ON THE GEOMETRY OF AXES OF COMPLEX CIRCLES OF PARTITION

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Abstract. In this paper we continue the development of the circles of partition by introducing a certain geometry of the axes of complex circles of partition. We use this geometry to verify the condition in the squeeze principle in special cases with regards to the orientation of the axes of complex circles of partition.

1. **Introduction**

In our seminal work [1], we introduced the method of *circles of partition* (CoP), a novel approach grounded in a combinatorial structure that encodes specific additive properties of subsets of integers. This structure is equipped with a geometric interpretation, whereby the elements are viewed as points in the plane, with their weights corresponding to elements of the underlying subset. Formally, we define the set of points as:

$$\mathcal{C}(n,\mathbb{M}) = \{[x] \mid x, n-x \in \mathbb{M}\}\,. \tag{1.1}$$

Each point in this setexcept the central pointmust be uniquely paired with another point, such that the two are joined by a line referred to as an axis of the CoP. We denote an axis of a CoP by $\mathbb{L}_{[x],[y]}$, and define an axis contained within the CoP as:

$$\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M}) \quad \text{which implies} \quad [x],[y] \in \mathcal{C}(n,\mathbb{M}) \quad \text{with} \quad x+y=n.$$

In [2], we extended the method of circles of partition to complex numbers, where the corresponding points are weighted by complex numbers and co-axis points are connected by a line. This leads to the notion of the *complex circle of partition* (cCoP), defined as:

$$\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}}) = \left\{[z] \mid z, n-z \in \mathbb{C}_{\mathbb{M}}, \Im(z)^2 = \Re(z)\,(n-\Re(z))\right\},$$

where

$$\mathbb{C}_{\mathbb{M}} := \{z = x+iy \mid x \in \mathbb{M}, y \in \mathbb{R}\} \subseteq \mathbb{C},$$

with $\mathbb{M} \subseteq \mathbb{N}$. This complex additive structure is abbreviated as *cCoP*. The condition $\Im(z)^2 = \Re(z)\,(n-\Re(z))$, referred to as the **circle condition**, ensures that all points on the cCoP lie on a circle in the complex plane. This circle, known as the *embedding circle* of the cCoP $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$, is denoted by $\mathfrak{C}_n$. These

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embedding circles have the property that they are fully contained within larger embedding circles, except at the origin, which serves as a common point [2]. For any axis, we assign the following relation:

$$\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_\mathbb{M}) \quad \text{which implies} \quad [z_1], [z_2] \in \mathcal{C}^o(n, \mathbb{C}_\mathbb{M}) \quad \text{with} \quad z_1 + z_2 = n.$$

The structure of complex circles of partition offers greater versatility, incorporating features absent in the standard CoPs. Notably, for each axis $\mathbb{L}_{[z],[n-z]}$ of a cCoP, there exists a corresponding *conjugate axis*:

$$\mathbb{L}_{\overline{[z]},\overline{[n-z]}},$$

where $\overline{[z]}$ and $\overline{[n-z]}$ denote the complex conjugate points. The geometric configuration of embedding circles, including the space outside the embedding circle, reveals an interesting ordering principle between points on interacting axes from distinct cCoPs. A key consequence of the circle condition is the following result:

$$|\mathbb{L}_{[z_1],[z_2]}| = n$$

for any axis $\mathbb{L}_{[z_1],[z_2]} \in \mathcal{C}^o(n, \mathbb{C}_\mathbb{M}) = \big\{[z] \mid z, n - z \in \mathbb{C}_\mathbb{M}, \Im(z)^2 = \Re(z)\,(n - \Re(z))\big\}$. The *squeeze principle* [3], introduced as a tool for studying the binary Goldbach conjecture, also finds a slightly modified version in [2]. For the readers convenience, we briefly revisit this elegant principle below.

Theorem 1.1 (The squeeze principle)**.** *Let $\mathbb{B} \subset \mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}^o(n, \mathbb{C}_\mathbb{M})$ and $\mathcal{C}^o(n + t, \mathbb{C}_\mathbb{M})$ with $t \geq 4$ be non–empty cCoPs with integers n, t, s of the same parity. If there exist an axis $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_\mathbb{M})$ with $z_2 \in \mathbb{C}_\mathbb{B}$ and an axis $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(n+t, \mathbb{C}_\mathbb{M})$ with $w_1 \in \mathbb{C}_\mathbb{B}$ such that*

$$\Re(z_1) < \Re(w_1) \text{ and } \Re(z_2) < \Re(w_2) \tag{1.2}$$

then there exists an axis $\mathbb{L}_{[u_1],[u_2]} \hat{\in} \mathcal{C}^o(n + s, \mathbb{C}_\mathbb{B})$ with $0 < s < t$. Hence also $\mathcal{C}^o(n + s, \mathbb{C}_\mathbb{M})$ is not empty.

Proof. From the existence of an axis $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_\mathbb{M})$ follows $\Re(z_2) = n - \Re(z_1)$. With the requirement (1.2) we get

$$\Re(z_2) > n - \Re(w_1). \tag{1.3}$$

On the other hand from the existence of an axis $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(n + t, \mathbb{C}_\mathbb{M})$ follows $\Re(w_2) = n + t - \Re(w_1)$ and with the requirement (1.2) and the result (1.3) we get

$$\begin{aligned} n - \Re(w_1) < \Re(z_2) < n + t - \Re(w_1) \mid +\Re(w_1) \\ n < \Re(z_2) + \Re(w_1) < n + t \\ n < n + s < n + t. \end{aligned}$$

By virtue of the requirements $z_2, w_1 \in \mathbb{C}_\mathbb{B}$ and $n + s = \Re(z_2) + \Re(w_1)$ there is an axis $\mathbb{L}_{[u_1],[u_2]} \hat{\in} \mathcal{C}^o(n + s, \mathbb{C}_\mathbb{B})$ with $\Re(u_1) = \Re(w_1)$ and $\Re(u_2) = \Re(z_2)$ with their imaginary parts determined by the circle condition. Hence $\mathcal{C}^o(n+s, \mathbb{C}_\mathbb{B}) \neq \emptyset$. Since $\mathbb{B} \subset \mathbb{M}$, it follows immediately that $\mathbb{C}_\mathbb{B} \subset \mathbb{C}_\mathbb{M}$ and hence $\mathcal{C}^o(n + s, \mathbb{C}_\mathbb{M}) \neq \emptyset$. This completes the proof of the squeeze principle. $\square$

Theorem 1.1, commonly referred to as the squeeze principle, serves as a fundamental framework for investigating the feasibility of partitioning integers of a particular parity using elements drawn from a specific subset of the integers. This principle operates by identifying a pair of non-empty complex circles of partition (cCoPs) that share a common base set. Subsequently, additional non-empty cCoPs with generators constrained within the interval defined by these two initial generators are determined. The squeeze principle can be applied effectively to examine the broader issue of partitioning numbers such that each summand belongs to the same subset of positive integers. It also prompts further exploration into the geometric conditions under which it holds, driven by the following fundamental questions:

Question 1. *How do the notions of interiors and exteriors with respect to cCoPs facilitate the proof of the squeeze principle?*

Question 2. *What role do the imaginary weights of members of cCoPs play in this context?*

Question 3. *Are the embedding circles of cCoPs key to proving the Binary Goldbach Conjecture (BGC)?*

2. Orientations of axes of Complex circles of partition and related Geometries

In this section we introduce and study the geometry of the axis of cCoPs. We launch the following languages as a precursor to our studies. In this section we consider only axes of distinct cCoPs $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ such that $\Re(z_1) < \Re(z_2)$ and $\Re(w_1) < \Re(w_2)$ with $\Re(z_1) \neq \Re(w_1)$ and $\Re(z_2) \neq \Re(w_2)$.

Definition 2.1. Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ be a non–empty cCoPs with $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$. We denote the **gradient** of the axis $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ with

$$\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) = \frac{\Im(z_2) - \Im(z_1)}{\Re(z_2) - \Re(z_1)}.$$

We say it is an axis of **positive orientation** if the gradient is positive. On the other hand if the gradient is negative, then we say it is an axis of the cCoP with a **negative orientation**.

Definition 2.2. Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ be non–empty cCoPs with $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$. We say the axes are of **homogeneous** orientation if they point to the same direction. We denote this relation with $\mathbb{L}_{[z_1],[z_2]} \mid\mid \mathbb{L}_{[w_1],[w_2]}$. If they point to different directions, then we say the axes are of **mixed** orientation. We denote the axes of distinct orientation that are perpendicular with the relation $\mathbb{L}_{[z_1],[z_2]} \perp \mathbb{L}_{[w_1],[w_2]}$. If they point to different directions and do not intersect, then we say the axes $\mathbb{L}_{[z_1],[z_2]}$ and $\mathbb{L}_{[w_1],[w_2]}$ are **skewed**.

Proposition 2.3. *Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ be non–empty cCoPs with $\mathbb{L}_{[z],[\overline{z}]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathbb{L}_{[w],[\overline{w}]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$. Then*

$$\mathbb{L}_{[z],[\overline{z}]} \;||\; \mathbb{L}_{[w],[\overline{w}]}.$$

Proof. The claim follows since $\mathbb{L}_{[z],[\overline{z}]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathbb{L}_{[w],[\overline{w}]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ are the degenerate axes of their corresponding cCoPs and each degenerate axis must be parallel to the imaginary axis. It follows by transitivity that the axes must be parallel to each other. □

Lemma 2.4. *Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ be non–empty cCoPs with $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ for $m > n$ with $\Re(z_1) < \Re(z_2)$ and $\Re(w_1) < \Re(w_2)$ such that the axes are of positive orientation. If $\mathbb{L}_{[z_1],[z_2]} \;||\; \mathbb{L}_{[w_1],[w_2]}$, then $\Re(z_1) < \Re(w_1)$ and $\Re(z_2) < \Re(w_2)$.*

Proof. We note that the axes $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ passes through the point $(\frac{n}{2}, 0)$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ also passes through the point $(\frac{m}{2}, 0)$ with $m > n$. It follows that $\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) = \mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]})$ so that we can write

$$\frac{\Im(z_2)}{\Re(z_2) - \frac{n}{2}} = \frac{\Im(w_2)}{\Re(w_2) - \frac{m}{2}}$$

since $\mathbb{L}_{[z_1],[z_2]} \;||\; \mathbb{L}_{[w_1],[w_2]}$. Since $m > n$, it follows that $\mathrm{Int}[\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})] \subset \mathrm{Int}[\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})]$. Since the axes are of positive orientation with $\mathbb{L}_{[z_1],[z_2]} \;||\; \mathbb{L}_{[w_1],[w_2]}$ then it implies that $\Im(z_2) < \Im(w_2)$ so that $\Re(z_2) - \frac{n}{2} < \Re(w_2) - \frac{m}{2} \iff \Re(z_2) < \Re(w_2)$. Let us join the point w_2 to the point z_2 by a straight line, then it is easy to see that the gradient of this line is given by

$$\frac{\Im(w_2) - \Im(z_2)}{\Re(w_2) - \Re(z_2)} > 0.$$

Similarly, let us join the point z_1 to the point z_2 by a straight line. We compute the gradient of this line as

$$\frac{\Im(z_1) - \Im(w_1)}{\Re(z_1) - \Re(w_1)} < 0.$$

Let us suppose that

$$\frac{\Im(z_1) - \Im(w_1)}{\Re(z_1) - \Re(w_1)} > 0$$

then $\Im(z_1) - \Im(w_1) > 0$ since $\mathbb{L}_{[z_1],[z_2]} \;||\; \mathbb{L}_{[w_1],[w_2]}$ with $m > n$ so that $\Re(z_1) > \Re(w_1)$. It follows that

$$\Re(w_1) = \Re(\overline{w_1}) < \Re(z_1) = \Re(\overline{z_1})$$

with

$$|w_1 - (\frac{m}{2}, 0)| = |\overline{w_1} - (\frac{m}{2}, 0)| > |z_1 - (\frac{m}{2}, 0)| = |\overline{z_1} - (\frac{m}{2}, 0)|.$$

Since

$$|\Im(w_1)| = |\Im(\overline{w_1})| > |\Im(z_1)| = |\Im(\overline{z_1})|$$

and the points $[w_1], [\overline{w_1}]$ are opposite points on the embedding circle $\mathfrak{C}_m$ and similarly the points $[z_1], [\overline{z_1}]$ on $\mathfrak{C}_n$ with $m > n$, it follows that $\mathfrak{C}_m$ and $\mathfrak{C}_n$ cannot have a common point at the origin. This violates the Big Bang theorem. □

We obtain a similar result of the natural ordering principle of the real part of axes of cCoPs in the case where the axes are all of negative orientation.

Lemma 2.5. *Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ be non–empty cCoPs with $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ for $m > n$ with $\Re(z_1) < \Re(z_2)$ and $\Re(w_1) < \Re(w_2)$ such that the axes are of negative orientation. If $\mathbb{L}_{[z_1],[z_2]} \mid\mid \mathbb{L}_{[w_1],[w_2]}$, then $\Re(z_1) < \Re(w_1)$ and $\Re(z_2) < \Re(w_2)$.*

Proof. We note that the axes $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ passes through the point $(\frac{n}{2}, 0)$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ also passes through the point $(\frac{m}{2}, 0)$ with $m > n$. It follows that $\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) = \mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]})$ so that we can write

$$\frac{\Im(z_2)}{\Re(z_2) - \frac{n}{2}} = \frac{\Im(w_2)}{\Re(w_2) - \frac{m}{2}}$$

since $\mathbb{L}_{[z_1],[z_2]} \mid\mid \mathbb{L}_{[w_1],[w_2]}$. Since $m > n$, it follows that $\mathrm{Int}[\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})] \subset \mathrm{Int}[\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})]$. Since the axes are of negative orientation with $\mathbb{L}_{[z_1],[z_2]} \mid\mid \mathbb{L}_{[w_1],[w_2]}$ then it implies that $\Im(z_1) = -\Im(z_2) < -\Im(w_2) = \Im(w_1)$ so that we have

$$\frac{-\Im(z_1)}{\Re(z_2) - \frac{n}{2}} = \frac{-\Im(w_1)}{\Re(w_2) - \frac{m}{2}} \iff \frac{\Im(z_1)}{\Re(z_2) - \frac{n}{2}} = \frac{\Im(w_1)}{\Re(w_2) - \frac{m}{2}}.$$

Since $\Im(w_1) > \Im(z_1)$, it follows that $\Re(z_2) - \frac{n}{2} < \Re(w_2) - \frac{m}{2} \iff \Re(z_2) < \Re(w_2)$ for $m > n$. Let us join the point w_2 to the point z_2 by a straight line, then it is easy to see that the gradient of this line is given by

$$\frac{\Im(w_2) - \Im(z_2)}{\Re(w_2) - \Re(z_2)} < 0$$

since $\Im(z_2) < \Im(w_2)$. Similarly, let us join the point z_1 to the point z_2 by a straight line. We compute the gradient of this line as

$$\frac{\Im(z_1) - \Im(w_1)}{\Re(z_1) - \Re(w_1)} > 0.$$

Let us suppose that

$$\frac{\Im(z_1) - \Im(w_1)}{\Re(z_1) - \Re(w_1)} > 0$$

then $\Im(z_1) - \Im(w_1) < 0$ since $\mathbb{L}_{[z_1],[z_2]} \mid\mid \mathbb{L}_{[w_1],[w_2]}$ with $m > n$ so that $\Re(z_1) > \Re(w_1)$. It follows that

$$\Re(w_1) = \Re(\overline{w_1}) < \Re(z_1) = \Re(\overline{z_1})$$

with

$$|w_1 - (\frac{m}{2}, 0)| = |\overline{w_1} - (\frac{m}{2}, 0)| > |z_1 - (\frac{m}{2}, 0)| = |\overline{z_1} - (\frac{m}{2}, 0)|.$$

Since

$$|\Im(w_1)| = |\Im(\overline{w_1})| > |\Im(z_1)| = |\Im(\overline{z_1})|$$

and the points $[w_1], [\overline{w_1}]$ are opposite points on the embedding circle $\mathfrak{C}_m$ and similarly the points $[z_1], [\overline{z_1}]$ on $\mathfrak{C}_n$ with $m > n$, it follows that $\mathfrak{C}_m$ and $\mathfrak{C}_n$ cannot have a common point at the origin. This violates the Big Bang theorem. $\square$

The lemma under discussion establishes a relationship between two geometric axes, each contained within distinct critical configurations of points (cCoPs) associated with different parameters. These axes, which are embedded within certain mathematical structures, are described as having negative orientation and being parallel to each other. The primary goal of the proof is to demonstrate that if these axes are parallel, specific relationships must hold between the real parts of their defining points.

The proof begins by identifying that the axes pass through specific points on the real line, which are determined by their respective parameters. Since the axes are parallel, they must share the same gradient. This geometric condition is expressed in terms of the real and imaginary components of the points that define the axes. The proof leverages this condition to relate the positions of the points on the two axes.

A crucial aspect of the proof is the negative orientation of the axes. This means that the imaginary parts of the points exhibit a specific symmetry, leading to certain inequalities between the imaginary components. These inequalities then directly impact the real components, allowing the proof to establish a relationship between the real parts of the points on the two axes.

To reinforce the result, the proof introduces a geometric argument, analyzing the slopes of lines connecting corresponding points on the two axes. This analysis confirms the earlier findings and ensures that the relationship between the real components holds consistently. However, the proof also encounters a potential contradiction with a previously established geometric theorem, referred to as the "Big Bang theorem." This contradiction serves to reinforce the conclusion, showing that the initial configuration must satisfy the proposed inequalities.

In summary, the proof combines geometric reasoning with analytic conditions to demonstrate that parallelism and negative orientation impose strict relationships between the real parts of points on the axes. The conclusion is drawn through a careful balance of geometric intuition and algebraic manipulation, highlighting the delicate interplay between the real and imaginary components of the points involved.

Next, we prove an important fact concerning the relationship between an axis of a cCoP and other axes of cCoPs with higher generators. It basically purports that the slope of cCoPs with higher generators must be relatively steeper so long as these axis intersect. We launch formally the following fact in the lemma below.

Lemma 2.6. *Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ be non–empty cCoPs with $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ for $m > n$ with $\Re(z_1) < \Re(z_2)$ and $\Re(w_1) < \Re(w_2)$. If the axis $\mathbb{L}_{[z_1],[z_2]}$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ intersect at a point above the real axis, then*

$$\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) > \mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}).$$

Proof. Suppose the axes $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ for $m > n$ with $\Re(z_1) < \Re(z_2)$ and $\Re(w_1) < \Re(w_2)$ intersect at the point $v \in \mathbb{C}$. We note that the axes $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ passes through the point $(\frac{n}{2}, 0)$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ also passes through the point $(\frac{m}{2}, 0)$ so that we can compute the gradient of the axes. We obtain

$$\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) = \frac{\Im(v)}{\Re(v) - \frac{m}{2}}$$

and

$$\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) = \frac{\Im(v)}{\Re(v) - \frac{n}{2}}.$$

Let us suppose that $\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) \leq \mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]})$, then it follows that

$$\frac{\Im(v)}{\Re(v) - \frac{m}{2}} \leq \frac{\Im(v)}{\Re(v) - \frac{n}{2}}.$$

Since the axes intersect at a point above the real axis, it must be that $\Im(v) > 0$ so that we obtain

$$\frac{1}{\Re(v) - \frac{m}{2}} \leq \frac{1}{\Re(v) - \frac{n}{2}} \iff \Re(v) - \frac{n}{2} \leq \Re(v) - \frac{m}{2} \iff m < n$$

which violates the inequality $m > n$. $\square$

We obtain an analogous result in the case the axes intersect below the real axis.

Lemma 2.7. *Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}^o(n, \mathbb{C}_\mathbb{M})$ and $\mathcal{C}^o(m, \mathbb{C}_\mathbb{M})$ be non–empty cCoPs with $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_\mathbb{M})$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_\mathbb{M})$ for $m > n$ with $\Re(z_1) < \Re(z_2)$ and $\Re(w_1) < \Re(w_2)$. If the axis $\mathbb{L}_{[z_1],[z_2]}$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ intersect at a point below the real axis, then*

$$\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) < \mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}).$$

Proof. Suppose the axes $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_\mathbb{M})$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_\mathbb{M})$ for $m > n$ with $\Re(z_1) < \Re(z_2)$ and $\Re(w_1) < \Re(w_2)$ intersect at the point $v \in \mathbb{C}$. We note that the axes $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_\mathbb{M})$ passes through the point $(\frac{n}{2}, 0)$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_\mathbb{M})$ also passes through the point $(\frac{m}{2}, 0)$ so that we can compute the gradient of the axes. We obtain

$$\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) = \frac{\Im(v)}{\Re(v) - \frac{m}{2}}$$

and

$$\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) = \frac{\Im(v)}{\Re(v) - \frac{n}{2}}.$$

Let us suppose that $\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) \geq \mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]})$, then it follows that

$$\frac{\Im(v)}{\Re(v) - \frac{m}{2}} \geq \frac{\Im(v)}{\Re(v) - \frac{n}{2}}.$$

Since the axes intersect at a point below the real axis, it must be that $\Im(v) < 0$ so that we obtain

$$\frac{1}{\Re(v) - \frac{m}{2}} \leq \frac{1}{\Re(v) - \frac{n}{2}} \iff \Re(v) - \frac{n}{2} \leq \Re(v) - \frac{m}{2} \iff m < n$$

which violates the inequality $m > n$. $\square$

We obtain a certain characterization of the gradient of axes of two interacting cCoPs. This is an immediate consequence of Lemma 2.6. It will also serve in many ways as a guiding principle for our further investigations.

Theorem 2.8. *Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}^o(n, \mathbb{C}_\mathbb{M})$ and $\mathcal{C}^o(m, \mathbb{C}_\mathbb{M})$ be non–empty cCoPs with $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_\mathbb{M})$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_\mathbb{M})$ for $m > n$ with $\Re(z_1) < \Re(z_2)$ and $\Re(w_1) < \Re(w_2)$. If the axis $\mathbb{L}_{[z_1],[z_2]}$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ intersect at a point above the real axis with $\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) < 0$, then*

$$\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) < 0.$$

Proof. Let us assume to the contrary that

$$\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) \geq 0$$

then $\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) > 0$, since $\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) \neq 0$. The axis $\mathbb{L}_{[z_1],[z_2]}$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ intersect at a point above the real axis so that

$$0 > \mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) > \mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) > 0$$

by virtue of Lemma 2.6, which is absurd. ◻

Theorem 2.9. *Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ be non–empty cCoPs with $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ for $m > n$ with $\Re(z_1) < \Re(z_2)$ and $\Re(w_1) < \Re(w_2)$. If the axis $\mathbb{L}_{[z_1],[z_2]}$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ intersect at a point below the real axis with $\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) > 0$, then*

$$\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) > 0.$$

Proof. Let us assume to the contrary that

$$\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) \leq 0$$

then $\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) < 0$, since $\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) \neq 0$. The axis $\mathbb{L}_{[z_1],[z_2]}$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ intersect at a point below the real axis so that

$$0 < \mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) < \mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) < 0$$

by virtue of Lemma 2.7, which is absurd. ◻

We obtain variants of Theorem 2.8 and Theorem 2.9 in the sequel.

Theorem 2.10. *Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ be non–empty cCoPs with $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ for $m > n$ with $\Re(z_1) < \Re(z_2)$ and $\Re(w_1) < \Re(w_2)$. If the axis $\mathbb{L}_{[z_1],[z_2]}$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ intersect at a point above the real axis with $\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) > 0$, then*

$$\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) > 0.$$

Proof. The axis $\mathbb{L}_{[z_1],[z_2]}$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ intersect at a point above the real axis so that

$$\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) > \mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) > 0$$

by virtue of Lemma 2.6, and the claim follows. ◻

Theorem 2.11. *Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ be non–empty cCoPs with $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ for $m > n$ with $\Re(z_1) < \Re(z_2)$ and $\Re(w_1) < \Re(w_2)$. If the axis $\mathbb{L}_{[z_1],[z_2]}$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ intersect at a point below the real axis with $\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) < 0$, then*

$$\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) < 0.$$

Proof. The axis $\mathbb{L}_{[z_1],[z_2]}$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ intersect at a point below the real axis so that

$$\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) < \mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) < 0$$

by virtue of Lemma 2.7, and the claim follows. ◻

The theorems in this work address the behaviour of two axes of cCoPs that intersect at points in the complex plane, with specific attention to their gradients. These axes belong to certain configurations of points (cCoPs) and are described by their real and imaginary components. The focus is on understanding how the gradients (slopes) of these axes relate to one another under different conditions of intersection, whether above or below the real axis.
The gradients (slopes) of the axes are influenced by where the intersection occurs (above or below the real axis). The proofs rely heavily on the structure of the axes and their geometric relationships, particularly through the use of lemmas that describe how gradients behave at intersections. The arguments draw on both contradiction and direct comparison of gradients to establish the necessary conditions for consistency in the behaviour of the axes. These theorems form a coherent framework for understanding how parallelism, intersection points, and gradients interact in these geometric configurations, offering a clear and structured view of their behaviour in different parts of the complex plane.

It is worthwhile noting that we have only confirmed the natural ordering principle of the real parts of the upper axes points of cCoPs in the case the corresponding axes of distinct cCoPs are parallel. We would like this behaviour to be propagated among the remaining configuration of the axes of cCoPs that we have not yet exhaust. It is possible that certain imagined configuration may not hold in this geometry. In the following sequel, we will examine this naturally exhibiting principle in the cases where any two axes of distinct non-empty cCoPs are skewed. We launch the following result.

Lemma 2.12. *Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ be non–empty cCoPs with $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ for $m > n$ with $\Re(z_1) < \Re(z_2)$ and $\Re(w_1) < \Re(w_2)$. If the axis $\mathbb{L}_{[z_1],[z_2]}$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ are skewed with $\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) > \mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) > 0$ then $\Re(z_1) < \Re(w_1)$ and $\Re(z_2) < \Re(w_2)$.*

Proof. Let axis $\mathbb{L}_{[z_1],[z_2]}$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ be skewed with $\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) > \mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) > 0$. Let us join z_2 to w_2 by a straight line. Then by the embedding $\mathrm{Int}[\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})] \subset \mathrm{Int}[\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})]$ with the axes $\mathbb{L}_{[z_1],[z_2]}$ and $\mathbb{L}_{[w_1],[w_2]}$ passing through the point $(\frac{n}{2}, 0)$ and $(\frac{m}{2}, 0)$, respectively with $m > n$, it follows that $\Im(w_2) > \Im(z_2)$. Let us suppose that the gradient of this line

$$\frac{\Im(w_2) - \Im(z_2)}{\Re(w_2) - \Re(z_2)} < 0.$$

Then it follows that $\Re(w_2) < \Re(z_2)$ so that the axes $\mathbb{L}_{[z_1],[z_2]}$ and $\mathbb{L}_{[w_1],[w_2]}$ must intersect at a point since $\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) > \mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) > 0$ and $\mathbb{L}_{[w_1],[w_2]}$ passes through the point $(\frac{m}{2}, 0)$ with $m > n$, contradicting the requirement that the axes $\mathbb{L}_{[z_1],[z_2]}$ and $\mathbb{L}_{[w_1],[w_2]}$ are skewed . Thus we must have $\Re(w_2) > \Re(z_2)$. Similarly let us join the point z_1 to the point w_1 by a straight line. Then by the embedding $\mathrm{Int}[\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})] \subset \mathrm{Int}[\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})]$ with the axes $\mathbb{L}_{[z_1],[z_2]}$ and $\mathbb{L}_{[w_1],[w_2]}$ passing through the point $(\frac{n}{2}, 0)$ and $(\frac{m}{2}, 0)$, respectively with $m > n$ and $\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) > \mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) > 0$, it follows that $\Im(w_1) < \Im(z_1) < 0$. Let us suppose that the gradient of this line

$$\frac{\Im(w_1) - \Im(z_1)}{\Re(w_1) - \Re(z_1)} > 0$$

then it implies that $\Re(z_1) > \Re(w_1)$. It follows that

$$\Re(w_1) = \Re(\overline{w_1}) < \Re(z_1) = \Re(\overline{z_1})$$

with

$$|w_1 - (\frac{m}{2}, 0)| = |\overline{w_1} - (\frac{m}{2}, 0)| > |z_1 - (\frac{m}{2}, 0)| = |\overline{z_1} - (\frac{m}{2}, 0)|.$$

Since

$$|\Im(w_1)| = |\Im(\overline{w_1})| > |\Im(z_1)| = |\Im(\overline{z_1})|$$

and the points $[w_1], [\overline{w_1}]$ are opposite points on the embedding circle $\mathfrak{C}_m$ and similarly the points $[z_1], [\overline{z_1}]$ on $\mathfrak{C}_n$ with $m > n$, it follows that $\mathfrak{C}_m$ and $\mathfrak{C}_n$ cannot have a common point at the origin. This violates the Big Bang theorem. □

We obtain an analogous result in the case all the axis are of negative orientation and slopes down negatively.

Lemma 2.13. *Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ be non–empty cCoPs with $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ for $m > n$ with $\Re(z_1) < \Re(z_2)$ and $\Re(w_1) < \Re(w_2)$. If the axis $\mathbb{L}_{[z_1],[z_2]}$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ are skewed with $\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) < \mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) < 0$ then $\Re(z_1) < \Re(w_1)$ and $\Re(z_2) < \Re(w_2)$.*

Proof. Let the axis $\mathbb{L}_{[z_1],[z_2]}$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ be skewed with $0 > \mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) > \mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]})$. Let us join z_1 to w_1 by a straight line. Then by the embedding $\mathrm{Int}[\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})] \subset \mathrm{Int}[\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})]$ with the axes $\mathbb{L}_{[z_1],[z_2]}$ and $\mathbb{L}_{[w_1],[w_2]}$ passing through the point $(\frac{n}{2}, 0)$ and $(\frac{m}{2}, 0)$, respectively with $m > n$, it follows that $\Im(w_1) < \Im(z_1) < 0$. Let us suppose that the gradient of this line

$$\frac{\Im(w_1) - \Im(z_1)}{\Re(w_1) - \Re(z_1)} < 0$$

then it follows that $\Re(w_1) < \Re(z_1)$. It follows that

$$\Re(w_1) = \Re(\overline{w_1}) < \Re(z_1) = \Re(\overline{z_1})$$

with

$$|w_1 - (\frac{m}{2}, 0)| = |\overline{w_1} - (\frac{m}{2}, 0)| > |z_1 - (\frac{m}{2}, 0)| = |\overline{z_1} - (\frac{m}{2}, 0)|.$$

Since

$$|\Im(w_1)| = |\Im(\overline{w_1})| > |\Im(z_1)| = |\Im(\overline{z_1})|$$

and the points $[w_1], [\overline{w_1}]$ are opposite points on the embedding circle $\mathfrak{C}_m$ and similarly the points $[z_1], [\overline{z_1}]$ on $\mathfrak{C}_n$ with $m > n$, it follows that $\mathfrak{C}_m$ and $\mathfrak{C}_n$ cannot have a common point at the origin. This violates the Big Bang theorem. Similarly, let us join z_2 to w_2 by a straight line. Then by the embedding $\mathrm{Int}[\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})] \subset \mathrm{Int}[\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})]$ with the axes $\mathbb{L}_{[z_1],[z_2]}$ and $\mathbb{L}_{[w_1],[w_2]}$ passing through the point $(\frac{n}{2}, 0)$ and $(\frac{m}{2}, 0)$, respectively with $m > n$, it follows that $\Im(w_2) < \Im(z_2) < 0$. Let us suppose that the gradient of this line

$$\frac{\Im(w_2) - \Im(z_2)}{\Re(w_2) - \Re(z_2)} > 0.$$

Then it follows that $\Re(w_2) < \Re(z_2)$ so that the axes $\mathbb{L}_{[z_1],[z_2]}$ and $\mathbb{L}_{[w_1],[w_2]}$ must intersect at a point since $\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) < \mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) < 0$ and $\mathbb{L}_{[w_1],[w_2]}$ passes through the point $(\frac{m}{2}, 0)$ with $m > n$, contradicting the requirement that the axes $\mathbb{L}_{[z_1],[z_2]}$ and $\mathbb{L}_{[w_1],[w_2]}$ are skewed . Thus we must have $\Re(w_2) > \Re(z_2)$. □

We examine the remaining skew case of interacting axes of distinct cCoPs in the scenario where they have gradient of opposite signs.

Lemma 2.14. *Let* $\mathbb{M} \subseteq \mathbb{N}$ *and* $\mathcal{C}^o(n, \mathbb{C}_\mathbb{M})$ *and* $\mathcal{C}^o(m, \mathbb{C}_\mathbb{M})$ *be non–empty cCoPs with* $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_\mathbb{M})$ *and* $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_\mathbb{M})$ *for* $m > n$ *with* $\Re(z_1) < \Re(z_2)$ *and* $\Re(w_1) < \Re(w_2)$*. If the axis* $\mathbb{L}_{[z_1],[z_2]}$ *and the axis* $\mathbb{L}_{[w_1],[w_2]}$ *are skewed with* $\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) > 0$ *and* $\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) < 0$ *such that*

$$|\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]})| > |\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]})|$$

then $\Re(z_1) < \Re(w_1)$ *and* $\Re(z_2) < \Re(w_2)$*.*

Proof. Under the requirement $|\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]})| > |\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]})|$ with the embedding

$$\mathrm{Int}[\mathcal{C}^o(n, \mathbb{C}_\mathbb{M})] \subset \mathrm{Int}[\mathcal{C}^o(m, \mathbb{C}_\mathbb{M})]$$

it implies that $\Im(z_1) > \Im(w_1)$ and $\Im(z_2) < \Im(w_2)$. Let us join the point z_1 to the point w_1 by a straight line and suppose for the gradient of this line

$$\frac{\Im(z_1) - \Im(w)}{\Re(z_1) - \Re(w_1)} > 0.$$

then it follows that $\Re(w_1) < \Re(z_1)$. It follows that

$$\Re(w_1) = \Re(\overline{w_1}) < \Re(z_1) = \Re(\overline{z_1})$$

with

$$|w_1 - (\frac{m}{2}, 0)| = |\overline{w_1} - (\frac{m}{2}, 0)| > |z_1 - (\frac{m}{2}, 0)| = |\overline{z_1} - (\frac{m}{2}, 0)|.$$

Since

$$|\Im(w_1)| = |\Im(\overline{w_1})| > |\Im(z_1)| = |\Im(\overline{z_1})|$$

and the points $[w_1], [\overline{w_1}]$ are opposite points on the embedding circle $\mathfrak{C}_m$ and similarly the points $[z_1], [\overline{z_1}]$ on $\mathfrak{C}_n$ with $m > n$, it follows that $\mathfrak{C}_m$ and $\mathfrak{C}_n$ cannot have a common point at the origin. This violates the Big Bang theorem. Similarly, let us join z_2 to w_2 by a straight line and suppose of the gradient of this line

$$\frac{\Im(z_2) - \Im(w_2)}{\Re(z_2) - \Re(w_2)} < 0.$$

Then it implies that $\Re(z_2) > \Re(w_2)$ since $\Im(z_2) - \Im(w_2) < 0$. Since $\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) > 0$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ must pass through the point $(\frac{m}{2}, 0)$ with $m > n$, it follows that both axes $\mathbb{L}_{[z_1],[z_2]}$ and $\mathbb{L}_{[w_1],[w_2]}$ must intersect at a point. This violates the requirement that the axes are skewed. □

Up to this point, we have nearly exhausted the investigation of the inherent ordering behavior of the real parts of the axis points for interacting cCoPs with distinct generators, specifically in the cases where the axes are either parallel or skewed. However, the case of interacting axes that intersect remains to be fully explored. It is important to note that attempting to replicate the same arguments in the scenario where the axes of distinct cCoPs intersect may lead to complications or potential deadlocks. We now turn our attention to examining the corresponding converses of Lemma 2.6 and Lemma 2.7.

Lemma 2.15. *Let* $\mathbb{M} \subseteq \mathbb{N}$ *and* $\mathcal{C}^o(n, \mathbb{C}_\mathbb{M})$ *and* $\mathcal{C}^o(m, \mathbb{C}_\mathbb{M})$ *be non–empty cCoPs with* $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_\mathbb{M})$ *and* $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_\mathbb{M})$ *for* $m > n$ *with* $\Re(z_1) < \Re(z_2)$ *and* $\Re(w_1) < \Re(w_2)$*. If* $\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) < \mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]})$*, then the axis* $\mathbb{L}_{[z_1],[z_2]}$ *and the axis* $\mathbb{L}_{[w_1],[w_2]}$ *cannot intersect at a point above the real axis.*

Proof. Suppose the axes $\mathbb{L}_{[z_1],[z_2]}$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ intersect at a point above the real axis with $\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) < \mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]})$. Now let $v \in \mathbb{C}$ be their point of intersection, then $\Im(v) > 0$ and we obtain

$$\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) = \frac{\Im(v)}{\Re(v) - \frac{m}{2}}$$

and

$$\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) = \frac{\Im(v)}{\Re(v) - \frac{n}{2}}.$$

It follows that

$$\frac{\Im(v)}{\Re(v) - \frac{m}{2}} < \frac{\Im(v)}{\Re(v) - \frac{n}{2}} \iff \Re(v) - \frac{n}{2} < \Re(v) - \frac{m}{2} \iff m < n$$

which violates the inequality $m > n$. □

Lemma 2.16. *Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ be non–empty cCoPs with $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ for $m > n$ with $\Re(z_1) < \Re(z_2)$ and $\Re(w_1) < \Re(w_2)$. If $\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) > \mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]})$, then the axis $\mathbb{L}_{[z_1],[z_2]}$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ cannot intersect at a point below the real axis.*

Proof. Suppose the axes $\mathbb{L}_{[z_1],[z_2]}$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ intersect at a point below the real axis with $\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) > \mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]})$. Now let $v \in \mathbb{C}$ be their point of intersection, then $\Im(v) < 0$ and we obtain

$$\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) = \frac{\Im(v)}{\Re(v) - \frac{m}{2}}$$

and

$$\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) = \frac{\Im(v)}{\Re(v) - \frac{n}{2}}.$$

It follows that

$$\frac{\Im(v)}{\Re(v) - \frac{m}{2}} > \frac{\Im(v)}{\Re(v) - \frac{n}{2}} \iff \Re(v) - \frac{n}{2} < \Re(v) - \frac{m}{2} \iff m < n$$

which violates the inequality $m > n$. □

It is important to note that by combining Lemma 2.6 with its converse, Lemma 2.16, one obtains an equivalent statement. A similar equivalence holds when Lemma 2.7 is paired with Lemma 2.15. These equivalences, in their own right, could serve as benchmarks for proving or disproving such configurations within the geometry. However, the arguments and methods employed in this paper do not sufficiently address cases where arbitrary axes of distinct cCoPs intersect.

In light of the preceding lemmas and their respective converses, we observe a natural synergy between the geometric configurations described. The interplay of gradients and axes of distinct cCoPs reveals important boundary conditions for the axes interaction below the real axis. While the conditions for non-intersection are clearly outlined, a more nuanced exploration of cases involving arbitrary intersections of axes will be reserved for future work. For the present study, we restrict our analysis to configurations where the axes exhibit certain symmetry and limiting behavior, particularly those governed by the Squeeze Principle. By doing so, we provide a focused framework for investigating the dynamics of cCoP axes

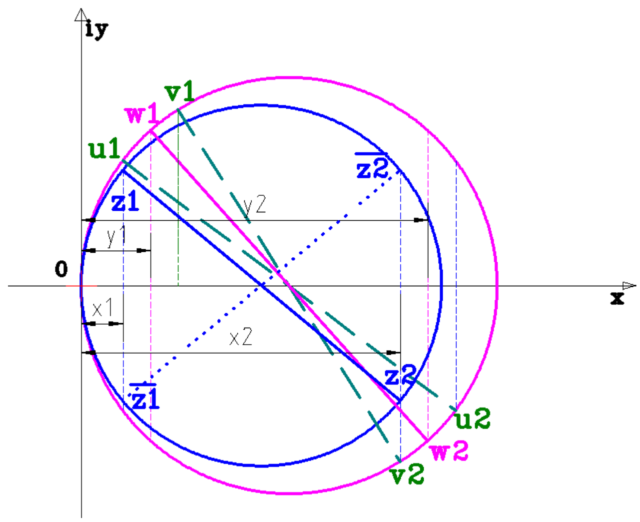

FIGURE 1. Limiting Axes for the Squeeze Principle

within specific boundary limits, which allows for more precise characterizations in subsequent theorems.

If a certain non–empty cCop $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$ with an axis $\mathbb{L}_{[z],[n-z]}$ is given, then for another non–empty cCoP $\mathcal{C}^o(n+t,\mathbb{C}_{\mathbb{M}})$ a lower and an upper limiting axis $\mathbb{L}_{[u],[n+t-u]}$ resp. $\mathbb{L}_{[v],[n+t-v]}$ for the validity of the Squeeze Principle can be determined analytically (see figure 1).

Lemma 2.17. *Let $\mathbb{M}\subseteq\mathbb{N}$ and $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(n+t,\mathbb{C}_{\mathbb{M}})$ with $t\geq 4$ be non–empty cCoPs with positive integers n,t of the same parity. If there is an axis $\mathbb{L}_{[z_1],[z_2]}$ of $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$, then there are a lower and an upper limiting axis $\mathbb{L}_{[u_1],[u_2]}$ resp. $\mathbb{L}_{[v_1],[v_2]}$ of the cCoP $\mathcal{C}^o(n+t,\mathbb{C}_{\mathbb{M}})$ with*

$$\Re(z_1)=\Re(u_1)\ \textit{and}\ \Re(v_2)=\Re(z_2)$$

such that

$$\mathrm{Grad}(\mathbb{L}_{[u_1],[u_2]})^2=\frac{4\Re(z_1)\,(n+t-\Re(z_1))}{(n+t-2\Re(z_1))^2}\ \textit{and}$$

$$\mathrm{Grad}(\mathbb{L}_{[v_1],[v_2]})^2=\frac{4\Re(z_2)\,(n+t-\Re(z_2))}{(n+t-\Re(z_2))^2}.$$

Proof. Appropriate of Definition 2.1 the gradient of the axis $\mathbb{L}_{[u_1],[u_2]}$ of the cCop $\mathcal{C}^o(n+t,\mathbb{C}_{\mathbb{M}})$ is defined as

$$\mathrm{Grad}(\mathbb{L}_{[u_1],[u_2]})=\frac{\Im(u_2)-\Im(u_1)}{\Re(u_2)-\Re(u_1)}.$$

Since $\Im(u_2)=-\Im(u_1)$, we have

$$\mathrm{Grad}(\mathbb{L}_{[u_1],[u_2]})=\frac{-2\Im(u_1)}{\Re(u_2)-\Re(u_1)}$$

and squared

$$\mathrm{Grad}(\mathbb{L}_{[u_1],[u_2]})^2 = \frac{4\Im(u_1)^2}{(\Re(u_2) - \Re(u_1))^2}.$$

In accordance with the *circle condition*, we get

$$\mathrm{Grad}(\mathbb{L}_{[u_1],[u_2]})^2 = \frac{4\Re(u_1)(n+t-\Re(u_1))}{(\Re(u_2) - \Re(u_1))^2}$$

and because $\Re(u_1) = \Re(z_1)$ and $\Re(u_2) - \Re(u_1) = n + t - 2\Re(u_1)$

$$\mathrm{Grad}(\mathbb{L}_{[u_1],[u_2]})^2 = \frac{4\Re(z_1)(n+t-\Re(z_1))}{(n+t-2\Re(z_1))^2}.$$

In an equivalent manner, we get for the axis $\mathbb{L}_{[v_1],[v_2]}$ of the same cCoP using the requirement $\Re(v_2) = \Re(z_2)$

$$\mathrm{Grad}(\mathbb{L}_{[v_1],[v_2]})^2 = \frac{4\Re(z_2)(n+t-\Re(z_2))}{(n+t-2\Re(z_2))^2}.$$

□

However if $\Re(z_2) < \frac{n+t}{2}$, then the gradient of the upper limiting axis changes its sign. In this case the verical diameter of the embedding circle $\mathfrak{C}_{n+t}$ plays the role of the upper limiting axis and its gradient goes to infinity.

Theorem 2.18. *Let the requirements of Lemma 2.17 be fulfilled. If for an axis $\mathbb{L}_{[w_1],[w_2]}$ of the cCoP $\mathcal{C}^o(n+t, \mathbb{C}_{\mathbb{M}})$ with $\Re(w_1) < \Re(w_2)$ holds*

$$\mathrm{Grad}(\mathbb{L}_{[u_1],[u_2]})^2 < \mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]})^2 < \mathrm{Grad}(\mathbb{L}_{[v_1],[v_2]})^2 \tag{2.1}$$

with the limiting axes like in Lemma 2.17, then and only then the requirements of the Squeeze Principle (see Theorem 1.1) $\Re(z_1) < \Re(w_1)$ and $\Re(z_2) < \Re(w_2)$ are fulfilled.

Proof. We set $x_1 := \Re(z_1), x_2 := \Re(z_2), y_1 := \Re(w_1), y_2 := \Re(w_2)$ and $m := n + t$. Using this in (2.1) and with Lemma 2.17 we get

$$\frac{4x_1(m-x_1)}{(m-2x_1)^2} < \frac{4y_1 \cdot y_2}{(y_2-y_1)^2} < \frac{4x_2(m-x_2)}{(m-2x_2)^2}. \tag{2.2}$$

For the divisor of the central term it holds

$$(y_2 - y_1)^2 = y_1^2 + y_2^2 - 2y_1y_2 = (y_1+y_2)^2 - 4y_1y_2 = m^2 - 4y_1y_2 > 0.$$

We consider at first the left inequality. By crosswise multiplication and dividing by 4 we get

$$x_1(m-x_1)(m^2 - 4y_1y_2) < y_1y_2(m-2x_1)^2$$
$$m^3x_1 - 4mx_1y_1y_2 - m^2x_1^2 + 4x_1^2y_1y_2 < m^2y_1y_2 - 4mx_1y_1y_2 + 4x_1^2y_1y_2.$$

We note on both sides $-4mx_1y_1y_2 + 4x_1^2y_1y_2$. Hence it remains

$$\begin{aligned} m^3x_1 - m^2x_1^2 &< m^2y_1y_2 \mid \div m^2 \\ x_1(m - x_1) &< y_1y_2 \\ mx_1 - x_1^2 &< my_2 - y_2^2 \\ y_2^2 - x_1^2 &< m(y_2 - x_1) \\ (y_2 + x_1)(y_2 - x_1) &< m(y_2 - x_1) \mid \div(y_2 - x_1) \\ y_2 + x_1 &< m \mid -y_2 \\ x_1 &< m - y_2 = y_1 \\ \Re(z_1) &< \Re(w_1). \end{aligned}$$

Now we consider the right inequality at first for the case $x_2 > \frac{m}{2}$. By crosswise multiplication and dividing by 4 we get now

$$\begin{aligned} y_1y_2(m - 2x_2)^2 &< (mx_2 - x_2^2)(m^2 - 4y_1y_2) \\ 4x_2^2y_1y_2 - 4mx_2y_1y_2 + m^2y_1y_2 &< m^3x_2 - 4mx_2y_1y_2 + 4x_2^2y_1y_2 - m^2x_2^2 \\ m^2y_1y_2 &< m^3x_2 - m^2x_2^2 \\ y_1y_2 &< mx_2 - x_2^2. \end{aligned}$$

And now with $y_2 = m - y_1$

$$\begin{aligned} y_1(m - y_1) = my_1 - y_1^2 &< mx_2 - x_2^2 \\ m(y_1 - x_2) &< y_1^2 - x_2^2 \text{ and since } x_2 > y_1 \\ m(x_2 - y_1) &> x_2^2 - y_1^2 = (x_2 - y_1)(x_2 + y_1) \\ m &> x_2 + y_1 \\ m - y_1 &> x_2 \\ y_2 &> x_2 \\ \Re(z_2) &< \Re(w_2). \end{aligned}$$

In the case $x_2 \le \frac{m}{2}$ we have the following situation. Because of the requirement $\Re(w_1) < \Re(w_2)$ we have $y_2 > \frac{m}{2}$ and finally

$$\Re(w_2) = y_2 > \frac{m}{2} \ge x_2 = \Re(z_2).$$

Since all derivation chains are reversible, the reversed direction is valid too. $\square$

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