There are no Extraordinary cycles in Collatz conjecture

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Abstract

In this paper, through the careful study and thinking of the Collatz conjecture, We give a short proof for any positive integer odd number x, when the odd number x is equal to 1, the cycle sequence is: $1, 1, 1, \cdots$, When the odd number x is greater than 1, there is no Extraordinary cyclic sequence.

1 Introduction

The Collatz conjecture problem is called "3x + 1", For any positive integer $x \in \mathbb{N}$, if x is even continuously divided by 2 such that it is equal to an odd number, if x is odd then multiply by 3 and add 1, repeat this operation over and over again, and you're bound to get 1 in a finite number of steps. Its title:

$$f(x) := \begin{cases} x/2, & \text{if } x \equiv 0 \pmod 2; \\ 3x+1, & \text{if } x \equiv 1 \pmod 2, \end{cases} \qquad \forall x \in \mathbb{N},$$

There is a positive integer $k \in \mathbb{N}$, There are always consequences $f^k(x) = 1$.
If the Collatz conjecture has a non-trivial cyclic sequence, the Collatz conjecture is not valid. So it's very important to prove that Collatz conjecture don't have nontrivial cyclic sequences.

2 Definition and conclusion

definition 2.1 *an odd number: An integer that cannot be divided by 2 is called an odd number. A positive integer with an odd number can be represented as* $2k + 1$, $k \in \mathbb{N}$.

definition 2.2 *Number of branches: An odd number of positive integers that can be divided by 3 is called a branch number The number of branches can be expressed as* $3 \cdot (2k + 1)$,$k \in \mathbb{N}$.

definition 2.3 *Formula for the General Term of the Positive Integer Odd Number x Collatz Transform Sequence:* $f^k(x)$ *is explicitly written as:* [1]

$$f^k(x) = \frac{3^k x + 3^{k-1} + 3^{k-2} \cdot 2^{p_1} + \cdots + 3 \cdot 2^{p_1+\cdots+p_{k-2}} + 2^{p_1+\cdots+p_{k-1}}}{2^{p_1+\cdots+p_k}}, \tag{1}$$

where p_j $(j \in \{1, \ldots, k\})$ *is the multiplicity of the factor* 2 *in the number* $3f^{j-1}(x)+1$. *The equivalent transformation as:*

$$2^{p_1+\cdots+p_k} \cdot f^k(x) - 3^k x = 3^{k-1} + 3^{k-2} \cdot 2^{p_1} + 3^{k-3} \cdot 2^{p_1+p_2} + \cdots + 3 \cdot 2^{p_1+\cdots+p_{k-2}} + 2^{p_1+\cdots+p_{k-1}}. \tag{2}$$

3 Correlation theorem

Theorem 3.1 *For any odd* x_0 *greater than 1, let the odd sequence of the odd* x_0 *Collatz transformation be:* $x_0, x_1, x_2, \ldots x_n$, *and let* $p_1, p_2, \ldots, p_n$ *be the number of consecutive divisions by 2 in the odd* x_0 *Collatz Odd transformation, and* $p_j \geq 1$, $p_j \in \mathbb{N}$. *If* $x_n \neq 1$, *there must be* $x_i \neq x_{i+1}$, $0 \leq i \leq n-1$, *that is, the two adjacent numbers of the odd number sequence of the Collatz transform are not equal.*
Proof. Using the method of proof by contradiction, assuming that two adjacent numbers are equal, then there is $3 \cdot x_i + 1 = x_i \cdot 2^{p_{i+1}}$.

If you divide the two sides by x_i, then you have $3+\frac{1}{x_i}=2^{p_{i+1}}$ .because x_i is an odd number greater than 1, the two sides of the equation are not equal, the result is contradictory, and the original proposition is true.

Theorem 3.2 *The number of branches does not circulate.*
Proof. *Using the method of proof by contradiction, it is assumed that there exists a certain number of branches that cause the odd sequence of the Collatz transformation to generate a cycle, that is, the number of branches is transformed into the number of branches. According to the equivalent transformation formula of the Collatz transformation general term formula, it can be inferred that:*
$2^{p_1+\cdots+p_k}\cdot f^k(x)-3^k x=3^{k-1}+3^{k-2}\cdot 2^{p_1}+3^{k-3}\cdot 2^{p_1+p_2}+\cdots\cdots+3\cdot 2^{p_1+\cdots+p_{k-2}}+2^{p_1+\cdots+p_{k-1}}$.
Obviously the left side of the equation can be divisible by 3, and the right side of the equation can not be divisible by 3. The original proposition stands in contradiction.

Theorem 3.3 *$\forall m\in\mathbb{N}$, $\forall k\in\mathbb{N}$, and $3^m+1=2^k$. Then there is only one set of solutions $m=1$, $k=2$.*
Proof. *By substituting $m=1$, $k=2$ into the equation, there is $1=1$, there is a set of solutions, the proposition is true.*
Now let's talk about $m\geq 2$.When m is even, let $m=2n$, $n\in\mathbb{N}$, according to the binomial theorem:
$3^m=3^{2n}=9^n=(8+1)^n=8^n+\binom{n}{1}\cdot 8^{n-1}+\binom{n}{2}\cdot 8^{n-2}+\cdots+\binom{n}{n-1}\cdot 8^1+1$
Obviously, such numbers can be represented as $8t+1$ odd numbers,$t\in\mathbb{N}$. When m is odd number, let $m=2n+1$, $n\in\mathbb{N}$. According to the binomial theorem, it can be obtained:
$3^m=3^{2n+1}=3\cdot 9^n=3\cdot(8+1)^n=3\cdot(8^n+\binom{n}{1}\cdot 8^{n-1}+\binom{n}{2}\cdot 8^{n-2}+\cdots+\binom{n}{n-1}\cdot 8^1+1)$
It can also be expressed as $24t+3$ odd numbers, $t\in\mathbb{N}$. Since $8t+1+1=8t+2$, $24t+3+1=24t+4$, $t\in\mathbb{N}$, dividing two numbers continuously by 2 results in $4t+1$ and $6t+1$ being odd numbers greater than 1. Obviously, they cannot be continuously divided by 2, and the original proposition holds.

Theorem 3.4 *For an odd number of positive integers greater than 1,the sequence of odd numbers in the collatz transform is : $x_0,x_1,x_2\cdots x_n$, let $p_1,p_2,...,p_n$ be the number of consecutive divisions by 2 in the odd x_0 Collatz Odd transformation, and $p_j\geq 1$, $p_j\in\mathbb{N}$, According to the equivalent transformation formula of the general term of collatz transformation, let $x_n\cdot 2^{p_1+\cdots+p_n}-x_0\cdot 3^n=3^{n-1}+3^{n-2}\cdot 2^{p_1}+3^{n-3}\cdot 2^{p_1+p_2}+\cdots+3\cdot 2^{p_1+\cdots+p_{n-2}}+2^{p_1+\cdots+p_{n-1}}=S_n$. If there is a positive integer n, $n\in\mathbb{N}$, such that makes the equation of the collatz transformation*
$x_n\cdot 2^{p_1+\cdots+p_n}-x_0\cdot 3^n=2^{p_1+\cdots+p_n}-3^n$.*There must be $x_n\neq x_0$*
Proof. *From the meaning of the title, $x_n\cdot 2^{p_1+\cdots+p_n}-x_0\cdot 3^n=2^{p_1+\cdots+p_n}-3^n$.*
To simplify an equation:

$$(x_n-1)\cdot 2^{p_1+\cdots+p_n}=(x_0-1)\cdot 3^n \tag{3}$$

Since x_0, x_n is an odd number of positive integers greater than 1, then there is:
$\frac{x_n-1}{x_0-1}=\frac{3^n}{2^{p_1+\cdots+p_n}}\neq 1$
So $x_n\neq x_0$. The above propositions stand.

Theorem 3.5 *For any odd number x_0 greater than 1, let the odd sequence of the odd number x_0 collatz transformation be $x_0,x_1,x_2,\cdots,x_n$, and let $p_1,p_2,\cdots,p_n$ be the number of times that the odd number x_0 collatz transformation is divided by 2 continuously. Then the odd number x_0 does not produce a cycle, that is, $x_0\neq x_n$. When $x_0=1$, a cycle occurs, and the cycle sequence is: $1,1,1,\cdots$*
Proof. *When $x_0=1$, a cycle is generated, and the cycle sequence is: $1,1,1,\cdots$. The proposition is obviously true.*
When $x_0>1$, If the odd number x_0 is the number of branches, theorem 3.2 shows that there is no cycle and the proposition holds.
When the odd number x_0 is not the number of branches, using the reduction to absurdity, it is assumed that there is a positive integer odd number x_0, and the odd number x_n and the odd number x_0 generate cycles, so there are $x_0=x_n,x_0\neq x_j,1\leq j\leq n-1$.
According to the general formula of the equivalent transformation of odd-numbered sequences, we can get:
$x_0\cdot(2^{p_1+p_2+\cdots+p_n}-3^n)=3^{n-1}+3^{n-2}\cdot 2^{p_1}+3^{n-3}\cdot 2^{p_1+p_2}+\cdots+3^1\cdot 2^{p_1+p_2+\cdots+p_{n-2}}+2^{p_1+p_2+\cdots+p_{n-1}}$.
Let the right side of the equation:
$3^{n-1}+3^{n-2}\cdot 2^{p_1}+3^{n-3}\cdot 2^{p_1+p_2}+\cdots+3^1\cdot 2^{p_1+p_2+\cdots+p_{n-2}}+2^{p_1+p_2+\cdots+p_{n-1}}=S_n$

If $2^{p_1+p_2+\cdots+p_n} - 3^n < 0$, The assumption is false, the original proposition is true.
If $2^{p_1+p_2+\cdots+p_n} - 3^n > 0$, The following situations are discussed:
1. *When S_n is a prime number, there are two cases according to the principle of unique decomposition of prime numbers:*
1.1 *The following equation holds.*

$$\begin{cases} x_0 = 1 \\ 2^{p_1+p_2+\cdots+p_n} - 3^n = S_n \end{cases} \tag{4}$$

1.2 *The following equation holds.*

$$\begin{cases} x_0 = S_n \\ 2^{p_1+p_2+\cdots+p_n} - 3^n = 1 \end{cases} \tag{5}$$

For the first case, It is obvious that $x_0 = 1$ and x_0 are positive integer odd numbers greater than 1, so the original proposition is true, so the original proposition is true.
For the second case, According to Lemma 3.3, we know that $n = 1, p_1 = 2$, and according to Lemma 3.1, we know that Collatz conjecture odd numbers are not equal to each other. the original proposition is true.
2. *If S_n is the product of many prime numbers, The following four scenarios are discussed:*
2.1 *The following equation holds.*

$$\begin{cases} x_0 = 1 \\ 2^{p_1+p_2+\cdots+p_n} - 3^n = S_n \end{cases} \tag{6}$$

2.2 *The following equation holds.*

$$\begin{cases} x_0 = S_n \\ 2^{p_1+p_2+\cdots+p_n} - 3^n = 1 \end{cases} \tag{7}$$

2.3 *S_n has no x_0 factor*
2.4 *There is a factor of x_0 in S_n, which means there is a positive integer c, then The following equation holds.*

$$\begin{cases} x_n = x_0 \cdot c \\ x_0 \cdot (2^{p_1+p_2+\cdots+p_n} - 3^n) = x_0 \cdot c \end{cases} \tag{8}$$

For the first case, same as 1.1, the original proposition holds.
For the second case, same as 1.2, the original proposition holds.
For the third case, S_n is not divisible by x_0, the original proposition holds.
For the fourth case, (8) is divided by x_0, its collatz equivalent transformation is:

$$2^{p_1+p_2+\cdots+p_n} - 3^n = c \tag{9}$$

For equation (9) , According to lemma 3.4, we know that $x_n \neq x_0$, so the assumption is not true, and the odd number x_0 does not produce a cycle, the original proposition is true.
To sum up, the original proposition is true.

4 corollary

In this paper, we know that for any odd number x_0 greater than 1, let the odd number x_0 collatz transform odd number sequence be $x_0, x_1, x_2, \cdots x_n$, There must be $x_0 \neq x_j, 1 \leq j \leq n$ that does not produce a cycle; when $x_0 = 1$, the cycle sequence is $1, 1, 1 \cdots$.

References

[1] Huang Guo lin Wu Jia bang. Elongation iteration in the $3n + 1$ conjecture [j]. *Journal of Hua zhong University of Science and Technology*, 29(09):112–114., 2001.