Two Types of Universal Arrows

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Abstract A universal arrow is a pair which consists of an object and a morphism. And an isomorphism is defined by a universal arrow. The isomorphism may be a composition of two morphisms. We may define two types of universal arrows, which is determined by the properties of the morphisms. A universal arrow is of the type I if the morphisms are not isomorphisms; And a universal arrow is of the type II if the morphisms are isomorphisms.

CONTENTS

1. Introduction

Suppose that $F\colon \mathcal{C} \to \mathcal{D}$ is a functor. Given a $D \in \mathcal{D}$. A universal arrow from F to D is a pair $\langle R, u\rangle$ consisting of an object $R \in \mathcal{C}$ and a morphism $u\colon F(R) \to D$ in $\mathcal{D}$ such that the equation (3.1) holds. See definition 3.1 for more details.

The equation (3.1) factors as $\tilde{u} \circ \vec{F}$. Then we define two types of universal arrows in definition 3.2. A universal arrow is of the type I if $\vec{F}$ and $\tilde{u}$ are not isomorphic; And a universal arrow is of the type II if $\vec{F}$ and $\tilde{u}$ are isomorphisms. See section 3.1 for more details.

A limit $\varprojlim F$ of a functor F is defined by a universal arrow from Δ to F. This universal arrow is of the type II, see proposition 3.2. There are universal arrows determined by an adjunction $\langle F, G, \phi\rangle$. These universal arrows are of the type I in general. But if some conditions are satisfied, then the universal arrow is of the type II, see proposition 3.3. Furthermore, other examples are given in section 3.2.

2. Preliminaries

Definition 2.1 ([4–6])**.** A **category** $\mathcal{C}$ consists of:

- a collect of **objects**;

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- for each pair $A, B \in \mathcal{C}$, a collect $\mathrm{Hom}_{\mathcal{C}}(A, B)$ of **morphisms** from A to B;
- for each triple $A, B, C \in \mathcal{C}$, a function

$$\mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$$

  given by

$$(g, f) \mapsto g \circ f,$$

  call **composition**;
- for each $A \in \mathcal{C}$, a morphism $id_A \in \mathrm{Hom}(A, A)$, called **identity** on A,

satisfying the following axioms:

associativity: for each $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, and $h \in \mathrm{Hom}_{\mathcal{C}}(C, D)$

$$h \circ (g \circ f) = (h \circ g) \circ f;$$

identity law: for each $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$,

$$id_B \circ f = f \circ id_A = f.$$

Definition 2.2 ([4–6])**.** Let $\mathcal{C}, \mathcal{D}$ be categories. A **functor** $F\colon \mathcal{C} \to \mathcal{D}$ is a morphism consisting of:

- assigning to each object $C \in \mathcal{C}$ an object $F(C) \in \mathcal{D}$;
- assigning to each morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ a morphism $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$,

satisfying the following axioms:

- $F(g \circ f) = F(g) \circ F(f)$ for each composition $g \circ f$;
- $F(id_A) = id_{F(A)}$ for each object $A \in \mathcal{C}$.

Definition 2.3 ([4–6])**.** Let $\mathcal{C}, \mathcal{D}$ be categories, and let $\mathcal{C} \overset{F}{\underset{G}{\rightrightarrows}} \mathcal{D}$ be functors. A morphism τ from F to G is called a **natural transformation**, written $\tau\colon F \stackrel{\cdot}{\to} G$, provided that τ is a function which assigns to each $C \in \mathcal{C}$ a morphism $\tau_C := \tau(C)\colon F(C) \to G(C)$ in $\mathcal{D}$ such that for each morphism $f\colon C \to C'$ in $\mathcal{C}$ the following diagram commutes in $\mathcal{D}$.

$$\begin{array}{ccc} F(C) & \xrightarrow{\tau_C} & G(C) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(C') & \xrightarrow[\tau_{C'}]{} & G(C') \end{array}$$

Definition 2.4 ([4])**.** Let $\mathcal{C}$, $\mathcal{D}$ be categories, and $S, T\colon \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ functors. A **dinatural transformation** $\tau\colon S \stackrel{\cdot\cdot}{\to} T$ is a function which assigns to each object $C \in \mathcal{C}$ a morphism $\tau_C := \tau(C)\colon S(C, C) \to T(C, C)$ of $\mathcal{D}$ in such a way that for every morphism $f\colon C \to C'$ in $\mathcal{C}$ the following diagram is commutative.

$$\begin{array}{ccccc} & S(C,C) & \xrightarrow{\tau_C} & T(C,C) & \\ {\scriptstyle S(f,1)}\nearrow & & & & \searrow{\scriptstyle T(1,f)} \\ S(C',C) & & & & T(C,C') \\ {\scriptstyle S(1,f)}\searrow & & & & \nearrow{\scriptstyle T(f,1)} \\ & S(C',C') & \xrightarrow[\tau_{C'}]{} & T(C',C') & \end{array}$$

3. Two Types of Universal Arrows

Recall the definition of a universal arrow.

Definition 3.1 ([4–6])**.** Let F be a functor from $\mathcal{C}$ to $\mathcal{D}$. Given an object $D \in \mathcal{D}$. a **universal arrow** from F to D is a pair $\langle R, u\rangle$ consisting of an object $R \in \mathcal{C}$ and a morphism u from $F(R)$ to D, such that for all object $C \in \mathcal{C}$ and every morphism $g: F(C) \to D$, there exists a unique morphism $f: C \to R$ with $g = u \circ F(f)$.

Furthermore, there is the dual concept of definition 3.1.

3.1. The Definition of Two Types.

It is clear that if the pair $\langle R, u\rangle$ is a universal arrow, then we have the following isomorphism[2, 4] for all $C \in \mathcal{C}$.

$$\mathrm{Hom}_{\mathcal{C}}(C, R) \cong \mathrm{Hom}_{\mathcal{D}}(F(C), D) \tag{3.1}$$

Let $\vec{F}$ denote the restriction of the functor F to the hom-sets, and let

$$\tilde{u}\colon \mathrm{Hom}_{\mathcal{D}}(F(C), F(R)) \to \mathrm{Hom}_{\mathcal{D}}(F(C), D)$$

be a morphism given by

$$h \mapsto u \circ h.$$

Then the equation (3.1) factors as $\tilde{u} \circ \vec{F}$:

$$\mathrm{Hom}_{\mathcal{C}}(C, R) \xrightarrow{\vec{F}} \mathrm{Hom}_{\mathcal{D}}(F(C), F(R)) \xrightarrow{\tilde{u}} \mathrm{Hom}_{\mathcal{D}}(F(C), D). \tag{3.2}$$

Observation 3.1. Since $\tilde{u} \circ \vec{F}$ is an isomorphism, we have that $\vec{F}$ is monic[4–6], and $\tilde{u}$ is epic[4–6].

And the restriction of $\tilde{u}$ to the image of $\mathrm{Hom}_{\mathcal{C}}(C, R)$ under $\tilde{F}$ is monic. Hence if

$$F(\mathrm{Hom}_{\mathcal{C}}(C, R)) = \mathrm{Hom}_{\mathcal{D}}(F(C), F(D)), \tag{3.3}$$

then $\vec{F}$ and $\tilde{u}$ are isomorphisms.

Observation 3.2. We have that $\vec{F}$ is an isomorphism if and only if $\tilde{u}$ is an isomorphism.

Furthermore, if the condition (3.3) is not satisfied, then we have that for every $h \in \mathrm{Hom}_{\mathcal{D}}(F(C), F(R))$ with $h \notin \mathrm{im}\,\vec{F}$, there exists a unique $f \in \mathrm{Hom}_{\mathcal{C}}(C, R)$ such that

$$u \circ h = u \circ F(f).$$

Therefore, we may define two types of universal arrows as follows.

Definition 3.2. Let the notations be as in equations (3.1) and (3.2) and definition 3.1.

I. The morphisms $\vec{F}$ and $\tilde{u}$ are not isomorphic;
II. The morphisms $\vec{F}$ and $\tilde{u}$ are isomorphisms.

Some examples will be given.

3.2. **Examples.**

Notation 3.1. For an arbitrary functor F, let $\vec{F}$ denote the restriction of F to the hom-sets, and let $\dot{F}$ denote the restriction of F to the objects. For an arbitrary category $\mathcal{C}$ with an arbitrary morphism $u\colon B \to C \in \mathcal{C}$, let $\tilde{u}$ denote the morphism defined as follows:

$$\tilde{u}\colon \begin{cases} \mathrm{Hom}(A,B) \to \mathrm{Hom}(A,C) & \text{given by } f \mapsto u \circ f, \text{ or,} \\ \mathrm{Hom}(C,A) \to \mathrm{Hom}(B,A) & \text{given by } g \mapsto g \circ u, \text{ but not both.} \end{cases}$$

Suppose that $F\colon \mathcal{C} \to \mathcal{D}$ is a functor such that $\dot{F}(C) = D$ for every object $C \in \mathcal{C}$ and a fixed object $D \in \mathcal{D}$. Let $D \neq D' \in \mathcal{D}$. We assume that the pair $\langle R, u\rangle$ is a universal arrow from F to D'. For all $C \in \mathcal{C}$, we have that

$$\mathrm{Hom}_{\mathcal{C}}(C,R) \cong \mathrm{Hom}_{\mathcal{D}}(D,D'),$$

and

$$\mathrm{Hom}_{\mathcal{C}}(C,R) \xrightarrow{\vec{F}} \mathrm{Hom}_{\mathcal{D}}(D,D) \xrightarrow{\tilde{u}} \mathrm{Hom}_{\mathcal{D}}(D,D').$$

Hence for all $C \in \mathcal{C}$,

$$\mathrm{Hom}_{\mathcal{C}}(C,R) \cong \mathrm{Hom}_{\mathcal{C}}(R,R) \cong \mathrm{Hom}_{\mathcal{D}}(D,D').$$

And if the identity morphism $id_D \in \mathrm{Hom}_{\mathcal{D}}(D,D)$ is not in $\vec{F}(\mathrm{Hom}_{\mathcal{C}}(C,R))$, then there exists a unique $f \in \mathrm{Hom}_{\mathcal{C}}(C,R)$ such that $u \circ F(f) = u \circ id_D = u$. This is possible. Furthermore, for all $C \in \mathcal{C}$, if the equation

$$\mathrm{Hom}_{\mathcal{C}}(R,R) \cong \mathrm{Hom}_{\mathcal{D}}(D,D)$$

holds, then the universal arrow is of the type II, otherwise the universal arrow is of the type I.

Let G be a directed graph, and $G' \subset G$ a subgraph of G. Suppose that F is an inclusion functor from G' to G. Given a vertex $g \in G$ with $g \notin G'$. We assume that a universal arrow $\langle r, u\rangle$ from F to g exists. Then we have that

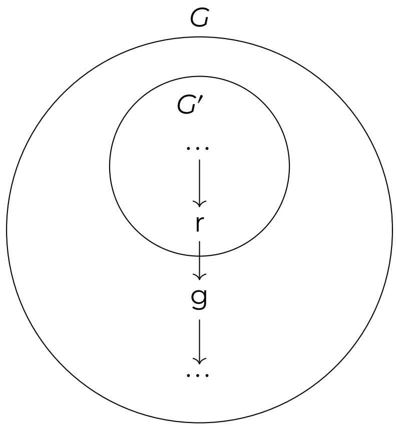
$$\mathrm{Hom}_{G'}(v,r) \cong \mathrm{Hom}_{G}(v,g),$$

and

$$\mathrm{Hom}_{G'}(v,r) \xrightarrow{\vec{F}} \mathrm{Hom}_{G}(v,r) \xrightarrow{\tilde{u}} \mathrm{Hom}_{G}(v,g),$$

for all $v \in G'$. Hence the morphism $u\colon r \to g$ is a unique edge from the subgraph G' to g if G' is finite*.

*The finiteness hypothesis is necessary.



Proposition 3.1. *The maps $\vec{F}$ and $\tilde{u}$ are isomorphisms. Thus the universal arrow is of the type II.*

Proof. It is evident. □

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor, and $\Delta\colon \mathcal{D} \to \mathcal{D}^{\mathcal{C}}$ a diagonal[4] functor. Then a **limit** of the functor F is a universal arrow $\langle R, \tau\rangle$ from Δ to F. The object $R \in \mathcal{D}$ is called **limit object**[4], written $\varprojlim F := R$, and for every natural transformation[4] $\sigma\colon \Delta(D) \dot{\to} F$, there exists a unique $f\colon D \to R$ in $\mathcal{D}$ such that σ factors through $\Delta(f)$ along $\tau\colon \Delta(R) \dot{\to} F$, cf. [4–6]. Hence we have that

$$\mathrm{Hom}_{\mathcal{D}}(D, R) \cong \mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}(\Delta(D), F),$$

for all $D \in \mathcal{D}$. Therefore, we have that

$$\mathrm{Hom}_{\mathcal{D}}(D, R) \xrightarrow{\vec{\Delta}} \mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}(\Delta(D), \Delta(R)) \xrightarrow{\tilde{\tau}} \mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}(\Delta(D), F).$$

The maps $\vec{\Delta}$ and $\tilde{\tau}$ are isomorphisms.

Proposition 3.2. *The universal arrow of every (co)limit is of the type II.*

Proof. This follows immediately from the definition of a diagonal functor. □

Let $\boldsymbol{A}$ be an abelian[2, 3] group, and $\boldsymbol{A} = M_0 \supset M_1 \supset M_2 \supset \cdots$ a sequence of subgroups. Suppose that $\mathcal{N}$ is a category consisting of

objects: nonnegative integers,
morphisms: $i \to j$ if $i \geq j$.

Let $F\colon \mathcal{N} \to \boldsymbol{Ab}$ be a functor, which assigns to a nonnegative number i a factor group[2, 3] $\boldsymbol{A}/M_i$, and assigns to a morphism $i \to j$ a canonical[2] epimorphism[2, 3] $\boldsymbol{A}/M_i \to \boldsymbol{A}/M_j$ given by $a + M_i \mapsto a + M_j$, and let $\Delta\colon \boldsymbol{Ab} \to \boldsymbol{Ab}^{\mathcal{N}}$ be a diagonal[4] functor. Then the pair $\langle \varprojlim F, \tau\rangle$ is a universal arrow from Δ to F. And We call $\varprojlim F$

the **completion**(denoted $\hat{\boldsymbol{A}} := \varprojlim F$) of $\boldsymbol{A}$ with respect to M_i, cf. [1, 7]. Hence we have that

$$\mathrm{Hom}_{\boldsymbol{Ab}}(\boldsymbol{B}, \hat{\boldsymbol{A}}) \cong \mathrm{Hom}_{\boldsymbol{Ab}^{\mathcal{N}}}(\Delta(\boldsymbol{B}), F),$$

and

$$\mathrm{Hom}_{\boldsymbol{Ab}}(\boldsymbol{B}, \hat{\boldsymbol{A}}) \xrightarrow{\vec{F}} \mathrm{Hom}_{\boldsymbol{Ab}^{\mathcal{N}}}(\Delta(\boldsymbol{B}), \Delta(\hat{\boldsymbol{A}})) \xrightarrow{\tilde{\tau}} \mathrm{Hom}_{\boldsymbol{Ab}^{\mathcal{N}}}(\Delta(\boldsymbol{B}), F),$$

for all $\boldsymbol{B} \in \boldsymbol{Ab}$. Furthermore, we have that

$$\varprojlim F := \{(a_0, a_1, \dots) \in \prod_i \boldsymbol{A}/m_i \mid a_i \equiv a_j \pmod{m_j} \text{ for all } i \geq j\}.$$

It is clear that $\vec{F}$ and $\tilde{\tau}$ are isomorphisms.

Let $\mathcal{X}$ and $\mathcal{Y}$ be categories. An **adjunction**[3, 4, 7] from $\mathcal{X}$ to $\mathcal{Y}$ is a triple $\langle F, G, \phi \rangle$ consisting of two functors

$$\mathcal{X} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{Y},$$

and a map ϕ which assigns to every pair $\langle X \in \mathcal{X}, Y \in \mathcal{Y} \rangle$ an isomorphism of hom-sets

$$\phi \colon (X, Y) \mapsto \phi_{X,Y} \colon \mathrm{Hom}_{\mathcal{X}}(X, G(Y) \cong \mathrm{Hom}_{\mathcal{Y}}(F(X), Y), \tag{3.4}$$

which is natural[4] in $\mathcal{X}$ and $\mathcal{Y}$. For all pair $\langle X \in \mathcal{X}, Y \in \mathcal{Y} \rangle$, we have that every morphism $f \colon X \to G(Y)$ makes the diagram (3.5) commute, cf. [4–6].

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{X}}(X, G(Y)) & \xrightarrow[\cong]{\phi_{X,Y}} & \mathrm{Hom}_{\mathcal{Y}}(F(X), Y) \\ \uparrow{\scriptstyle \tilde{f}} & & \uparrow{\scriptstyle \widetilde{F(f)}} \\ \mathrm{Hom}_{\mathcal{X}}(G(Y), G(Y)) & \xrightarrow[\phi_{G(Y),Y}]{\cong} & \mathrm{Hom}_{\mathcal{Y}}(F \circ G(Y), Y) \end{array} \tag{3.5}$$

Observe that an identity morphism $id_{G(Y)} \in \mathrm{Hom}_{\mathcal{X}}(G(Y), G(Y))$. Hence for every $v \in \mathrm{Hom}_{\mathcal{Y}}(F(X), Y)$, there exists a unique morphism $f \in \mathrm{Hom}_{\mathcal{X}}(X, G(Y))$ such that $v = F(f) \circ u$, where

$$u := (\phi_{G(Y),Y}(id_{G(Y)}) \colon F \circ G(Y) \to Y),$$

that is, the morphism u is the image of the identity morphism $id_{G(Y)}$ under the map $\phi_{G(Y),Y}$.

Given a $Y \in \mathcal{Y}$. Then we have that the pair $\langle G(Y), u \rangle$ is a universal arrow from F to Y by diagram (3.5). Hence we have that

$$\mathrm{Hom}_{\mathcal{X}}(X, G(Y)) \xrightarrow{\vec{F}} \mathrm{Hom}_{\mathcal{Y}}(F(X), F \circ G(Y)) \xrightarrow{\tilde{u}} \mathrm{Hom}_{\mathcal{Y}}(F(X), Y),$$

for all $X \in \mathcal{X}$. In general $\vec{F}$ and $\tilde{u}$ need not be isomorphisms. Hence the universal arrow is of the type II if further conditions are satisfied.

Proposition 3.3. *The universal arrow $\langle G(Y), u \rangle$ is of the type II if and only if u is a monomorphism.*

Proof. By observation 3.1, $\tilde{u}$ is an epimorphism. Hence we have that $\tilde{u}$ is an isomorphism if and only if u is a monomorphism. Therefore, $\vec{F}$ is an isomorphism if and only if u is a monomorphism by observation 3.2. □

Remark. By equation (3.4), we have that the diagram (3.6) is commutative.

(3.6)
$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{X}}(X, G(Y)) & \xrightarrow[\cong]{\phi_{X,Y}} & \mathrm{Hom}_{\mathcal{Y}}(F(X), Y) \\ {\scriptstyle \widetilde{G(u)}}\uparrow & & \uparrow{\scriptstyle \tilde{u}} \\ \mathrm{Hom}_{\mathcal{X}}(X, G \circ F \circ G(Y)) & \xrightarrow[\phi_{X,F\circ G(Y)}]{\cong} & \mathrm{Hom}_{\mathcal{Y}}(F(X), F \circ G(Y)) \end{array}$$

Therefore, it is clear that if $\tilde{u}$ is an isomrphism, then we have that

$$\mathrm{Hom}_{\mathcal{X}}(X, G(Y)) \cong \mathrm{Hom}_{\mathcal{Y}}(F(X), F \circ G(Y)).$$

Of course, the dual statements hold by the dual arguments. We shall give some examples.

Let $H\colon \boldsymbol{Grp} \to \boldsymbol{Set}$ be a forgetful[2, 4, 7] functor which assigns to a group $\boldsymbol{G}$ the underlying[2, 7] set of $\boldsymbol{G}$ and assigns to a homomorphism of groups a map of sets, and let $F\colon \boldsymbol{Set} \to \boldsymbol{Grp}$ be a functor which assigns to a set X a free group[2, 3] $F(X)$ generated by X and assigns to a map $f\colon X \to Y$ a homomorphism[2, 3] $F(f)\colon F(X) \to F(Y)$ induced by f.

Remark. For a group $\boldsymbol{G} \in \boldsymbol{Grp}$ with an identity member $id \in \boldsymbol{G}$, the member $id \in H(\boldsymbol{G})$ is a normal element of the set $H(\boldsymbol{G})$. Hence the member $id \in F \circ H(\boldsymbol{G})$ is *not* the identity member of the group $F \circ H(\boldsymbol{G})$.

For every pair $\langle X \in \boldsymbol{Set}, \boldsymbol{G} \in \boldsymbol{Grp}\rangle$, we have that

(3.7)
$$\mathrm{Hom}_{\boldsymbol{Set}}(X, H(\boldsymbol{G})) \cong \mathrm{Hom}_{\boldsymbol{Grp}}(F(X), \boldsymbol{G}).$$

Hence the functor F is the adjoint of H, cf. [2, 4, 7].

Given a nonempty set $X \in \boldsymbol{Set}$. Then we have that the pair $\langle F(X), \iota\rangle$ is a universal arrow from the set X to the functor H, where $\iota\colon X \to H \circ F(X)$ is an inclusion map. Therefore, we have that

$$\mathrm{Hom}_{\boldsymbol{Grp}}(F(X), \boldsymbol{G}) \xrightarrow{\vec{H}} \mathrm{Hom}_{\boldsymbol{Set}}(H \circ F(X), H(\boldsymbol{G})) \xrightarrow{\tilde{\iota}} \mathrm{Hom}_{\boldsymbol{Set}}(X, H(\boldsymbol{G})),$$

for all $\boldsymbol{G} \in \boldsymbol{Grp}$.

Proposition 3.4. *The map $\vec{H}$ and $\tilde{\iota}$ are isomorphisms if and only if $X \cong H \circ F(X)$.*

Proof. Observe that ι is a monomorphism, and $\tilde{\iota}$ is an epimorphism. It follows that $\tilde{\iota}$ is an isomorphism if and only if ι is an epimorphism. Hence $\vec{H}$ is an isomorphism if and only if ι is an epimorphism. □

Remark. If the set X is finite, then $\vec{H}$ and $\tilde{\iota}$ are not isomorphisms. And the converse does not hold. If $X \cong H \circ F(X)$, then the set X should be a denumerable[8] set.

On the other hand, given a group $\boldsymbol{G} \in \boldsymbol{Grp}$, we have that the pair $\langle H(\boldsymbol{G}), \pi \rangle$ is a universal arrow from the functor F to the group $\boldsymbol{G}$, where $\pi\colon F \circ H(\boldsymbol{G}) \to \boldsymbol{G}$ is a canonical epimorphism[2, 3]. Therefore, we have that

$$\mathrm{Hom}_{\boldsymbol{Set}}(X, H(\boldsymbol{G})) \xrightarrow{\vec{F}} \mathrm{Hom}_{\boldsymbol{Grp}}(F(X), F \circ H(\boldsymbol{G})) \xrightarrow{\tilde{\pi}} \mathrm{Hom}_{\boldsymbol{Grp}}(F(X), \boldsymbol{G}),$$

for all $X \in \boldsymbol{Set}$.

Proposition 3.5. *The maps $\vec{F}$ and $\tilde{\pi}$ are isomorphisms if and only if $F \circ H(\boldsymbol{G}) \cong \boldsymbol{G}$.*

Proof. Observe that π and $\tilde{\pi}$ are epimorphisms. Thus the morphism $\tilde{\pi}$ is an isomorphism if and only if π is a monomorphism. This implies that $\vec{F}$ is an isomorphism if and only if π is a monomorphism. □

Remark. if the group $\boldsymbol{G}$ is finite, then $\vec{F}$ and $\tilde{\pi}$ are not isomorphisms. But the converse is not true.

Let $\boldsymbol{R}, \boldsymbol{S}$ be rings[2, 3], and ${}_{\boldsymbol{R}}\boldsymbol{M}_{\boldsymbol{S}}$ a bimodule[2, 7]. Suppose that $\mathcal{S}$ and $\mathcal{R}$ are categories of right $\boldsymbol{S}$-modules and right $\boldsymbol{R}$-modules, respectively. Then the functor $- \otimes_{\boldsymbol{R}} \boldsymbol{M}$ is the adjoint of the functor $\mathrm{Hom}_{\mathcal{S}}(\boldsymbol{M}, -)$, cf. [2, 4, 7].

Let F denote $\mathrm{Hom}_{\mathcal{S}}(\boldsymbol{M}, -)$, and let G denote $- \otimes_{\boldsymbol{R}} \boldsymbol{M}$. Hence we have that

$$\mathrm{Hom}_{\mathcal{S}}(\boldsymbol{A} \otimes_{\boldsymbol{R}} \boldsymbol{M}, \boldsymbol{B}) \cong \mathrm{Hom}_{\mathcal{R}}(\boldsymbol{A}, \mathrm{Hom}_{\mathcal{S}}(\boldsymbol{M}, \boldsymbol{B})),$$

and

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{S}}(\boldsymbol{A} \otimes_{\boldsymbol{R}} \boldsymbol{M}, \boldsymbol{B}) & \xrightarrow[\cong]{\phi_{\boldsymbol{A},\boldsymbol{B}}} & \mathrm{Hom}_{\mathcal{R}}(\boldsymbol{A}, \mathrm{Hom}_{\mathcal{S}}(\boldsymbol{M}, \boldsymbol{B})) \\ \uparrow{\scriptstyle \tilde{f}} & & \uparrow{\scriptstyle \widetilde{F(f)}} \\ \mathrm{Hom}_{\mathcal{S}}(\boldsymbol{A} \otimes_{\boldsymbol{R}} \boldsymbol{M}, \boldsymbol{A} \otimes_{\boldsymbol{R}} \boldsymbol{M}) & \xrightarrow[\phi_{G(\boldsymbol{A}),\boldsymbol{B}}]{\cong} & \mathrm{Hom}_{\mathcal{R}}(\boldsymbol{A}, \mathrm{Hom}_{\mathcal{S}}(\boldsymbol{M}, \boldsymbol{A} \otimes_{\boldsymbol{R}} \boldsymbol{M})) \end{array}$$

for every triple $\langle \boldsymbol{A} \in \mathcal{R}, \boldsymbol{B} \in \mathcal{S}, f\colon \boldsymbol{A} \otimes_{\boldsymbol{R}} \boldsymbol{M} \to \boldsymbol{B} \rangle$.

Given an $\boldsymbol{A} \in \mathcal{R}$. The pair $\langle \boldsymbol{A} \otimes_{\boldsymbol{R}} \boldsymbol{M}, u \rangle$ is a universal arrow from $\boldsymbol{A}$ to $\mathrm{Hom}_{\mathcal{S}}(\boldsymbol{M}, -)$ where

$$u := \phi_{G(\boldsymbol{A}),\boldsymbol{B}}(id_{\boldsymbol{A} \otimes_{\boldsymbol{R}} \boldsymbol{M}})\colon \boldsymbol{A} \to \mathrm{Hom}_{\mathcal{S}}(\boldsymbol{M}, \boldsymbol{A} \otimes_{\boldsymbol{R}} \boldsymbol{M}),$$

that is, the image of the identity morphism $id_{\boldsymbol{A} \otimes_{\boldsymbol{R}} \boldsymbol{M}} \in \mathrm{Hom}_{\mathcal{S}}(\boldsymbol{A} \otimes_{\boldsymbol{R}} \boldsymbol{M}, \boldsymbol{A} \otimes_{\boldsymbol{R}} \boldsymbol{M})$ under $\phi_{G(\boldsymbol{A}),\boldsymbol{B}}$. Hence we have that

$$\begin{aligned} \mathrm{Hom}_{\mathcal{S}}(\boldsymbol{A} \otimes_{\boldsymbol{R}} \boldsymbol{M}, \boldsymbol{B}) &\xrightarrow{\vec{F}} \mathrm{Hom}_{\mathcal{R}}(\mathrm{Hom}_{\mathcal{S}}(\boldsymbol{M}, \boldsymbol{A} \otimes_{\boldsymbol{R}} \boldsymbol{M}), \mathrm{Hom}_{\mathcal{S}}(\boldsymbol{M}, \boldsymbol{B})) \\ &\xrightarrow{\tilde{u}} \mathrm{Hom}_{\mathcal{R}}(\boldsymbol{A}, \mathrm{Hom}_{\mathcal{S}}(\boldsymbol{M}, \boldsymbol{B})), \end{aligned}$$

for all $\boldsymbol{B} \in \mathcal{S}$.

Proposition 3.6. *The maps $\vec{F}$ and $\tilde{u}$ are isomorphisms if and only if u is an epimorphism.*

Proof. We have that $\tilde{u}$ is an epimorphism by observation 3.1. Hence we have that $\tilde{u}$ is an isomorphism if and only if u is an epimorphism. Hence $\vec{F}$ and $\tilde{u}$ are isomorphisms if and only if u is an epimorphism. □

Dually, given a $\boldsymbol{B} \in \mathcal{S}$, the pair $\langle \mathrm{Hom}_{\mathcal{S}}(\boldsymbol{M}, \boldsymbol{B}), v \rangle$ is a universal arrow from $\boldsymbol{—} \otimes_{\boldsymbol{R}} \boldsymbol{M}$ to $\boldsymbol{B}$, where $v \colon \mathrm{Hom}_{\mathcal{S}}(\boldsymbol{M}, \boldsymbol{B}) \otimes_{\boldsymbol{R}} \boldsymbol{M} \to \boldsymbol{B}$. Hence we have that

$$\mathrm{Hom}_{\mathcal{R}}(\boldsymbol{A}, \mathrm{Hom}_{\mathcal{S}}(\boldsymbol{M}, \boldsymbol{B})) \xrightarrow{\vec{G}} \mathrm{Hom}_{\mathcal{S}}(\boldsymbol{A} \otimes_{\boldsymbol{R}} \boldsymbol{M}, \mathrm{Hom}_{\mathcal{S}}(\boldsymbol{M}, \boldsymbol{B}) \otimes_{\boldsymbol{R}} \boldsymbol{M})$$
$$\xrightarrow{\tilde{v}} \mathrm{Hom}_{\mathcal{S}}(\boldsymbol{A} \otimes_{\boldsymbol{R}} \boldsymbol{M}, \boldsymbol{B}),$$

for all $\boldsymbol{A} \in \mathcal{R}$.

Proposition 3.7. *The maps $\vec{G}$ and $\tilde{v}$ are isomorphisms if and only if v is a monomorphism.*

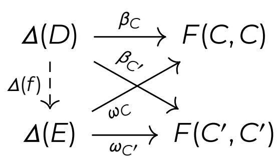
Proof. Observe that $\tilde{u}$ is an epimorphism. Hence we have that $\tilde{v}$ is an isomorphism if and only if v is a monomorphism. Therefore, $\vec{G}$ and $\tilde{v}$ are isomorphisms if and only if v is a monomorphism. □

Suppose that $\mathcal{C}$ and $\mathcal{D}$ are categories. Let $F \colon \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be a functor, and

$$\Delta \colon \mathcal{D} \to \mathcal{D}^{\mathcal{C}^{op} \times \mathcal{C}}$$

a diagonal[4] functor. An **end**[4] of the functor F is a universal dinatural[4] transformation $\langle E, \omega \rangle$ from Δ to F, where the object $E \in \mathcal{D}$, and $\omega \colon \Delta(E) \xrightarrow{\bullet\bullet} F$ is a dinatural transformation such that to every dinatural transformation $\beta \colon \Delta(D) \xrightarrow{\bullet\bullet} F$ there exists a unique morphism $f \colon D \to E$ which makes the diagram (3.8) commute, for all $C, C' \in \mathcal{C}$, cf. [4].

(3.8)



Hence we have that

$$\mathrm{Hom}_{\mathcal{D}}(D, E) \cong \mathrm{Hom}_{\mathcal{D}^{\mathcal{C}^{op} \times \mathcal{C}}}(\Delta(D), F),$$

and

$$\mathrm{Hom}_{\mathcal{D}}(D, E) \xrightarrow{\vec{\Delta}} \mathrm{Hom}_{\mathcal{D}^{\mathcal{C}^{op} \times \mathcal{C}}}(\Delta(D), \Delta(E)) \xrightarrow{\tilde{\omega}} \mathrm{Hom}_{\mathcal{D}^{\mathcal{C}^{op} \times \mathcal{C}}}(\Delta(D), F),$$

for all $D \in \mathcal{D}$. An end of a functor is regarded as a limit of the functor. Therefore, we have that $\vec{\Delta}$ and $\tilde{\omega}$ are isomorphisms, and the universal arrow of every end is of the type II.

References

[1] David Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Springer, 2004.
[2] Thomas W. Hungerford, *Algebra*, Springer, 1974.
[3] Nathan Jacobson, *Basic algebra*, 2nd ed., Dover Publications, 2009.
[4] Saunders Mac Lane, *Categories for the working mathematician*, 2nd ed., Springer, 1971.
[5] Tom Leinster, *Basic category theory*, Cambridge University Press, 2014.
[6] Emily Riehl, *Category theory in context*, 2014.
[7] Joseph J. Rotman, *An introduction to homological algebra*, 2nd ed., Springer, 2009.

[8] Robert L. Vaught, *Set theory an introduction*, 2nd ed., Birkhäuser Boston, 2001.

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