A New Property of the Sheldon Prime

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Abstract

In this article, the author inquires into a new property for a set of numbers containing the Sheldon prime as the first term that allows to calculate summations just by solving a simple multiplication.

1 Introduction

In [1] the authors propose that 73 is a unique number due to the rare properties it possesses, creating the concept of a "Sheldon Prime" and conjecturing it was the only number with all these properties. In [2] Carl Pomerance and Chris Spicer proved this conjecture, thus establishing that 73 is the only number that fulfill these properties, and consequently, that it is the only Sheldon prime.
In the current article, it is shown an interesting new property of the Sheldon prime, which is also shared with a set of numbers.

The idea of this new property comes from analyzing the result of $\sum_{i=1}^{73} i$ and noticing that 2701 is, surprisingly, 73×37; which is interesting due to the fact that 37 is $rev(73)$, where rev(n) is the integer which has the same digits as n but in the reverse order.

2 How does the property work?

To begin with, let us define a generic natural number n with b digits where a_i is the i^{th} digit as

$$n = 10^b a_0 + 10^{b-1} a_1 + 10^{b-2} a_2 + ... + 10^2 a_{b-2} + 10 a_{b-1} + a_b. \tag{1}$$

The numbers that have this new property satisfy that $\sum_{i=1}^{n} i = n \times rev(n)$
In the previous example where $n = 73$, $\sum_{i=1}^{73} i = 73 \times 37$
We know that 73 fulfills this property, but how many other numbers than 73 have this property

Theorem: There is an infinite set of numbers that have this property

Demonstration:
Using the same notation as above, allow us to denote rev(n) as

$$rev(n) = 10^b a_b + 10^{b-1} a_{b-1} + 10^{b-2} a_{b-2} + ... + 10^2 a_2 + 10 a_1 + a_0 \tag{2}$$

To obtain an equivalent expression, we can use the "Gauss's summation trick" which allows us to rewrite $\sum_{i=1}^{n} i$ as $\frac{n+1}{2}$. This gives us (3), showing that this property only works for numbers that are equal to two times its reverse number minus 1.

$$\sum_{i=1}^{n} i = n \times rev(n) \rightarrow \frac{n(n+1)}{2} = n \times rev(n) \rightarrow rev(n) = \frac{n+1}{2} \rightarrow n = 2rev(n) - 1 \tag{3}$$

Now, we can analyze which numbers verify this property.

In the first instance, without doing any calculations, we can ensure that n cannot be a palindromic number because *rev*(n)=n.

We also discard number 1 because although $rev(1) = \frac{1+1}{2}$, its reverse is equal to itself $rev(1) = 1$

The rest of one-digit numbers are all discarded because, as number 1, their reverses are themselves (and obviously not equal to the double of themselves minus 1).

With the information we have at this point, it is possible to assure that the numbers with this property have at least two digits.

3 The first digit

Let us start by focusing on the first digit:

As it is written above being $n = 10^b a_0 + ... + a_b \rightarrow rev(n) = 10^b a_b + ... + a_0$

Then, according to the property mentioned $rev(n) = \frac{n+1}{2}$, therefore, for the last digit of $rev(n)$ which is a_0 we can say that $a_0 = \frac{a_b+1}{2} \rightarrow 2a_0 - 1 = a_b$.

Considering that every a_i can only adopt integer values from 0 to 9, if $a_0 > 5 \rightarrow a_b$ will have two digits, which is not allowed. To avoid this case, we can eliminate the last digit and isolate the first digit using this formula: $\lfloor \frac{2a_0-1}{10} \rfloor \times 10$

Then we can subtract the first digit to ensure that a_b will have just one digit:

$$2a_0 - 1 - \lfloor \frac{2a_0 - 1}{10} \rfloor \times 10 = a_b \tag{4}$$

Let us use a table to analyze the possible values:

Table 1: Calculation of the possible a_b values depending on the a_0 values.

a_0	calculation	a_b
0	$2 \times 0 - 1 - \lfloor \frac{0-1}{10} \rfloor \times 10 = 2 \times 0 - 1 - 0$	-1
1	$2 \times 1 - 1 - \lfloor \frac{1-1}{10} \rfloor \times 10 = 2 \times 1 - 1 - 0$	1
2	$2 \times 2 - 1 - \lfloor \frac{2-1}{10} \rfloor \times 10 = 2 \times 2 - 1 - 0$	3
3	$2 \times 3 - 1 - \lfloor \frac{3-1}{10} \rfloor \times 10 = 2 \times 3 - 1 - 0$	5
4	$2 \times 4 - 1 - \lfloor \frac{4-1}{10} \rfloor \times 10 = 2 \times 4 - 1 - 0$	7
5	$2 \times 5 - 1 - \lfloor \frac{5-1}{10} \rfloor \times 10 = 2 \times 5 - 1 - 0$	9
6	$2 \times 6 - 1 - \lfloor \frac{6-1}{10} \rfloor \times 10 = 2 \times 6 - 1 - 10$	1
7	$2 \times 7 - 1 - \lfloor \frac{7-1}{10} \rfloor \times 10 = 2 \times 7 - 1 - 10$	3
8	$2 \times 8 - 1 - \lfloor \frac{8-1}{10} \rfloor \times 10 = 2 \times 8 - 1 - 10$	5
9	$2 \times 9 - 1 - \lfloor \frac{9-1}{10} \rfloor \times 10 = 2 \times 9 - 1 - 10$	7

As we can see in the table, a_b is always an odd number.

4 The last digit

Let us study now the last digit.

Returning again to the property which sets up that being $n = 10^b a_0 + ... + a_b$ and $rev(n) = 10^b a_b ... a_0 \rightarrow rev(n) = \frac{n+1}{2}$, it is easy to see that the first digit of n (a_0) it would be the double of the first digit of $rev(n)$ (a_b), so permit us write $a_0 = 2 \times a_b$.

As before, a_i can only adopt integer values from 0 to 9, and if $a_b > 5 \rightarrow a_0$ will have two digits, which is not allowed. To prevent this, we use formula 4, but adapted to this case, which looks like this:

$$a_0 = 2a_b - \lfloor \frac{2a_b}{10} \rfloor \times 10 \tag{5}$$

It also has to be considered that the previous digit of a_0 (a_1) could have more than one digit due to the multiplication, which will carry over a digit to a_0. As a_i has to be lower than 10, the maximum carry it would be 1. To include this in our consideration, allow us to write the formula as $2a_b - \lfloor \frac{2a_b}{10} \rfloor \times 10 = a_0 + c | c$ is the possible carry.

Now, as it is done before, we create a table but taking into account that a_b has to be an odd number between 1 and 9 and that the carry can be 0 or 1.

Table 2: Verification for the possible values of a_b.

a_0	calculation without carry	$a_b + 0$	calculation with carry	$a_b + 1$
1	$2 \times 1 - \lfloor \frac{2}{10} \rfloor \times 10 = 2 - 0$	2	$2 \times 1 + 1 - \lfloor \frac{3}{10} \rfloor \times 10 = 2 \times 2 + 1 - 0$	3
3	$2 \times 3 - \lfloor \frac{6}{10} \rfloor \times 10 = 6 - 0$	6	$2 \times 3 + 1 - \lfloor \frac{7}{10} \rfloor \times 10 = 6 + 1 - 0$	7
5	$2 \times 5 - \lfloor \frac{10}{10} \rfloor \times 10 = 10 - 10$	0	$2 \times 5 + 1 - \lfloor \frac{11}{10} \rfloor \times 10 = 10 + 1 - 10$	1
7	$2 \times 7 - \lfloor \frac{14}{10} \rfloor \times 10 = 14 - 10$	4	$2 \times 7 + 1 - \lfloor \frac{15}{10} \rfloor \times 10 = 14 + 1 - 10$	5
9	$2 \times 9 - \lfloor \frac{18}{10} \rfloor \times 10 = 18 - 10$	8	$2 \times 9 + 1 - \lfloor \frac{19}{10} \rfloor \times 10 = 18 + 1 - 10$	9

5 Comparing the results

As a_0 and a_b have the same value in both numbers, we can join the tables to analyze the results:

Table 3: Comparison between supposed values of a_0 and the theoretical values it would have.

Supposed a_0	theoretical a_b	theoretical a_0 without carry	theoretical a_0 with carry
0	-1	-2	-1
1	1	2	3
2	3	6	7
3	5	0	1
4	7	**4**	5
5	9	8	9
6	1	2	3
7	3	6	**7**
8	5	0	1
9	7	4	5

As we can see, there are only two values for a supposed a_0 that match with the theoretical a_0. This automatically discards the rest of the values due to an incongruence.

This means that if n has two digits, it has to be either 47 or 73.

- If $n = 47 \rightarrow \frac{47+1}{2} = rev(47)$
  $\frac{47+1}{2} = \frac{48}{2} = 24$
  $rev(47) = 74$
- If $n = 73 \rightarrow \frac{73+1}{2} = rev(73)$
  $\frac{73+1}{2} = \frac{74}{2} = 37$
  $rev(73) = 37$

After doing the verification, it seems that 47 does not fulfill this property, therefore, we can conclude that all the numbers with this property will look like $7 \times 10^b + 10^{b-1}a_{b-1} + 10^{b-2}a_{b-2} + ... + 10^2 a_2 + 10a_1 + 3$

6 The middle terms

To start with an easy number to analyze, let us choose a number n with three digits (b = 3).

In that case, n would be $7 \times 100 + 10a_{b-1} + 3$ and rev(n) would be $3 \times 100 + 10a_{b-1} + 7$

If n satisfies the property, then $300 + 10a_{b-1} + 7 \times 2 + 1 = 7 \times 100 + 10a_{b-1} + 3$

As can be seen, a_{b-1} does not change when multiplying it by two and adding 1 (because, as it is proved before, there will be a carry 1).

Knowing this, we can say that $2a_{b-1} + 1 = a_{b-1}$.

We also know that a_{b-1} might transfer the carry to the next number because the $2 \times 3 + c = 7$ where c is the carry.

This means that a_{b-1} has to be a number from 5 to 10 (because if it was lower, there will be no carrying).

Because of this, we know that a_{b-1} will have two digits. Thus, we are going to exclude the first digit using the same method used before. This gives us this formula:

$$2 \times a_{b-1} + 1 - \lfloor \frac{2 \times a_{b-1} + 1}{10} \rfloor \times 10 = a_{b-1} \tag{6}$$

If we make a table to analyze the results, we get this:

Table 4: Verification for the possible values of a_{b-1}.

a_{b-1}	calculation	result
5	$2 \times 5 + 1 - \lfloor \frac{2\times5+1}{10} \rfloor \times 10 = 11 - 10$	1
6	$2 \times 6 + 1 - \lfloor \frac{2\times6+1}{10} \rfloor \times 10 = 13 - 10$	3
7	$2 \times 7 + 1 - \lfloor \frac{2\times7+1}{10} \rfloor \times 10 = 15 - 10$	5
8	$2 \times 8 + 1 - \lfloor \frac{2\times8+1}{10} \rfloor \times 10 = 17 - 10$	7
9	$2 \times 9 + 1 - \lfloor \frac{2\times9+1}{10} \rfloor \times 10 = 19 - 10$	**9**

As it can be observed, the only digit that has the property of transferring the carry to the next term without changing itself is the 9. This means that not only 73 and 793 fulfill the property of being $\sum_{i=1}^{n} i$ equal to $n \times rev(n)$, but it is also extended to every number whose first digit is $7(a_0 = 7)$, last digit is $3(a_b = 3)$, and all other digits are $9(a_i = 9 \ \ \forall a_i | i{\neq}0$ and i≠b).

From this information, we define two sets of numbers: the set that contains the upper limit value of the summations, which will be called "Bigger numbers' set" ($\mathbb{B}$) and the one that contains its reverses called from now on "Smaller numbers' set" ($\mathbb{S}$).

$$\sum_{i=1}^{n} i = n \times rev(n) \quad \forall n \in \mathbb{B} \tag{7}$$

$$\sum_{i=1}^{rev(n)} i = n \times rev(n) \quad \forall n \in \mathbb{S} \tag{8}$$

Table 5: Representation of the first ten terms of both sets

$\mathbb{B}$	$\mathbb{S}$
73	37
793	397
7993	3997
79993	39997
799993	399997
7999993	3999997
79999993	39999997
799999993	399999997
7999999993	3999999997
79999999993	39999999997

Although some of these numbers are not prime, all can be expressed as Pythagorean primes $(4n + 1)$.

In the case of a number m that belongs to $\mathbb{S}$, it can be written as $m = 4 \times (10^{\lfloor \log_{10} m \rfloor} - 1) + 1$

In the case of a number n that belongs to $\mathbb{B}$, it can be written as $n = 8 \times (10^{\lfloor \log_{10} n \rfloor} - 1) + 1$

Using this last expression, we can affirm that for every $n > 1$

$$\sum_{i=1}^{n} i = n \times rev(n) \quad \forall n = 8 \times (10^{\lfloor \log_{10} n \rfloor} - 1) + 1 \tag{9}$$

7 Conclusion

We have defined a set of numbers with infinitely many terms for which the summation from 1 to one of the numbers in this set can be solved just by multiplying the number by its reverse; which can significantly simplify calculations when doing mental arithmetic.

References

[1] Jessie Byrnes, Chris Spicer, and Alyssa Turnquist. The Sheldon Conjecture. *Math Horizons*, 23(2): 12–15, 2015.

[2] Carl Pomerance and Chris Spicer. Proof of the Sheldon Conjecture. *The American Mathematical Monthly*, 126(8): 688–698, 2019.