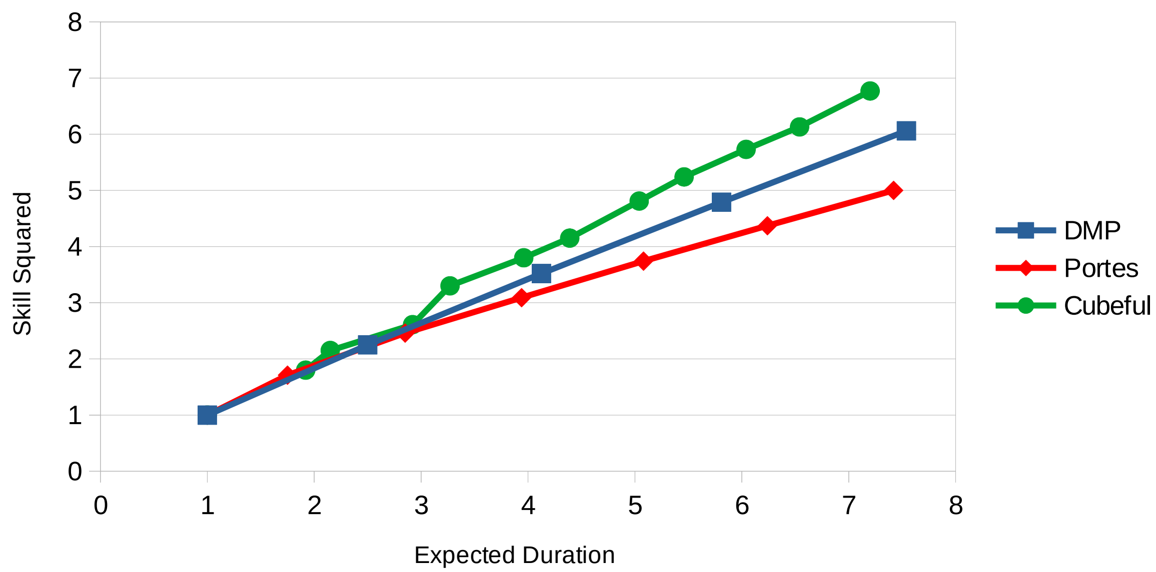
Table 8B. Skill in Cubeful Backgammon Matches up to 13 points

Length	1	2	3	4	5	6	7	8	9	10	11	12	13
$\sqrt{Length}$	1.00	1.41	1.73	2.00	2.24	2.45	2.65	2.83	3.00	3.16	3.32	3.46	3.61
"Rolls"	1.00	–	1.33	–	1.63	–	1.89	–	2.16	–	2.34	–	–
Zoidis Math	1.00	1.00	1.31	1.37	1.59	1.67	1.80	1.87	1.99	2.06	2.16	2.23	2.31
Zoidis GNU	1.00	1.00	1.34	1.47	1.62	1.82	1.95	2.04	2.19	2.29	2.39	2.48	2.60

This time the theoretical approach is much closer to the GNUbg rollouts. The reason they end up diverging is because of the unrealistic assumption of perfectly efficient doubling required to compute a MET mathematically. In fact, the skill values of odd length matches are much closer to the skill values of Portes matches. How can we explain Heled's results though? Unfortunately, he didn't go into a lot of details about his methodology, but I suspect that the choice of players is the culprit. For example, using noise to obtain a weaker player than 0ply leads to nonsensical results (like longer matches containing less skill than shorter ones) because of the unnatural (random) mistakes it makes.

If we compare the correct skill values, cubeful matches dominate DMP matches which in turn dominate Portes matches (of length greater than 2). We say that one format dominates another if matches of the former type are shorter in duration AND contain more skill than matches of the latter type of approximately equivalent length. For example, the 6-point Portes match has a skill value of 2.13 and lasts about 6.24 games, whereas both the 4-point DMP match and the 9-point cubeful match have a skill value of 2.19 but last only 5.81 and 5.04 games respectively. For short matches though, the skill values of equivalent lengths are much closer and so we must plot them against the duration to see what's going on. Essentially, we want a graph of the function $S \circ D^{-1}$ for the various formats, where S and D are the skill and expected duration functions with respect to the match length. Since the square of this function is close to being a straight line – for reasons that will become apparent in the next chapter – we actually plotted the squares of the skill values to get a better picture.

Figure 1.



3.5. Skill and Duration Formulas

It would be useful to have formulas (even approximate) for the skill functions of the various formats to use in the ELO system. For DMP matches in particular, we can actually find explicit formulas for both the skill and the expected duration which can be shown to be related. Check the Appendix for derivations of the following 2 formulas.

$$S(N) = \binom{2N}{N} \frac{2N}{2^{2N}} \tag{6}$$

$$D(N) = 2N - S(N) \tag{7}$$

This relationship might seem unexpected at first, but it actually makes perfect sense. Assuming luck was evenly distributed, the difference in points at the end of a match represents how much better a player the winner is and can thus be used as a measure of skill. Now simply observe that, if the difference in points at the end of an N-point match is S, the duration of the match would be

$$D = N + (N - S) = 2N - S$$

Similar relationships between skill and expected duration exist for all types of backgammon matches. Specifically, the sum of the duration and skill functions seems to always be a linear function of the match length. Since by definition D(1)=S(1)=1 the line is of the following form:

$$D(N) + S(N) = B \cdot N - B + 2$$

B is approximately equal to **1.26 in Portes matches** and **0.64 in cubeful matches**. Furthermore, the skill values can be fitted with a square root function of the following form:

$$S(N) = \sqrt{A \cdot N - A + 1}$$

Since by definition S(1)=1 this formula also has only one degree of freedom, namely A which is approximately equal to **0.68 in Portes matches** and **0.46 in cubeful matches**. For reference, we also fitted the skill values of DMP matches and got $A \approx 1.27$ which of course agrees with the coefficient of N we get when we apply Stirling’s approximation:

$$S(N) \sim \sqrt{\frac{4N}{\pi}}$$

This comeback of the square root is not a coincidence. As we noted above, skill can be represented by the difference in points at the end of a match. In a DMP match, that difference corresponds to the distance from the origin in a random walk. And since the average distance form the origin is proportional to the square root of the number of steps, it shouldn't be surprising that the square root makes an appearance. After all, this is precisely the reason why it was chosen in the first place. Approximate formulas for all formats examined are shown in the following table.

Table 9. Useful Approximations for the ELO System

Format	$S(N)$	$D(N)+S(N)$
DMP	$\sqrt{1.27N - 0.27}$	$2N$
Portes	$\sqrt{0.68N + 0.32}$	$1.26N + 0.74$
Cubeful	$\sqrt{0.46N + 0.54}$	$0.64N + 1.36$

4. Conclusion

We examined various formats using GNUbg rollouts between unequally skilled players in an effort to find out which one favors the better player most. Use of the doubling cube does indeed favor the better player but only in match play. It turns out that the most skillful form of money play is DMP. This is the exact opposite of what I expected to find. I thought that in match play the weaker player can shorten the duration of the match (and thus the number of decisions) by doubling aggressively, but the stronger player can make similar adjustments and with greater accuracy.

5. References

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A.1. Unequal Skill Cubeful MET

	PC	-1	-2	-3	-4	-5	-6	-7	-8	-9	-10	-11	-12	-13
PC	51.35		52.58	69.93	71.21	82.67	83.64	89.83	90.52	94.07	94.52	96.53	96.83	
-1		51.35	69.93	76.33	83.13	85.37	90.11	91.57	94.27	95.05	96.66	97.12	98.05	98.31
-2	50.12	33.40	51.35	61.68	69.09	76.44	81.85	85.91	89.12	91.51	93.52	94.94	96.14	96.97
-3	33.40	26.37	41.91	51.81	59.57	67.12	73.57	78.41	82.78	85.97	88.97	91.08	93.01	94.41
-4	32.12	19.62	34.61	44.28	51.98	59.87	66.58	72.25	76.97	81.04	84.50	87.33	89.75	91.65
-5	20.08	17.15	27.44	36.78	44.39	52.18	59.22	65.18	70.56	75.14	79.23	82.68	85.63	88.15
-6	19.11	12.05	21.38	30.41	37.66	45.62	52.45	58.86	64.36	69.55	74.12	77.99	81.46	84.38
-7	12.33	10.31	17.10	25.35	31.96	39.55	46.18	52.63	58.31	63.72	68.66	72.94	76.79	80.21
-8	11.64	7.33	13.51	20.89	27.09	34.23	40.62	47.08	52.75	58.35	63.36	67.97	72.19	75.92
-9	7.53	6.33	10.83	17.40	22.85	29.45	35.45	41.74	47.36	52.96	58.12	62.91	67.35	71.48
-10	7.08	4.46	8.46	14.11	19.10	25.12	30.85	36.85	42.32	47.93	53.09	58.07	62.59	66.87
-11	4.59	3.87	6.79	11.68	16.06	21.52	26.67	32.36	37.67	43.09	48.30	53.23	57.92	62.32
-12	4.29	2.72	5.32	9.45	13.36	18.24	22.99	28.30	33.37	38.68	43.64	48.69	53.34	57.89
-13		2.36	4.27	7.77	11.14	15.50	19.80	24.62	29.48	34.43	39.41	44.24	48.93	53.51

This table can be used in matches between players with an ELO difference of approximately 47. Since everyone uses PR as a rating system though, we can assume an average of 20 decisions per player in a cubeful game and use the approximately 0.06ppg edge of 1ply over 0ply we found in Chapter 2.2 to calculate the corresponding PR difference of 1.5 (millipoints per decision). Scaling up these figures, we arrive at the following equivalence:

$$2.5\,PR \approx 0.1\,ppg = 80\,ELO$$

Finally, it’s worth noting that because the edge of the better player in cubeful games is about twice as big compared to cubeless games, a factor of 2 needs to be applied to the PR in cubeless games and all Crawford scores. At the moment, GNUbg makes no such adjustment, whereas XG uses a factor of 1.5 but only at DMP.

A.2. The DMP Skill Formula

$$S(N) = \lim_{W \to 1/2} S(N,W) = \lim_{W \to 1/2} \frac{\log\left(\frac{1}{P(N,W)} - 1\right)}{\log\left(\frac{1}{W} - 1\right)}$$

Since an N-point cubeless DMP match is equivalent to a best of 2N-1 match, we can use the binomial distribution to calculate the probability of the better player winning:

$$P(N,W) = \sum_{K=N}^{2N-1} \binom{2N-1}{K} W^K (1-W)^{2N-1-K}$$

Using L' Hopital's rule we get

$$S(N) = \lim_{W \to 1/2} \frac{(1-W) \cdot W}{[1-P(N,W)] \cdot P(N,W)} \sum_{K=N}^{2N-1} \binom{2N-1}{K} \cdot W^{K-1} \cdot (1-W)^{2N-2-K} \cdot [K-(2N-1) \cdot W]$$

Since $P(N,1/2)=1/2$ we have

$\sum_{K=N}^{2N-1} \binom{2N-1}{K} = P(N,1/2) \cdot 2^{2N-1} = 2^{2N-2}$ and our limit becomes

$$S(N) = 2^{3-2N} \cdot \sum_{K=N}^{2N-1} \binom{2N-1}{K} \cdot K - \frac{2N-1}{2^{2N-2}} \sum_{K=N}^{2N-1} \binom{2N-1}{K} \Leftrightarrow$$

$$S(N) = (2N-1) \cdot 2^{3-2N} \cdot \sum_{K=N}^{2N-1} \binom{2N-2}{K-1} - (2N-1)$$

Let $A = \sum_{K=N}^{2N-1} \binom{2N-2}{K-1} = \sum_{K=N-1}^{2N-2} \binom{2N-2}{K} = \sum_{K=0}^{N-1} \binom{2N-2}{K} \Rightarrow$

$$2A = \sum_{K=0}^{N-1} \binom{2N-2}{K} + \sum_{K=N-1}^{2N-2} \binom{2N-2}{K} = \binom{2N-2}{N-1} + \sum_{K=0}^{2N-2} \binom{2N-2}{K} = \binom{2N}{N} \frac{N}{2 \cdot (2N-1)} + 2^{2N-2} \Rightarrow$$

$$S(N) = \binom{2N}{N} \frac{2N}{2^{2N}} \quad \blacksquare$$

A.3. The DMP Duration Formula

Since an N-point cubeless DMP match will end after a minimum of N games and a maximum of 2N-1 games, the expected duration can be expressed as:

$$D(N)=\sum_{K=N}^{2N-1} K\cdot P(K)$$

where P(K) is the probability of the match lasting exactly K games. In order for the match to end after K games, the winner of game K must win N-1 of the first K-1 games. There are C(K-1,N-1) combinations in which this happens. Assuming both players are equally likely to win a single game, the probability of each combination is 2^{1-K} and thus we have:

$$D(N)=\sum_{K=N}^{2N-1} K\cdot\binom{K-1}{N-1}\cdot 2^{1-K}=N\sum_{K=N}^{2N-1}\binom{K}{N}\cdot 2^{1-K} \Leftrightarrow$$

$$D(N)=N\sum_{K=0}^{N-1}\binom{K+N}{N}2^{1-K-N}=N\cdot 2^{1-2N}\sum_{K=0}^{N-1}\binom{N+K}{N}\cdot 2^{N-K} \Leftrightarrow$$

$$D(N)=N\cdot 2^{1-2N}\left\{\sum_{K=0}^{N}\binom{N+K}{N}\cdot 2^{N-K}-\binom{2N}{N}\right\}$$

To simplify our calculations, let's consider a best of 2N+1 match, equivalent to an (N+1)-point match. Let K be the score of the loser of the match. The game that decides the match is preceded by exactly N games won by the winner and K games won by the loser. These N+K games can occur in any order. Now imagine that the maximum of 2N+1 games are played even if the winner's already decided. Thus, there remain N-K games which can go either way. By symmetry – assuming both players are equally likely to win – the total number of sequences of 2N+1 games is equal to twice the total number of sequences that decide the outcome in the (N+K+1)-st game. Therefore, we have:

$$2\sum_{K=0}^{N}\binom{N+K}{K}\cdot 2^{N-K}=2^{2N+1} \Leftrightarrow \sum_{K=0}^{N}\binom{N+K}{N}\cdot 2^{N-K}=2^{2N} \Rightarrow$$

$$D(N)=2N\cdot\left\{1-\binom{2N}{N}\cdot 2^{-2N}\right\} \Leftrightarrow$$

$$D(N)=2N-S(N) \quad \blacksquare$$