# Fractional order system identification with state delay

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ABSTRACT

In this paper, we consider the continuous time fractional order system with unknown state . The fractional integral operational matrix of the block pulse functions(BPFs) is upper triangular Toeplitz. Using the commutativity and nilpotent property of upper triangular Toeplitz, we propose an efficient identification method in which the nonlinear parameters. The accuracy of the proposed method is illustrated by several simulations.

Keyword: block pulse function, delay, fractional order system, parameter identification

## 1. Introduction

Due to non-local and historical memory properties of fractional order system(FOS), it has been widely studied by many researchers, see [1-7]. See also books [8-10]. The time delay is one of the natural problem in the practical engineering applications. FOSs with time delays can be found in various engineering systems such as chemical process, nuclear reactor, and the dynamic behavior of HIV infection of $CD^+$ T-cells, see [11]. In practice, FOSs with state delays arise from the dynamic behavior of viscoelastic materials, dynamical processes in self-similar and porous structures, control theory of mechanical system, strongly porous materials, some amorphous semiconductors, see [12-18].

In particular, they arise from communication lags, feedback delay in measurement, delay in closed loop systems. In [17], the authors studied the control problem of SISO system which is presented as FOS with state delay when there exists a time delay in the state feedback loop. In [18], the authors use the control signal composed of state delay signals to stabilize the vibration system of single degree of freedom. The stabilization of FOSs is greatly affected by state delay, see [13, 16]. So this is an important non-negligible factor in the construction of model for FOS with state delay.

Because of the lack of acceptable geometric or physical explanations of fractional calculus, the identification for FOSs has been the best approach for building FO models of physical systems. It is well known that the time domain identification method is one of the identifications of FOSs.

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This is based on the minimization of both equation error and output error and hence several least square algorithms have been used. For example, we refer to a recursive error prediction approach [19], modulating function method [20, 21], Haar wavelet-based method [22, 23], BPFs-based method [11, 24-28].

Within our knowledge, for FOSs with state delay there seems to be no work concerning parameter identification method in which linear and nonlinear parameters are individually estimated. So the aim of this paper is to develop such an identification method for FOSs with unknown state and input delays. The main novelty is to derive a new method, by which the linear and nonlinear parameters are separably estimated by using the commutativity and nilpotent property of the fractional integral operational matrix(FIOM) of BPFs.

The paper consists of 5 sections. In section 2, we give the definitions of fractional calculus, BPFs and explain the commutativity and nilpotent property of the FIOM of BPFs, which is upper triangular Toeplitz. Section 3 is devoted to construction of a separable recursive model for FOSs with unknown state and input delays. In section 4, we give some simulation examples for verifying the effectiveness of the proposed method. Section 5 concludes this paper.

## 2 Preliminaries

### 2.1 Definitions of the fractional order integral and derivative

The Riemann-Liouville fractional order integral is defined by

$$ {}_{t_0}^{R}I_t^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s) ds, \tag{2.1}$$

where $\alpha > 0$, $(t_0, t)$ is the integral interval and $\Gamma(\cdot)$ is the Gamma function, that is, $\Gamma(\alpha) := \int_0^{\infty} x^{\alpha-1} e^{-x} dx$. For the sake of simplicity, we set $t_0 = 0$ and denote ${}_{t_0}^{R}I_t^{\alpha}$ by $I^{\alpha}$. Let $\ell \in \mathbb{N}, \ell - 1 < \alpha < \ell$. Then the Riemann-Liouville fractional order derivative is defined as

$$\begin{aligned} D^{\alpha} f(t) &= I^{-\alpha} f(t) = D^{\ell} I^{\ell} I^{-\alpha} f(t) = D^{\ell} I^{\ell-\alpha} f(t) \\ &= \frac{d^{\ell}}{dt^{\ell}} \left[ \frac{1}{\Gamma(\ell-\alpha)} \int_{t_0}^{t} (t-s)^{-\alpha+\ell-1} f(s) ds \right]. \end{aligned} \tag{2.2}$$

### 2.2 The FIOM of BPFs

As in [28], a set of BPFs in the semi-open interval $[0, T)$ is defined by

$$ B_n(t) := \begin{cases} 1, & \frac{n-1}{N} T \le t < \frac{n}{N} T, \\ 0, & otherwise, \end{cases} \tag{2.3}$$

where $n = 1, \cdots, N$, and $N$ is the number of BPFs in the set.

We call $B(t):=[B_1(t),\cdots,B_N(t)]^T$ by block pulse vector. If we define the elements of $X=[x_1,\ldots,x_N]^T$ by

$$x_n=\frac{N}{T}\int_{\frac{n-1}{N}T}^{\frac{n}{N}T}x(t)dt\approx x(\frac{n-1}{N}T),$$

then any absolute integrable function defined on interval $[0,T)$ can be approximated by a linear combination of BPFs as follow:

$$x(t)=\sum_{n=1}^{N}\left[\frac{N}{T}\int_{\frac{n-1}{N}T}^{\frac{n}{N}T}x(t)dt\right]B_n(t)\approx\sum_{n=1}^{N}x_nB_n(t)=X^TB(t). \tag{2.4}$$

By Eq. (2.1), we have

$$I^{\alpha}B(t)=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}*B(t), \tag{2.5}$$

where the symbol $*$ means convolution. In [30], it was shown that Eq. (2.5) can be represented as

$$I^{\alpha}B(t)\approx P_{\alpha}B(t) \tag{2.6}$$

where

$$P_{\alpha}:=\frac{h^{\alpha}}{\Gamma(\alpha+2)}\begin{pmatrix}\xi_1&\xi_2&\xi_3&\cdots&\xi_N\\0&\xi_1&\xi_2&\cdots&\xi_{N-1}\\0&0&\xi_1&\cdots&\xi_{N-2}\\\cdots&\cdots&\cdots&\cdots&\cdots\\0&0&0&0&\xi_1\end{pmatrix}, \tag{2.7}$$

$$h=\frac{T}{N},\ \xi_1=1,\ \xi_l=l^{\alpha+1}-2(l-1)^{\alpha+1}+(l-2)^{\alpha+1},\ l=2,\ldots,N.$$

We call the matrix $P_{\alpha}$ by FIOM for BPFs. By Eqs. (2.4), (2.6), the fractional order integral of the function $x(t)$ is given by

$$I^{\alpha}x(t)\approx X^TP_{\alpha}B(t). \tag{2.8}$$

### 2.3 Commutativity and nilpotent property of the FIOM

By Eq. (2.4), the function $x(t-\tau)$ is expanded by BPFs as follow:

$$x(t-\tau)\approx X^TB(t-\tau)=\sum_{i=1}^{\mathrm{N}}x_iB_i(t-\tau), \tag{2.9}$$

where

$$x_i=\frac{1}{h}\int_0^T x(t-\tau)B_i(t-\tau)dt. \tag{2.10}$$

In [11], it was pointed out that for $\tau=kh$, the delayed function $B(t-\tau)$ is represented as

$$B(t-\tau) = E_k \cdot B(t),\ t > \tau,\ 0 \le t \le T, \tag{2.11}$$

where for the Kronecker delta function $\delta_{ij}$

$$E_k = \left(\delta_{i,j+k}\right)_{i,j=1}^{n}, k = 0,1,\cdots,n-1. \tag{2.12}$$

The matrix $E_k$ is called $k$ order delay operational one. It was shown in [28] that for non-negative integers $i, j$,

$$E_i E_j = E_j E_i = \begin{cases} I, & \text{if } i+j=0 \\ E_{i+j}, & \text{if } i+j<N \\ 0, & \text{if } i+j \ge N. \end{cases} \tag{2.13}$$

As in [11], the Riemann-Liouville fractional order integral $B(t-\tau)$ is rewritten as

$$(I^\alpha B)(t-\tau) \approx P_\alpha B(t-\tau) = P_\alpha E_k B(t). \tag{2.14}$$

By Eq. (2.12), the FIOM from Eq. (2.7) is decomposed into

$$P_\alpha = \sum_{i=0}^{N-1} \frac{h^\alpha}{\Gamma(\alpha+2)} \xi_{i+1} E_i. \tag{2.15}$$

From Eq. (2.15), we can see that the FIOM from Eq. (2.7) is upper triangular Toeplitz which belongs to the subspace $T_N = \text{Span}\{E_0, E_1, \cdots, E_{N-1}\}$ .Next, let us consider the commutativity of FIOMs. In the Appendix A below, we show the following properties: for any two FIOMs, $A, B \in T_N$ and $i, j = 0,1,2,\cdots$,

$$\begin{aligned} &\text{i) } A^i B^j \in T_N, \\ &\text{ii) } A^i B^j = B^j A^i, \\ &\text{iii) } (AB)^i = A^i B^i. \end{aligned} \tag{2.16}$$

In [29], it was shown that if a matrix $A$ belongs to the strict upper triangular Toeplitz space $ST_N = \text{Span}\{E_1, \cdots, E_{N-1}\}$, then $A$ is nilpotent. Indeed, if $A \in ST_N$, then by (2.13) for all $k(\ge N)$

$$A^k = 0. \tag{2.17}$$

**Remark 1** *Eq. (2.17) is the main key in the transformation of identification of FOS with state delay onto the separable least square problem. In particular, by Eqs. (2.16) and (2.17), we get that for* $B = \sum_{k=0}^{N-1} b_{k+1} E_k \in T_N$ *and* $Q = \sum_{k=1}^{N-1} b_{k+1} E_k$,

$$B^{-1} = (b_1 I + Q)^{-1} = \sum_{k=0}^{N-1} \frac{Q^k}{(-b_1)^{k+1}}. \tag{2.18}$$

# 3 Identification algorithm for FOSs with unknown state

Here we construct a new identification algorithm for FOSs with unknown state and input delays using separable least square problem. At first, the identification method is proposed for FOS from [14] and then for FOS from [12, 13, 16].

Let us consider the FOS from [14]

$$D^{\alpha}x(t) + ax(t-\tau_a) = bu(t-\tau_b),$$
$$y(t) = x(t) + w(t), \tag{3.1}$$

where $\tau_a = kh$, $\tau_b = lh$ and $N$ is the number of BPFs and $T$ is the simulation time, and $u(t)$, $x(t)$, $y(t)$, $w(t)$ are input signal, state, output signal and the Gauss white noise, respectively. From Eqs. (2.8) and (2.14) the first equation in Eq. (3.1) can be rewritten as

$$X^T(I + aP_{\alpha}E_k)B(t) = bU^TP_{\alpha}E_lB(t).$$

Hence by (2.4)

$$x(t) = bU^TP_{\alpha}E_l(I + aP_{\alpha}E_k)^{-1}B(t). \tag{3.2}$$

By Eq. (2.18), the matrix $(I + aP^{\alpha}E_k)^{-1}$ with $k \geq 1$ is the following:

$$(I + aP_{\alpha}E_k)^{-1} = \sum_{i=0}^{N-1} (-1)^i a^i P_{\alpha}^i E_{ki}. \tag{3.3}$$

Inserting Eq. (3.3) into Eq. (3.2) implies that

$$x(t) = bU^TP_{\alpha}E_l\left[\sum_{i=0}^{N-1} (-1)^i a^i P_{\alpha}^i E_{ki}\right]B(t). \tag{3.4}$$

Now we set

$$\theta_L = (\theta_1, \theta_2, \cdots, \theta_N)^T,\ \theta_i = (-1)^{i-1}ba^{i-1},\ \theta_{NL} = (k, l)^T,$$

$$\varphi(\theta_{NL}, t) =$$
$$\left(U^TP_{\alpha}E_lB(t), U^TP_{\alpha}^2E_{k+l}B(t), U^TP_{\alpha}^3E_{2k+l}B(t), \cdots, U^TP_{\alpha}^NE_{(N-1)k+l}B(t)\right)_{1\times N}.$$

If $Mk + l < N \leq (M+1)k + l$, then by the previous notation Eq. (3.4) can be rewritten as

$$x(t) = \varphi(\theta_{NL}, t)\theta_L, \tag{3.5}$$

where

$$\varphi(\theta_{NL}, t) = (U^TP_{\alpha}E_lB(t), U^TP_{\alpha}^2E_{k+l}B(t), \cdots, U^TP_{\alpha}^{M+1}E_{Mk+l}B(t), 0, \cdots, 0). \tag{3.6}$$

By Eq. (3.6) the data matrix $\Phi(\theta_{NL})$ for $N$ sampling points can be written as

$$\Phi(\theta_{NL}) = \{\varphi_{ij}(\theta_{NL})\}_{N\times N}, \tag{3.7}$$

where

$$\varphi_{ij}(\theta_{NL}) = \begin{cases} U^TP_{\alpha}^jE_{(j-1)k+l}B(i), & \text{if } i = 1, \cdots, N, j = 1, \cdots, M+1, \\ 0, & \text{if } j > M+1. \end{cases} \tag{3.8}$$

Then by Eqs. (3.7) and (3.8) the identification for (3.1) is reduced to the separable least square problem as

$$R(\theta_L, \theta_{NL}) = \|Y - \Phi(\theta_{NL})\theta_L\|_2^2 : \min, \tag{3.9}$$

where $Y$ is sampling value vector for $y(t)$.

# 4 Simulation examples

We consider

$$\begin{aligned} D^\alpha x(t) + ax(t - \tau_a) &= bu(t - \tau_b), \\ y(t) &= x(t) + w(t), \end{aligned} \tag{4.1}$$

where $a = 1.1, b = 1.5, \tau_a = 1.2, \tau_b = 0.8, \alpha = 0.7$ and $w(t)$ is the Gauss white noise. The simulation time is $T = 20s$. Fig. 1 shows the step response for the FOS (4.1) with noise or without noise.

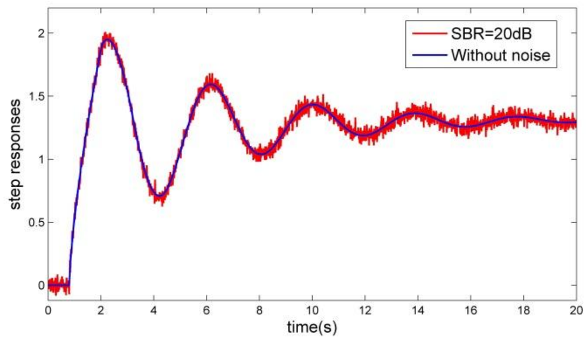


Figure 1: The step response for example 1

To show the dependence of output response to input or state delays, we present Fig. 2.

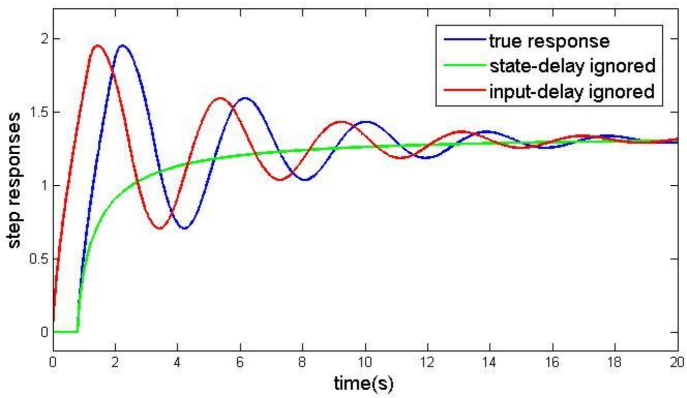


Figure 2: The step response for example 1 with ignored state or input delay

As we can see in Fig. 2, the output response for the system with ignored state delay is greatly different from real response than the one for the system with ignored input delay. This shows that the state delay is an important non-negligible factor.

Table 1 lists the identification results of nonlinear parameters $\tau_a, \tau_b$ for the FOS (4.1) with noise or without noise. Here the algorithm 1 is used.

Table 1. Identification results of nonlinear parameters for FOS (1)

| nonlinear parameter | true(s) | without noise | with noise (SNR) | |
|---|---|---|---|---|
| | | | 10dB | 20dB |
| state delay time | 1.2 | 1.2000 | 1.2000 | 1.2000 |
| input delay time | 0.8 | 0.8000 | 0.8000 | 0.8000 |

Table 1 shows that if the state and input delay times are integer times of the sampling interval $T/N$, then there does not exist the identification error for nonlinear parameters.

Based on this identification result, we can estimate the linear parameters. In Table 2, we present the identification accuracy by the relative error between real parameter $\theta = (a, b)^T$ and estimated one $\hat{\theta} = \left(\hat{a}, \hat{b}\right)^T$, i.e.,

$$\delta = \frac{\|\theta - \hat{\theta}\|}{\|\theta\|} \times 100(\%). \tag{4.2}$$

Table 2. The identification result according to various noise and numbers of BPFs.

| linear parameter | true | BPFs' number =100 | | | BPFs' number =500 | | |
|---|---|---|---|---|---|---|---|
| | | SNR=10dB | SNR=20dB | SNR=50dB | SNR=10dB | SNR=20dB | SNR=50dB |
| $a$ | 1.1 | 1.1313 | 1.0898 | 1.1001 | 1.0986 | 1.0998 | 1.1000 |
| $b$ | 1.5 | 1.6241 | 1.4997 | 1.4999 | 1.5014 | 1.5001 | 1.5000 |
| $\delta$ | - | 6.88 | 0.54 | 7.6e-03 | 0.100 | 0.010 | 0.000 |

From Table 2, we can see that the more increase the number of BPFs, the more improve the identification accuracy.

## 5 Conclusion

In this paper, the identification method for linear continuous FOS with unknown state is proposed. Simulation results show that the proposed method has a high identification accuracy for FOS with unknown state and input delays. In particular, Simulation result shows that the state delay is an important non-negligible factor.

Our further study is to develop an identification method for fractional order systems with unknown state and input delays, in which not only the delays and efficient but also differential order are estimated.

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