A connection between the Darwin term and a non-spherical charge distribution

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Abstract

A derivation of the Darwin term is given based on assuming a non-spherical charge distribution. The total translational energy of the system is obtained for a static electric field and the corresponding quantum equation is found to contain the Darwin term under certain conditions.

I. Introduction.

If we make a non-relativistic expansion of the Dirac equation, one of the terms in the expansion is the Darwin term, (for example see Sakurai[1]). It is named for C. G. Darwin[2] who first investigated it. It takes the form $-\frac{q\hbar^2}{8m^2c^2}\boldsymbol{\nabla}\cdot\boldsymbol{E}$ where $\boldsymbol{E}$ is the electric field, m is the mass of the particle, q its charge, and c is the speed of light. A bold symbol represents a vector.

The most common explanation of the Darwin term is that it is an effect due to Zitterbewegung, (for example see Sakurai[1]). Wilson[3] extends this approach by representing the electron as an oscillator. Yu, Henneberger[4] propose that it is an extension of the spin-orbit term while Fushchych et al.[5] show that it can thought of as a non-relativistic effect by expanding the Levi- Leblond equation. Khripolovich and Milstein[6] show that the Darwin term can be considered of the same origin as the spin-orbit term. Faber[7] shows that it can be associated with a random walk. We show that by using a non-spherical charge distribution it is possible to derive it in a way different from the above sources.

II. Charge Distribution

Consider a rotating cylindrically symmetric charge distribution in a frame where the angular velocity **ω** aligns with the z axis in an x, y, z rectangular coordinate system. Also take the charge density ρ to be centered at the center of the coordinate system and symmetric with respect to z. Call this the primed frame. We need $\int \rho dV = q$ where the spatial integral is over the particle. Because of the charge symmetry we also have

$$\int \rho x' dV = \int \rho y' dV = \int \rho z' dV = 0 \tag{1a}$$

$$\int \rho x'y' dV = \int \rho x'z' dV = \int \rho y'z' dV = 0 \tag{1b}$$

In this frame we then define the values $Q_1$and Q_2 by the relations

$$Q_1 = \int \rho z'^2 dV \tag{1c}$$

$$Q_2 = \int \rho x'^2 dV = \int \rho y'^2 dV \tag{1d}$$

Note that $Q_1$and Q_2 are similar to the moment of inertia for a mass distribution, but are defined in terms of the charge density instead of the mass density. In general for a rotating object the charge distribution will not be spherically symmetric but will depend upon the angular velocity ω. As a result of this Q_1 and Q_2 will also depend upon ω.

Now we need $Q_1 = Q_2$ as ω goes to zero, so to second order set $Q_1 = Q_0(1 + \alpha_1\omega + \alpha_2\omega^2)$ and $Q_2 = Q_0(1 + \beta_1\omega + \beta_2\omega^2)$ for some constants Q_0 , α_1 , α_2 , β_1 and β_2 . We want this to be unchanged if we replace ω by -ω, so we need $\alpha_1 = \beta_1 = 0$. To make the units work out correctly we can set $\alpha_2 = \alpha\frac{I}{mc^2}$ and $\beta_2 = \beta\frac{I}{mc^2}$ where I is the moment of inertia of the particle and α and β are dimensionless constants. I is the standard moment of inertia for a mass distribution and has units of grams-cm^2 (for example see Goldstein[8]). Thus we have

$$Q_1 = Q_0(1 + \alpha\frac{I}{mc^2}\omega^2) \tag{2a}$$

$$Q_2 = Q_0(1 + \beta\frac{I}{mc^2}\omega^2) \tag{2b}$$

In general rectangular coordinates x^i, where i = 1,2,3, using eqs. (2a,b), eqs. (1a-d) take the form

$$\int \rho\delta x^i dV = 0 \tag{3a}$$

$$\int \rho\delta x^i \delta x^j dV = Q_0((1 + \beta\frac{I}{mc^2}\omega^2)\delta^{ij} + (\alpha - \beta)\frac{I}{mc^2}\omega^i\omega^j) \tag{3b}$$

where δx^i represents the coordinate distance from the center of the charge.

We will consider a current density $\boldsymbol{j} = \rho(\boldsymbol{v} + \boldsymbol{\omega} \times \delta\boldsymbol{x})$ where **v** is the velocity of the particle. The magnetic moment μ and g factor are defined by (for example see Jackson[9])

$$\boldsymbol{\mu} = \frac{1}{2c}\int \delta\boldsymbol{x} \times \boldsymbol{j} dV \tag{4}$$

$$\boldsymbol{\mu} = \frac{gq}{2mc}\boldsymbol{s} \tag{5}$$

where **s** is the interior angular momentum of the particle. If we take $\boldsymbol{s} = I\boldsymbol{\omega}$, and use our relation for **j** given above along with eq. (4) and eqs. (3a,b) in eq. (5) we obtain

$$Q_0 = \frac{gq}{2m}I(1 - \beta\frac{I}{mc^2}\omega^2)$$

to order $1/c^2$. The α term cancels out. The moment of inertia can also be a function of ω^2 so set $I = I_0(1 + \gamma\frac{\omega^2}{c^2})$ for some constants γ and I_0, so to order $1/c^2$

$$Q_0 = \frac{gq}{2m}I_0(1 + \gamma\frac{\omega^2}{c^2} - \beta\frac{I_0}{mc^2}\omega^2)$$

In order for Q_0 and I_0 to be independent of ω we need $\gamma = \beta\frac{I_0}{m}$ so that

$$Q_0 = \frac{gq}{2m}I_0 \tag{6}$$

III. Equations of Motion and Quantization

Now consider the translational equation of motion with only a static electric field **E**.

$$m\frac{d\boldsymbol{v}}{dt} = \int \rho\boldsymbol{E}dV \tag{7}$$

Expanding **E** in a Taylor series about the center of the particle we have

$$\boldsymbol{E} = \boldsymbol{E}_0 + (\delta\boldsymbol{x}\cdot\boldsymbol{\nabla})\boldsymbol{E}_0 + \frac{1}{2}(\delta\boldsymbol{x}\cdot\boldsymbol{\nabla})^2\boldsymbol{E}_0 \tag{8}$$

where $\boldsymbol{E}_0$ is **E** and its derivatives evaluated at the center of the particle, and we have ignored terms higher than quadratic in $\delta\mathbf{x}$.

Using eq. (8), eqs. (3a,b), and eq. (6) along with the condition $\int \rho dV = q$ in eq. (7) we obtain the relation

$$m\frac{d\boldsymbol{v}}{dt} = q\{\boldsymbol{E}_0 + \frac{I_0}{2m}((1+\ \beta\frac{I_0}{mc^2}\omega^2)\nabla^2\boldsymbol{E}_0 + (\alpha-\beta)\frac{I_0}{mc^2}(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})^{\mathbf{2}}\boldsymbol{E}_0)\} \tag{9}$$

where we have ignored terms higher than quadratic in $\delta\mathbf{x}$ and dropped terms higher than $1/c^2$. We have also set g = 2.

Since we are only considering static electric fields, we can set $\boldsymbol{E}_0 = -\boldsymbol{\nabla}\phi$ where ϕ is the scalar potential. Expressing $\boldsymbol{E}_0$ in this form eq. (9) becomes

$$m\frac{d\boldsymbol{v}}{dt} = -q\boldsymbol{\nabla}\{\phi + \frac{I_0}{2m}((1+\ \beta\frac{I_0}{mc^2}\omega^2)\nabla^2\phi + (\alpha-\beta)\frac{I_0}{mc^2}(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})^{\mathbf{2}}\phi)\} \tag{10}$$

The right hand side can be viewed as the gradient of a potential so the total energy E of the system can be written as

$$E = \frac{1}{2}mv^2 + q\{\phi + \frac{I_0}{2m}((1+\ \beta\frac{I_0}{mc^2}\omega^2)\nabla^2\phi + (\alpha-\beta)\frac{I_0}{mc^2}(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})^{\mathbf{2}}\phi)\}$$

$$= \frac{1}{2m}p^2 + q\{\phi + \frac{I_0}{2m}\nabla^2\phi + \frac{1}{2m^2c^2}(\beta s^2\nabla^2 + (\alpha-\beta)(\boldsymbol{s}\cdot\boldsymbol{\nabla})^{\mathbf{2}})\phi\} \tag{11}$$

where **p** is the momentum, and again ignoring terms higher than $1/c^2$.

We will quantize the system by expressing the energy in eq. (11) as an operator by replacing **p** by $-i\hbar\boldsymbol{\nabla}$ and **s** by $\frac{1}{2}\hbar\boldsymbol{\sigma}$ where **σ** are the Pauli spin matrices in vector form (for example see Saxon[10]) so that our Schrodinger type equation takes the form

$$i\hbar\frac{\partial\psi}{\partial t} = [-\frac{\hbar^2}{2m}\nabla^2 + q\{\phi + \frac{I_0}{2m}\nabla^2\phi + \frac{\hbar^2}{8m^2c^2}(\beta\sigma^2\nabla^2 + (\alpha-\beta)(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})^{\mathbf{2}})\phi\}]\psi$$

$$= [-\frac{\hbar^2}{2m}\nabla^2 + q\{\phi + \frac{\hbar^2}{8m^2c^2}(2\beta+\alpha)\nabla^2\phi\}]\psi \qquad (12)$$

We have used the properties of the **σ** matrices so that $\sigma^2 = 3$ and $(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})^{\mathbf{2}} = \nabla^2$ and have taken the limit of I_0 going to zero. If we think of a rotating object then as ω increases the equator moment should expand and the moment parallel to **ω** should reduce in size. Therefore β should be positive and α negative. If we set $-\alpha = \beta = 1$ we obtain the Darwin term.

Conclusion.

One interesting thing about this derivation is that the 1/8 in front of the Darwin term comes out naturally, although there is no apparent reason why 2β+α should be 1. The other c^{-2} corrections in the non-relativistic expansion of the Dirac eq., the spin-orbit and relativistic mass correction terms, are due to a relativistic correction to the velocity while in our case the Darwin term appears to be a relativistic correction to the spin.

Instead of including the rotational equations and using a finding a Lagrangian for the whole system, the translational energy has just been used. It turns out that if we try to find a Lagrangian for the translational and rotational equations we run into problems.

In spite of these assumptions it is interesting that the Darwin term can be obtained by using a non-spherical charge distribution, and perhaps a more sophisticated derivation will lead to a better understanding.

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