An IMC-like Design of LQR-PID Controller Using Optimal Weight of Sensitivity Function

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ABSTRACT

Proportional-Integral-Derivative controller is remained as the most widely used controller for many industrial applications even though it was developed decades ago. This is because of its simplicity, satisfactory control performance and robustness. The classical and empirical PID tuning rules are applicable only to the FOPDT and cannot guarantee its optimality. The Ziegler-Nichols frequency response method is regarded as a basis of relay feedback auto-tuning because it uses the only information at the phase crossover frequency. The Ziegler-Nichols tunings are quite aggressive for lag- dominant processes but sluggish for delay-dominant processes. The IMC-PID settings yield good servo performance and robustness, they result in poor LD (load disturbance) rejection for lag-dominant plants.

We have proposed a new method of determining LQ index in consideration of dominant pole placement for desired performance and derive a simple LQR-PID tuning formula via IMC-like H_∞ approach for first order plus dead-time systems. We have determined the weight for sensitivity function so that LQ optimization is equivalent to the H_∞ optimization of the weighted sensitivity function. The proposed PID controller has the same performance as LQR controller and the tuning method is simple, since it does not need to solve the Riccati algebraic equation. We also present other two tuning methods for PID controller: LQR-like and pole-placement-like ones. The new contributions in this paper are: determination of LQ index for dominant poles placement and optimal weight for sensitivity, and derivation of LQR-PID tuning methods via IMC-like H_∞ approach, LQR and pole-placement approaches. The effectiveness of the proposed methodology and the identity of the PID parameters tuned by those three methods have been demonstrated via simulation.

Keywords: LQR-PID controller, optimal weight, Linear Quadratic Regulator, Internal Model Control

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1. Introduction

PID controller is remained as the most widely used controller for many industrial applications even though it was developed decades ago. This is because of its simplicity, satisfactory control performance and robustness.

The classical Ziegler-Nichols method [1] is widely used in control practice for its good control effect.

Especially, the Ziegler-Nichols frequency response method is regarded as a basis of relay feedback auto-tuning because it uses the only information at the phase crossover frequency [2-5]. However, the Ziegler-Nichols tunings are quite aggressive for lag-dominant processes but sluggish for delay-dominant processes.

The SP (set point) overshoot method guarantees the good performance and robustness for delay-dominant processes [6]. The classical and empirical PID tuning rules are applicable only to the FOPDT and cannot guarantee its optimality. Some researchers have proposed the inverse of maximum of absolute real part λ of loop transfer function as a tuning parameter for both of the gain margin and phase margin specifications, and have proposed the PI tuning method for IE optimality of system [7]. This method can be applied to process with arbitrary transfer function and guarantees of both robustness and optimality. They have demonstrated that the closed-loop responses are similar for a given value of λ, although the processes have large differences in the dynamics. This is convenient because the desired response of the system can be specified by setting only one parameter λ.

In the method, called MIGO (M-constrained integral gain optimization), they have introduced the maximum sensitivity M as a specification for both of gain and phase margin and have proposed a PI tuning formula to maximize integral gain subject to the sensitivity constraint [8]. A simple tuning rule for PID controllers called AMIGO (approximate MIGO) has been proposed [9]. AMIGO and MIGO well work for a wide range of processes including ones with integration and pure time delay.

The IMC design method has been proposed by many researchers[10-11] and was first applied to PID control of stable plants [12]. This method has gained remarkable industrial application due to its simple yet effective procedure [13-14]. Although the IMC-PID settings yield good servo performance and robustness, they result in poor LD (load disturbance) rejection for lag-dominant plants [15-16]. In order to improve LD rejection performance, based on min–max model matching theory, a methodology to Servo/ Regulator tradeoff tuning [17] and a robust PID design for smooth SP tracking [18] have been proposed. The frequency weight with two tuning parameters, two zeros, for the sensitivity function has been introduced: one zero adjusts the robustness/performance trade-off as in the IMC procedure; the other zero balances the servo and regulatory performance [19-20]. The pole-placement method together with LQR was used to obtain the PID parameters for second order systems [21]. They have determined the LQ cost function and desired dominant closed loop poles so that the system has appropriate oscillatory response.

The new contributions in this paper are: determination of LQ index for dominant poles placement and optimal weight for sensitivity, and derivation of LQR-PID tuning methods via IMC-like H_∞ approach, LQR and pole-placement approaches.

2. Problem statement

The process is assumed to be modeled by a first order plus dead-time (FOPDT) transfer function of the form

$$\tilde{G}_0(s) = \frac{k_0}{Ts+1} e^{-\tau s} \tag{1}$$

where k_0 is the static gain, T is the time constant and τ is the time delay.

The PID controller is represented as

$$G_c(s) = K_p\left(1 + \frac{1}{T_i s} + T_d s\right) = K_p + \frac{K_i}{s} + K_d s. \tag{2}$$

For the IMC-like design, the following internal model obtained by the Padé approximation of Eq. 1 is used.

$$G_0(s) = \frac{k_0}{Ts+1} \cdot \frac{1-\frac{\tau s}{2}}{1+\frac{\tau s}{2}} \tag{3}$$

Fig. 1 shows the structures of standard IMC system and its equivalent feedback control system.

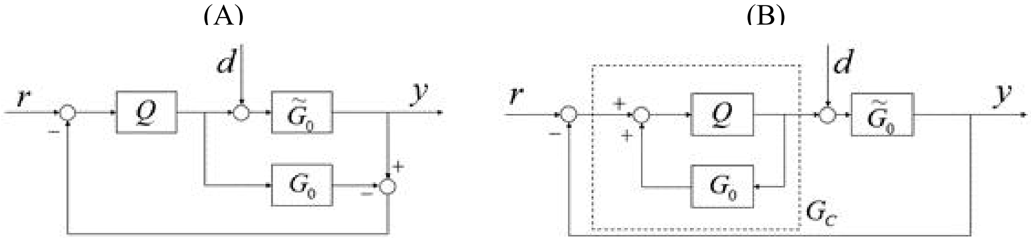

Fig. 1. IMC system and its equivalent feedback system

In the equivalent feedback system, the feedback controller is described as

$$G_c(s) = \frac{Q(s)}{1-G_0(s)Q(s)}$$

, and the transfer function from the LD to the output is written as

$$T_{yd}(s) = \frac{G_0(s)}{1+G_c(s)G_0(s)} = G_0(s)S(s)$$

In order to reject the LD as much as possible, we specify the cost function for determination of $G_c(s)$ as

$$\begin{aligned} J &= \min_{G_c} \left\|W_0(s)T_{yd}(s)\right\|_\infty = \min_{G_c} \left\|W_0(s)\frac{G_0(s)}{1+G_c(s)G_0(s)}\right\|_\infty \\ &= \min_{G_c} \|W_0(s)G_0(s)S(s)\|_\infty = \min_{G_c} \left\|W_0(s)\frac{k_0}{Ts+1}S(s)\right\|_\infty \\ &= \min_{G_c} k_0\|W(s)S(s)\|_\infty \end{aligned} \tag{4}$$

or

$$\rho = \min_{G_c} \|W(s)S(s)\|_\infty$$

where $W(s) = W_0(s)/(Ts+1)$ is the weight of the sensitivity function $S(s)$.

As shown in Eq. 4, the performance of IMC-PID controller mainly depends on the weight $W(s)$ of the sensitivity function. Therefore, it is very important to determine the weight for the sensitivity function so as to be able to maximally reject the LD.

3. IMC-like design of LQR-PID controller

The LQR is known for good control effect and robustness, so we use the following LQ index as a criterion for the performance

$$J = \int_0^\infty f^2(t)dt = \int_0^\infty \left(\ddot{e}(t) + f_1\dot{e}(t) + f_0e(t)\right)^2 dt. \tag{5}$$

This criterion is to be determined so that the closed-loop system has the appropriate oscillatory response i.e. desired dominant poles as follows

$$f(t) = \ddot{e}(t) + f_1\dot{e}(t) + f_0e(t) = 0 \tag{6}$$

where $f_1 = 2\xi\omega_n,\ f_0 = \omega_n^2,\ \xi < 1$.

By applying the Parceval's theorem, the LQ index can be rewritten as

$$\begin{aligned} J &= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} F(s)F(-s)ds \\ &= \int_0^\infty \left(\ddot{e}(t) + f_1\dot{e}(t) + f_0e(t)\right)^2 dt \\ &= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} (s^2 + f_1s + f_0)(s^2 - f_1s + f_0)E(s)E(-s)ds \end{aligned}$$

$$= \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} (s^4 - q_1 s^2 + q_0) E(s) E(-s) ds .$$

As a result, the LQ index Eq. 5 is expressed as

$$J = \int_0^\infty \left(q_0 e^2(t) + q_1 \dot{e}^2(t) + \ddot{e}^2(t) \right) dt \tag{7}$$

where $q_0 = f_0^2 = \omega_n^4$, $q_1 = f_1^2 - 2f_0 = 2\omega_n^2(2\xi^2 - 1)$.

Using LQ index Eq. 7, we can tune the PID controller based on the LQR approach, however, the LQR-like design requires solving of the Riccati algebraic equation. In this paper, we propose a simple IMC-like design for LQR-PID controller without solving of Riccati algebraic equation. By taking into account that the system performance depends on only the normalized dead-time $\theta = \tau / T$ and time-scaling with the time constant T, we rewrite the Eq. 6 as

$$f(t) = T^2 \ddot{e}(t) + f_1 T \dot{e}(t) + f_0 e(t) = 0. \tag{8}$$

Also, substituting $\frac{1}{\omega_n} = \lambda$ into Eq. 8, the following equation is obtained.

$$e(t) + 2\xi\lambda T \dot{e}(t) + T^2 \lambda^2 \ddot{e}(t) = 0 \tag{9}$$

Proposition. For the non-minimum phase process

$$G_0(s) = \frac{k_0}{Ts+1} \cdot \frac{1 - \frac{\tau s}{2}}{1 + \frac{\tau s}{2}} \tag{10}$$

, the LQ optimization with the LQ index below

$$J = \int_0^\infty \left(e(t) + 2\xi\lambda T \dot{e}(t) + T^2 \lambda^2 \ddot{e}(t) \right)^2 dt \tag{11}$$

is equivalent to the following H_∞ optimization for the weighted sensitivity

$$\min_{G_c \in RH_\infty} \| W(s) S(s) \|_\infty \tag{12}$$

where $S(s)$ is the sensitivity function and $W(s)$ is optimal weight of the sensitivity function, and these are represented, respectively, as

$$S(s) = \frac{1}{1 + G_c(s) G_0(s)} \tag{13}$$

$$W(s) = \frac{\lambda^2 s^2 + 2\xi\lambda\omega_t s + \omega_t^2}{s(s+\omega_t)},\ \lambda = \frac{1}{\omega_n},\ \omega_t = \frac{1}{T}. \tag{14}$$

The sensitivity function and the transfer function between LD d and output y for the optimal system are obtained, respectively, as

$$S^*(s) = \frac{4\lambda^2 + 4\xi\lambda\theta + \theta^2}{2(2+\theta)} \cdot \frac{s(s+\omega_t)}{\lambda^2 s^2 + 2\xi\lambda\omega_t s + \omega_t^2} \tag{15}$$

$$T_{yd}^*(s) = \frac{K_{yd} s}{\lambda^2 s^2 + 2\xi\lambda\omega_t s + \omega_t^2} \cdot \frac{1 - \frac{\tau}{2} s}{1 + \frac{\tau}{2} s} \tag{16}$$

where $K_{yd} = \frac{k_0 \omega_t (4\lambda^2 + 4\xi\lambda\theta + \theta^2)}{2(2+\theta)}$.

Proof. See Appendix A. □

Using the above proposition, the complementary sensitivity function corresponding to the sensitivity function Eq. 15 is determined as

$$T^*(s) = 1 - S^*(s) = \frac{(4\xi\lambda+\theta-2\lambda^2)s+(2+\theta)\omega_t}{(2+\theta)(\lambda^2 s^2+2\xi\lambda\omega_t s+\omega_t^2)}\left(1-\frac{\tau s}{2}\right)\omega_t. \tag{17}$$

Thus, the optimal IMC-LQR PID tuning rule is simply derived as

$$\begin{aligned} G_c(s) &= \frac{T^*(s)}{S^*(s)}\left(G_0(s)\right)^{-1} \\ &= \frac{(\tau s+2)\left((4\xi\lambda+\theta-2\lambda^2)s+(2+\theta)\omega_t\right)}{k_0(4\lambda^2+4\xi\lambda\theta+\theta^2)s} \\ &= K_c\frac{(T_1 s+1)(T_2 s+1)}{s} = K_p + \frac{K_i}{s} + K_d s \end{aligned} \tag{18}$$

where

$$K_c = \frac{2(2+\theta)\omega_t}{k_0(4\lambda^2+4\xi\lambda\theta+\theta^2)},\ T_1 = \frac{4\xi\lambda+\theta-2\lambda^2}{(2+\theta)\omega_t},\ T_2 = \frac{\tau}{2},$$

$$K_p = K_c(T_1+T_2),\ K_i = K_c,\ K_d = K_c T_1 T_2, \tag{19}$$

$$K_d = K_c T_1 T_2.$$

The step load disturbance response $Y_d(s)$ is

$$Y_d(s) = \frac{K_{yd}}{\lambda^2 s^2+2\xi\lambda\omega_t s+\omega_t^2}\cdot\frac{1-\frac{\tau s}{2}}{1+\frac{\tau s}{2}}. \tag{20}$$

The characteristic polynomial corresponding to oscillatory mode of Eq. 20 is identical with the one of the desired free response Eq. 9 specified in LQ index. Since Eq. 20 is the second order plus Padé approximated delay, the oscillatory mode is dominant. From this, the LQ index can be determined directly in demand of desired dominant poles.

The optimal PID controller with the desired performance measure can be easily obtained from Eq. 19 by properly setting the parameters ξ and ω_n of LQ index Eq. 11. The parameter ξ is related to the dampness, while another parameter $\lambda = 1/\omega_n$ is related to the swiftness of the response. By setting the damping ratio ξ in the tuning rule Eq. 19, we can let the system possible to have the oscillatory response appropriate to the LD rejection.

The response becomes faster as λ decreases, but it also causes poor robustness.

Hence, λ must be determined to satisfy robustness of the system and we use the maximum sensitivity as a robustness index. The maximum sensitivity is defined as

$$\begin{aligned} M_s &= \max_{\omega}|S^*(j\omega)| \\ &= \max_{\omega}\left\{\frac{4\lambda^2+4\xi\lambda\theta+\theta^2}{2(2+\theta)}\cdot\left|\frac{s(s+\omega_t)}{\lambda^2 s^2+2\xi\lambda\omega_t s+\omega_t^2}\right|_{s=j\omega}\right\} \\ &= \max_{\omega}\left\{\frac{4\lambda^2+4\xi\lambda\theta+\theta^2}{2(2+\theta)}\cdot|I(s)|_{s=j\omega}\right\}. \end{aligned} \tag{21}$$

The frequency corresponding to the maximum sensitivity is obtained as

$$\omega = \omega_t\sqrt{x}$$

$$x = \frac{-1-\sqrt{1-\lambda^2(4\xi^2-2-\lambda^2)}}{\lambda^2(4\xi^2-2-\lambda^2)} = \frac{1+\sqrt{1+\lambda^2(\lambda^2+2-4\xi^2)}}{\lambda^2(\lambda^2+2-4\xi^2)}. \quad (22)$$

The following condition must be satisfied for the solution above.

$$\lambda^2(\lambda^2+2-4\xi^2) > 0$$

This inequality is satisfied for all $\lambda > 0$ if the system is oscillatory i.e. $\xi \leq 1/\sqrt{2} \approx 0.707$. But if $\lambda < 1$, x becomes too big and the extreme point does not exist. Thus, for calculation of λ, we use the value of sensitivity function when ω is infinity instead of maximum sensitivity

$$M_s = |S^*(j\infty)| = \frac{4\lambda^2+4\xi\lambda\theta+\theta^2}{2(2+\theta)\lambda^2}. \quad (23)$$

Eq. 18 shows that the bigger λ gets, the smaller maximum sensitivity becomes i.e. the better robustness is obtained. As normalized dead time θ gets bigger, λ should be bigger for the same sensitivity. Because in most cases the extreme point does not exist, by using Eq. 18, we determine λ as

$$\lambda = \frac{\xi\theta+\theta\sqrt{\xi^2+\left(\left(1+\frac{\theta}{2}\right)M_s-1\right)}}{2\left(\left(1+\frac{\theta}{2}\right)M_s-1\right)} \quad (24)$$

Although this method aims at LD rejection performance, we can obtain also the good set-point tracking performance by properly setting the damping ratio ξ. Additionally, using the proposed LQ index, we have derived the tuning formulas of PID controller via LQR-like and pole-placement-like approach. (Refer to Appendix B, C.)

3. Simulation examples

In order to demonstrate the effectiveness of the PID tuning methodology proposed in this paper, we now present simulation results for different processes. In Example , we have compared the proposed method (IMC-LQR) with the other methods such as the MMA (model matching approach)[17] and the AMIGO (Approximate M-constrained integral gain optimization)[9]. Table 1 shows nominal models and real processes for simulation. The tuning results for nominal model G_i are shown in Table 2.

The MMA tuning rule [17] is

$$K_p = \frac{0.53T_i}{k_0\tau}, \; T_i = T + 0.25\tau, \; T_d = 0.258\tau N, \; N = 1.94\frac{T}{T_i} - 1$$

and the AMIGO tuning rule [9] is

$$K_p = \frac{1}{k_0}\left(0.2 + 0.45\frac{T}{\tau}\right),\ T_i = \frac{0.4\tau + 0.8T}{\tau + 0.1T}\tau,\ T_d = \frac{0.5\tau T}{0.3\tau + T},\ K_i = \frac{K_p}{T_i},\ K_d = K_p T_d.$$

In the IMC-LQR, setting damping ratio ξ and M_s respectively as 0.7 and 1.3, λ is calculated by Eq. 24. In the simulation, we obtained the responses of nominal model and real processes to show the robustness of the IMC-LQR. Figs. 2, 3, and 4 show the responses of control systems designed by IMC-LQR, MMA and AMIGO.

The comparison of the results obtained by MMA, AMIGO and IMC-LQR is shown in Table 3. As performance indices, the maximum error (e_m), the settling time (t_s) and the integrated absolute error (IAE) of step disturbance response were used. As shown in Figs. 2, 3, 4 and Table 3, most values of the performance indices for IMC-LQR are smaller than ones of other methods; especially its performance is approximate to AMIGO well-known for good LD rejection. Also, the proposed controller works well, whether the process is a delay-dominant or a lag-dominant, and has good robustness. The best performance of IMC-LQR is known also from the fact that its integrating gain K_i is the biggest.

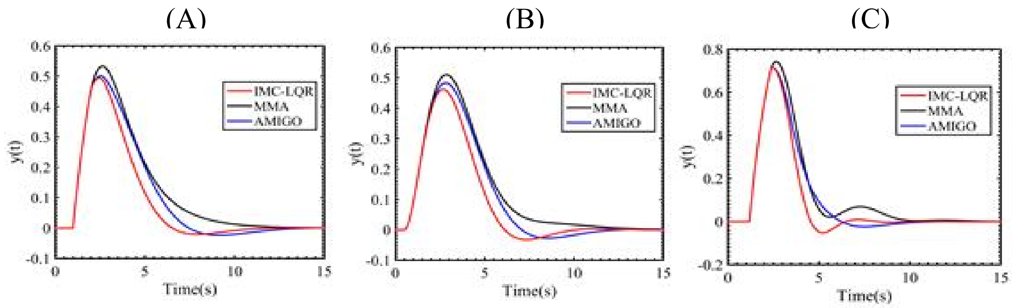

Fig. 2. Step disturbance response for processes (A) G_1 (B) G_{11} (C) G_{12}

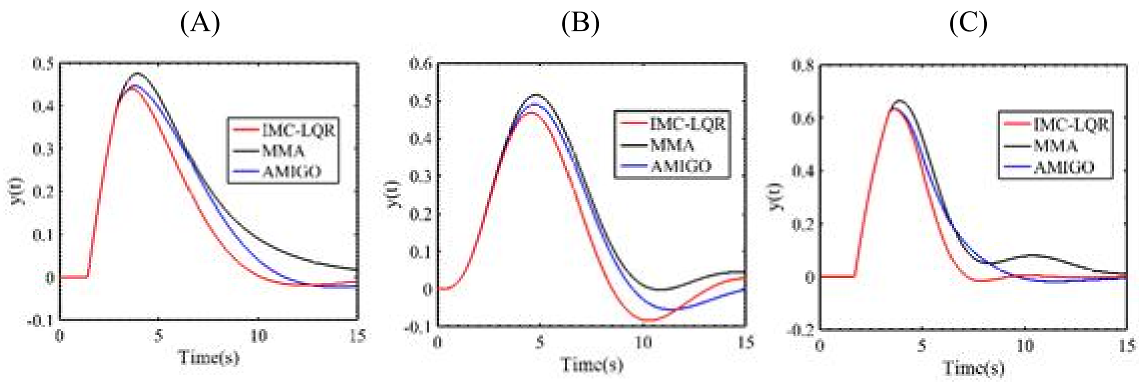

Fig. 3. Step disturbance response for processes (A) G_2 (B) G_{21} (C) G_{22}

(A) (B) (C)

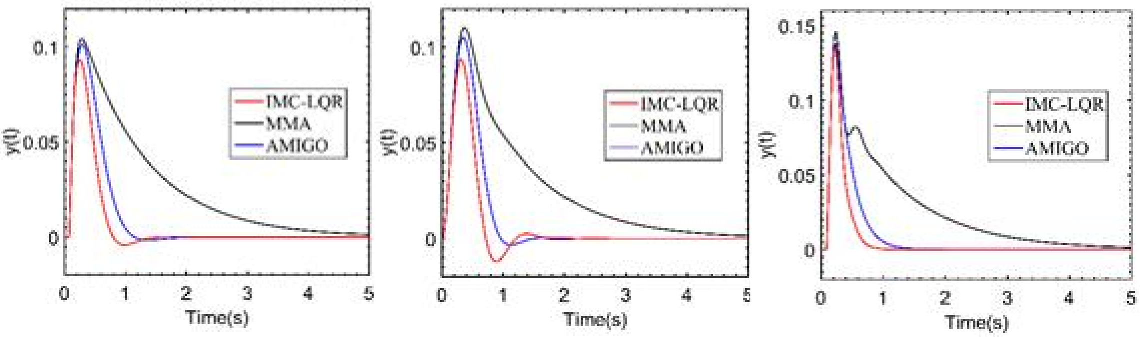

Fig. 4. Step disturbance response for processes (A) G_3 (B) G_{31} (C) G_{32}

Conclusion

In this paper, we have proposed a new form of LQ index for maximum LD rejection and have derived a tuning formula of LQR-PID controller for FOPDT systems.

The maximum load disturbance rejection is achieved by minimizing the LQ index. Robustness is guaranteed by requiring that the maximum sensitivity is less than a specified value M_s. The primary parameters of controller are damping ratio ξ and natural frequency ω_n. These parameters are independently determined in demand of the disturbance response damping and the robustness respectively, thus it provides the convenience of design. The proposed controller gives a second order oscillatory response specified by the parameters ξ and ω_n, and the similar responses for the FOPDT processes with a wide range of θ. This method is applicable to any process that can be modeled as or approximated to FOPDT, while not to oscillatory process. Research effort should be provided in the future in order to derive tuning rules of PID controllers for wide range of process.

Appendix A. Proof of proposition

By applying the Parseval's theorem, the LQ index Eq. 11 can be rewritten as

$$J = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}(\lambda^2T^2s^2 + 2\xi\lambda Ts + 1)(\lambda^2T^2s^2 - 2\xi\lambda Ts + 1)E(s)E(-s)ds$$

$$= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}F(s)F(-s)ds \tag{A.1}$$

and optimization of Eq. (A.1) is equivalent to the following minimization.

$$\min_{G_c\in RH_\infty}\|F(s)\|_\infty = \min_{G_c\in RH_\infty}\left\|(\lambda^2T^2s^2 + 2\xi\lambda Ts + 1)E(s)\right\|_\infty \tag{A.2}$$

On the other side, the error response to the step LD is

$$E(s) = -\frac{1}{s}\cdot\frac{G_0(s)}{1+G_c(s)G_0(s)} = -\frac{1}{s}G_0(s)S(s) \quad \text{(A.3)}$$

where $S(s) = \frac{G_0(s)}{1+G_c(s)G_0(s)}$ is the sensitivity function.

By substituting Eq. (A.3) into Eq. (A.2), we obtain

$$\min_{G_c \in RH_\infty} \|F(s)\|_\infty = \min_{G_c \in RH_\infty} \left\| \frac{k_0(\lambda^2T^2s^2+2\xi\lambda Ts+1)}{\omega_t Ts(Ts+1)} \cdot \frac{1-\frac{\tau s}{2}}{1+\frac{\tau s}{2}} \cdot S(s) \right\|_\infty$$

$$= \min_{G_c \in RH_\infty} \left\| \frac{\lambda^2T^2s^2+2\xi\lambda Ts+1}{Ts(Ts+1)} \cdot S(s) \right\|_\infty \quad \text{(A.4)}$$

$$= \min_{G_c \in RH_\infty} \|W(s)S(s)\|_\infty$$

where $W(s)$ is the weight of the sensitivity function and it is represented as

$$W(s) = \frac{\lambda^2T^2s^2+2\xi\lambda Ts+1}{Ts(Ts+1)} = \frac{\lambda^2s^2+2\xi\lambda\omega_t s+\omega_t^2}{s(s+\omega_t)}. \quad \text{(A.5)}$$

By the lemma,[17], the optimum value of the cost function Eq. (A.4) is

$$\rho^* = e_0 = W(2/\tau = 2\omega_t/\theta) = \frac{4\lambda^2+4\xi\lambda\theta+\theta^2}{2(2+\theta)}.$$

Therefore, the sensitivity function and the transfer function from LD d to output y of the optimal system are obtained, respectively, as

$$S^*(s) = \frac{\rho^*}{W(s)} = \frac{4\lambda^2+4\xi\lambda\theta+\theta^2}{2(2+\theta)} \cdot \frac{s(s+\omega_t)}{\lambda^2s^2+2\xi\lambda\omega_t s+\omega_t^2} \quad \text{(A.6)}$$

$$T_{yd}^*(s) = G_0(s)S^*(s) = \frac{K_{yd}s}{\lambda^2s^2+2\xi\lambda\omega_t s+\omega_t^2} \cdot \frac{1-\frac{\tau s}{2}}{1+\frac{\tau s}{2}} \quad \text{(A.7)}$$

Where $K_{yd} = k_0\omega_t\rho^* = k_0\omega_t W(2/\tau) = \frac{k_0\omega_t(4\lambda^2+4\xi\lambda\theta+\theta^2)}{2(2+\theta)}$.

Appendix B. LQR-like design of PID controller

The tuning rule derived here is used to demonstrate the validity of the IMC-PID tuning formula Eq. 19. For the convenience, we use the following normalized process

$$\widetilde{G}_0(s) = \frac{Y(s)}{U(s)} = \frac{1}{s+1}e^{-\theta s},\ \theta = \frac{\tau}{T} \quad \text{(B.1)}$$

and the PID controller is described as

$$G_c(s) = \frac{U(s)}{E(s)} = k_p\left(1 + \frac{1}{\tau_i s} + \tau_d s\right) = k_p + \frac{k_i}{s} + k_d s. \tag{B.2}$$

The controller for the process Eq. 1 is obtained by scaling the controller Eq. (B.2) as

$$K_p = \frac{k_p}{k_0},\ T_i = T\tau_i = \frac{Tk_p}{k_i},\ T_d = T\tau_d = \frac{Tk_d}{k_p},\ K_i = \frac{K_p}{T_i} = \frac{k_i}{Tk_0},\ K_d = K_p T_d = \frac{Tk_d}{k_0}.$$

By using the Padé approximation, the internal model can be described as

$$G_0(s) = \frac{1-\frac{\theta s}{2}}{(s+1)\left(1+\frac{\theta s}{2}\right)} = \frac{\frac{2}{\theta}-s}{s^2+\left(1+\frac{2}{\theta}\right)s+\frac{2}{\theta}} = \frac{a_0-s}{s^2+a_1 s+a_0} \tag{B.3}$$

and the state equation is expressed as

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= Cx \end{aligned} \tag{B.4}$$

where $x^T = (x_1 \quad x_2)$, $A = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $C = (a_0 \quad -1)$, $a_0 = 2/\theta$, $a_1 = (1 + 2/\theta)$.

Now, in order to design the LQR-PID controller, we, first, construct the augmented system equation for the state variables $\tilde{x}^T = (\dot{x}_1 \quad \dot{x}_2 \quad e)$

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{A}\tilde{x} + \tilde{b}\dot{u} \\ \dot{y} &= -\dot{e} = C\dot{x} = \tilde{C}\tilde{x} \end{aligned} \tag{B.5}$$

where $\tilde{A} = \begin{pmatrix} A & 0 \\ -C & 0 \end{pmatrix}$, $\tilde{b} = \begin{pmatrix} b \\ 0 \end{pmatrix}$, $\tilde{C} = (C \quad 0)$.

Then, each term of the LQ index Eq. 7 is represented as follows

$$e^2(t) = \tilde{x}^T Q_1 \tilde{x},$$

$$\dot{e}^2(t) = \dot{y}^2(t) = \tilde{x}^T Q_2 \tilde{x},$$

$$\ddot{e}^2(t) = \tilde{x}^T Q_3 \tilde{x} + 2\tilde{x}^T N\dot{u} + r\dot{u}^2$$

where $Q_2 = \tilde{C}^T\tilde{C}$, $Q_3 = \tilde{A}^T Q_2 \tilde{A}$, $N = \tilde{A}^T Q_2 \tilde{b}$, $r = \tilde{b}^T Q_2 \tilde{b}$.

As a result, the LQ index Eq. 7 is changed to the following form

$$J = \int_0^\infty (\tilde{x}^T Q \tilde{x} + 2\tilde{x}^T N\dot{u} + r\dot{u}^2)dt$$

where $Q = q_0 Q_1 + q_1 Q_2 + Q_3$.

Using a solution P of the Riccati algebraic equation

$$\widetilde{A}^T P + P\widetilde{A} - \left(P\tilde{b} + N\right)r^{-1}\left(\tilde{b}^T P + N^T\right) + Q = 0$$

and state feedback gain

$$K = (k_1 \quad k_2 \quad k_3) = r^{-1}(\tilde{b}^T P + N^T)$$

, LQ optimal control law is represented as

$$\dot{u}(t) =- K\tilde{x}(t) =- \left(k_1\dot{x}_1(t) + k_2\dot{x}_2(t) + k_3 e(t)\right)$$

$$u(t) =- \left(k_1 x_1(t) + k_2 x_2(t) + k_3 \int_0^t e(\tau)d\tau\right)$$

$$= \left(k_{12}x(t) + k_3 x_0(t)\right) \qquad \text{(B.6)}$$

where $k_{12} = (k_1 \quad k_2)$, $x = (x_1 \quad x_2)$, $x_0(t) = \int_0^t e(\tau)d\tau$.

Now, in order to derive the LQR-PID controller, we rewrite Eq. (B.6) as

$$\begin{aligned} u(t) &= k_p e(t) + k_d \dot{e}(t) + k_i \int_0^t e(\tau)d\tau \\ &=- k_p y(t) - k_d \dot{y}(t) + k_i x_0(t) \\ &=- k_p Cx(t) - k_d C\dot{x}(t) + k_i x_0(t) \\ &=- k_p Cx(t) - k_d C(Ax + bu) + k_i x_0(t) \end{aligned} \qquad \text{(B.7)}$$

$$u(t) =- \frac{1}{1+k_d Cb}\left((k_p \quad k_d)\begin{pmatrix} C \\ CA \end{pmatrix} x - k_i x_0\right).$$

By taking account that $Cb =- 1$, the following PID tuning formula is derived.

$$\left.\begin{aligned} k_d &= \frac{k_1 + a_0 k_2}{(k_1 + a_0(2 + 2a_0 + k_2))} \\ k_p &=- k_2 + k_d(1 + 2a_0 + k_2) \\ k_i &=- k_3(1 - k_d) \end{aligned}\right\} \qquad \text{(B.8)}$$

Appendix C. Pole-placement-like design of the PID controller

Here derives the tuning rule used to demonstrate the validity of characteristic polynomial of the optimal system in the proposition. The tuning rule is derived for the normalized process Eq. (B.1). From Eq. 16, the characteristic polynomial of optimal system is expressed as

$$D^*(s) = (\lambda^2 s^2 + 2\xi\lambda\omega_t s + \omega_t^2)\left(1 + \frac{\tau}{2}s\right). \qquad \text{(C.1)}$$

For normalized process,

$$\omega_t = \frac{1}{T} = 1,\ \tau = T\theta = \theta,\ \lambda = \frac{1}{\omega_n}.$$

Substituting these relations into Eq. (C.1), we obtain

$$D^*(s) = (s^2 + 2\xi\omega_n s + \omega_n^2)(s + a_0)$$

$$= (s^2 + f_1 s + f_0)(s + a_0),\ a_0 = \frac{2}{\theta}. \quad \text{(C.2)}$$

The characteristic polynomial of the closed-loop system consisting of the controller Eq. (B.2) and the process Eq. (B.3) is

$$D(s) = s(s+1)(s+a_0) + (a_0 - s)\left(k_i + k_p s + k_d s^2\right). \quad \text{(C.3)}$$

By taking account of one pole, $s = -a_0$, we obtain

$$k_i - k_p a_0 + k_d a_0^2 = 0 \Longrightarrow k_i = k_p a_0 - k_d a_0^2. \quad \text{(C.4)}$$

Using Eq. (C.4), Eq. (C.3) is rewritten as

$$D(s) = s(s+1)(s+a_0) + (a_0 - s)\left(k_p(s+a_0) + k_d\left(s^2 - a_0^2\right)\right)$$

$$= (s+a_0)\left(s(s+1) + (a_0 - s)\left(k_p + k_d(s - a_0)\right)\right). \quad \text{(C.5)}$$

Therefore, $D(s) = (1 - k_d)D^*(s)$.

As a result, the tuning formula via pole-placement-like approach is obtained as

$$\left.\begin{aligned} k_d &= \frac{f_1 a_0 - a_0 + f_0}{a_0^2 + f_1 a_0 + f_0} \\ k_p &= k_d(2a_0 + f_1) - f_1 a_0 + 1 \\ k_i &= k_p a_0 - k_d a_0^2 \end{aligned}\right\}$$

where $f_1 = 2\xi\omega_n,\ f_0 = \omega_n^2$.

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