Relativistic Effects on a Damped Harmonic Oscillator

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Abstract: In this study, we have attempted to study the damped harmonic oscillator and apply relativistic effects to it. We have expressed the equations in a format which can might be of interest to the reader to attempt a solution for the various cases such as overdamped, critically damped and other possible cases. Since this is preliminary study so we have not attempted to working out a solution, but we have expressed a compact equation.

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1 Introduction

In this section we will discus briefly about the harmonic oscilloator and its physical equations. These equations give a mathematical treatment to the harmonic oscillator.

1.1 Fundamentals of a Harmonic Oscillator

Regular Oscillator: The harmonic oscillator is one of the most useful concepts in theoretical and experimental physics equally. The applications of a harmonic oscillator have been widely used in all fields such as physics, engineering, chemistry, biology and even in fields such as biomechanics [1, 2]. The interdisciplinary uses have been numerous. Even though we understand the physics behind the harmonic oscillator pretty well, there is still so much more to understand and apply this system. Some examples where the understanding of a harmonic oscillator is being applied other than physics are as a thermostat trying to adjust a temperature, interactions involved and driving chemical reactions, growth of bacterial colonies and how they survive and evolve, and even as diverse as foxes eating rabbits and rabits eating plants [3]. The list is endless, but all of this can be explained on the basis of the simplest system which represents a regular harmonic oscillator, that is, a mass on a spring which is shown in the diagram below.
where the red color shows compression, blue color shows elongation and green shows any

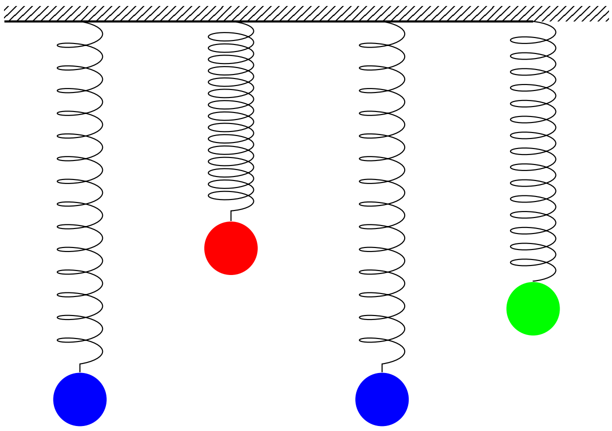


Figure 1. Harmonic oscillator diagram

intermediate position between maximum and minimum stretch. This system as shown here is not under the effect of any external forces.
Taking the simple case of a linear spring so that the force pulling the string back when it is stretched, is proportional to the amount of stretch. The force equation is written as,

$$m\frac{d^2x}{dt^2} = -kx \tag{1.1}$$

where m is the mass and k is the spring constant. The intitial conditions of such a system are $x(t = 0) = x_0$ and $\frac{dx}{dt}(t = 0) = v_0$. The general solution of eqn.(2.1) can be written as,

$$x = x_0cos(\omega t) + \frac{v_0}{\omega}sin(\omega t) \tag{1.2}$$

where $\omega = \sqrt{k/m}$.

Forced Oscillator: In this case we will discuss about an oscillator under the effect of an external force. So there is an external force making changes in the way the oscillations happen. The equation for a forced oscillator can be written as,

$$m\frac{d^2x}{dt^2} = -kx + F(t) \tag{1.3}$$

here $F(t)$ is the force which is causing bthe change and deviation from the regular motion of an oscillator. In this situation, m oscillates at the frequency that the force makes it to and also upon the frequency of the regular motion of the oscillator. There have been numerous studies on the types of forced oscillations and the applications of it in interdisciplinary fields are numerous. Some of these studies are given in [4–7]. However, the scope of this article is not to go into the details of forced vibrations.

1.2 Relativistic Description of Harmonic Oscillator

The equation for the harmonic oscillator can also be written as [8],

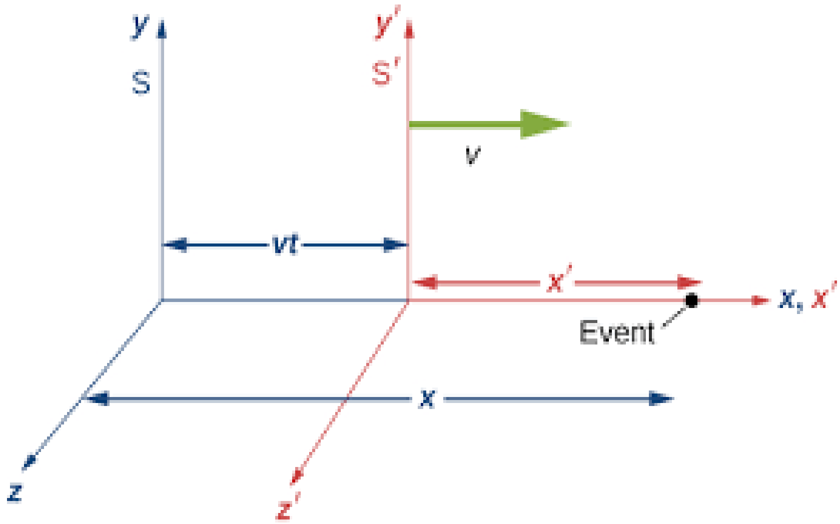
$$\frac{dp}{dt} + kx = 0 \tag{1.4}$$

where k is an arbitrary constant and p is the momentum. In the approximation of non-relativistic case equation (1.4) reduces to,

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \tag{1.5}$$

where ω is the angular frequency. Let us take two reference frames as shown in Figure 2. with axes (x, y, z) denoted by S, and (x', y', z') denoted by S', with S' moving with a velocity v with repect to S [9]. Here we are considering motion along the x-axis only.

Figure 2. Frame of reference in relativity



The relativistic momentum is defined by

$$p = \gamma m_0 v \tag{1.6}$$

where $\gamma = \left(\sqrt{1-\frac{v^2}{c^2}}\right)^{-1}$, m_0 is the rest mass. We can write equation of motion as

$$\frac{dp}{dt} + kx = 0 \tag{1.7}$$

Using equation (1.6) in equation (1.7) we get

$$\gamma^3 m_0 \frac{d^2x}{dt^2} + kx = 0 \tag{1.8}$$

using $\omega^2 = k/m_0$, we get

$$\gamma^3 \frac{d^2x}{dt^2} + \omega^2 x = 0 \tag{1.9}$$

using the boundary conditions $\frac{dx}{dt} = 0$ at $x = \pm X_0$, we get the solution as [8],

$$\dot{x} = c\left[1 - \frac{c^4}{(A - \frac{\omega^2 x^2}{2})^2}\right]^{\frac{1}{2}} \tag{1.10}$$

where $A = c^2 + \frac{\omega^2 x_0{}^2}{2}$. The time period of the oscillation is given by,

$$T = \frac{4}{c}\int_0^{x_0} \left[1 - \frac{c^4}{(A - \frac{\omega^2 x^2}{2})^2}\right]^{-\frac{1}{2}} dx$$

Depending on the type of harmonic oscillator, the time period can be calculated. If there is not a closed solution then we will have to use numerical techniques to calculate it. The aim of this article is not to calculate the integral for any specific harmonic oscillator, but to show express an idea that this type of relativistic effects can be included.

2 Relativistic Effects Applied to a Damped Harmonic Oscillator

A damped harmonic oscillator is one which is subjected to or is under the influence of an external force[10]. The mathematical equation governing a damped harmonic oscillator is given as,

$$m_0\ddot{x} + \alpha\dot{x} + kx = 0 \tag{2.1}$$

where α is the damping term. when $\alpha = 0$ we get the harmonic oscillator without damping. Equation(2.1) can be written as

$$\frac{dp}{dt} + \frac{\alpha}{m_0}p + kx = 0 \tag{2.2}$$

Using p from equation (1.6), we get

$$\gamma^3 m_0 \frac{d^2x}{dt^2} + (\frac{\alpha}{m_0})\gamma m_0 \frac{dx}{dt} + kx = 0 \tag{2.3}$$

Simplifying, we get,

$$\gamma^3 \frac{d^2x}{dt^2} + \gamma\sigma\frac{dx}{dt} + \omega^2 x = 0 \tag{2.4}$$

where $\sigma = \alpha/m_0$ and $\omega = k/m_0$.
This equation can be solved in multiple ways depending on the type of of the damping. The autnors aim is to express the equations and show that this can be done and the equations can be formulated in the relativistic case.

3 Conclusion

We have studied the harmonic oscillator in the regular non-damped situation and a damped situation applied to the relativistic case. Since this is a preliminary study, so we have not attempted at a specific solution for the harmonic oscillator. Our aim behind this study is to show that the equations for a relativistic damped oscillator can be expressed in a certain format. To undertsna dthe solution to this relativistic damped harmonic oscillator, we have to study the solutions of a damped oscillator in a non-relativistic case and apply it to a relativistic scenario. This is a long term study and the we are planning to continue it in the future.

4 Conflict of Interest

The authors declare no conflict of interest.

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