Proof of Collatz Conjecture

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Abstract:

Any positive integer can be expressed as k*2^n, where k is an odd number and n is a natural number. Each operation of the Collatz conjecture can be represented by (3k+1)*2^n, regardless of whether it is an odd or even number. The distribution type of k belongs to deterministic random distribution. Let 2^t be a perfect square number that is just less than 3k, and the cumulative probability value of (3k+1) being a perfect square number after each operation in the Collatz conjecture is conservatively estimated as Σ1/2^t. By comparing with the harmonic function Σ1/n, it is proved that when the number of operations gradually increases, the cumulative probability function value Σ1/2^t of (3k+1) being a perfect square number is much larger than 1, and tends to infinity when the number of operations is infinitely large. This result shows that the occurrence of (3k+1) being a perfect square is inevitable, thus proving the Collatz conjecture.

Keywords:

Collatz conjecture, deterministic random distribution, cumulative probability, harmonic function, lower bound function

The Collatz conjecture, also known as the 3x+1 conjecture, refers to the fact that for a natural number x, if it is odd, it is multiplied by 3 and then added by 1. If it is even, it is divided by the even factor 2^n. After several operations, it always returns to 1. No matter how large the value in this process is, it will fall rapidly like a waterfall. Even if other numbers are not, after several transformations, they will inevitably reach a pure even number and fall into a cycle of 16-8-4-2-1. This conjecture was proposed by the German mathematician Lothar Collatz in 1937 and has not yet been thoroughly proven. [1] [2]

1.Deterministic Random Distribution and Cumulative Probability

Any positive integer can be expressed as k*2^n, where k is an odd number and n is a natural number. Every operation of the Collatz conjecture, whether odd or even, can be represented by (3k+1)*2^n. The proof of the Collatz conjecture is only to prove that (3k+1)*2^n is a perfect square number or a pure even number after a finite number of operations. Since 2^n is a perfect square, it suffices to prove that (3k+1) is a perfect square after a finite number of operations to prove the conjecture.

The random distribution can be divided into deterministic random distribution and non-deterministic random distribution. The distribution of prime numbers belongs to deterministic random distribution, while the distribution of dice rolls belongs to non-deterministic random distribution. For example, the number of prime numbers less than the integer 100 is the same for different people, which is deterministic; The results of 100 different people operating the dice will not be exactly the same, and there is uncertainty.

For deterministic random distributions, the total probability of a specific event occurring is determined by summing the probabilities of each individual event, which is known as the cumulative probability. For example, it is known that the probability of each integer being a prime number is 1/Ln(x), and the number of prime numbers less than the integer x is calculated as the sum of x 1/Ln(x), which is x/Ln(x).

The distribution of odd factor k in the Collatz conjecture belongs to a deterministic random distribution. After the determination of any positive integer, different people can operate on it, and the odd factors after each operation are completely the same, which is deterministic. Therefore, the total probability of the specific event that (3k+1) is a perfect square is the sum of the probabilities of each individual event, which is the cumulative probability.

2. Proof of the conjecture

Let 2^t be a perfect square number just less than 3k, then 2^t < (3k+1)≤2*2^t. Since k is an arbitrary odd number, the value of (3k+1) fluctuates between greater than 2^t and less than or equal to 2*2^t, and is randomly distributed in the area close to 2*2^t. Since (3k+1) does not exist in the area greater than 2^t and less than 3k, and (3k+1) is an even number, the probability that (3k+1) is equal to 2*2^t after each operation is conservatively estimated to be 1/(2*2^t-2^t) = 1/2^t. Function Σ1/2^t is the lower bound function of the number of times that (3k+1) is a perfect square number.

As the number of operations gradually increases, if we can prove that the cumulative probability value Σ1/2^t >> 1 after each operation that (3k+1) is a perfect square number, and it tends to infinity when the number of operations is infinite, it means that the event that (3k+1) is a perfect square number is inevitable, which proves the conjecture.

In fact, k is any odd number, and (3k+1) must be an even number after each operation. Even numbers exist in many forms, because n is an arbitrary positive integer in the even number $k*2^n$, and the number of operations to extract an even number is not equal. As a result, the k value at the next operation fluctuates, and there is no monotonous increase trend. Therefore, the probability value $1/2^t$ of (3k+1) being a perfect square number is always fluctuating up and down in the initial stage. When (3k+1) is a perfect square number, each item of the probability value is 1/2, because the odd factor k is 1 at this time. The probability value of continuing the operation (3k+1) to be a perfect square is $1/(2^2-2^1)=1/2$, and there is no trend of gradually decreasing with the increase in the number of operations. The value of the cumulative probability function $\Sigma 1/2^t$ that (3k+1) is a perfect square is monotonically increasing, but the rising slope of the function curve is sometimes high and sometimes low in the initial stage. When (3k+1) is a perfect square, the rising slope of the function curve is fixed and there is no trend of gradually decreasing.

Comparing the cumulative probability function $\Sigma 1/2^t$ with the harmonic function $\Sigma 1/n$, each term $1/n$ of the harmonic function $\Sigma 1/n$ has a trend of gradually decreasing value as the number of terms n increases. This causes the rising slope of the harmonic function $\Sigma 1/n$ curve to gradually decrease.

No matter how large any positive integer $k*2^n$ is, the odd factor k in it is a fixed value. Therefore, the t value of the perfect square number 2^t just less than 3k is a finite value. When the value of n in the harmonic function $\Sigma 1/n$ is greater than 2^t, the value of the corresponding function $1/2^t$ will gradually become greater than the value of the function $1/n$. Therefore, when the number of operations approaches infinity, the value of the cumulative probability function $\Sigma 1/2^t$ is greater than the value of the corresponding number of harmonic functions $\Sigma 1/n$. Because the value of the harmonic function $\Sigma 1/n$ tends to infinity when n tends to infinity, when the number of operations gradually increases, the cumulative probability function value $\Sigma 1/2^t >>1$, of (3k+1) being a perfect square number, and it also tends to infinity when the number of operations is infinite, and does not converge.

The actual situation is that when the cumulative probability function value $\Sigma 1/2^t$ does not exceed the value 1, the result of (3k+1) being a perfect square number will appear. For example, for the positive integer 27 with complex operations, the operation process forms a total of 41 odd factors, and the cumulative probability (stop probability) of (3k+1) being a perfect square number when the operation stops is only 0.29. Continue the operation, and the $\Sigma 1/2^t$ function curve will maintain a stable monotonically rising trend with a fixed slope.

3.Conclusion

The cumulative probability function value $\Sigma 1/2^t$ (lower bound function) represents the lower bound of the number of times (3k+1) is a perfect square number. When the number of operations gradually increases, it is much larger than the value 1. When the number of operations approaches infinity, the cumulative probability function value $\Sigma 1/2^t$ is infinite and does not converge. This shows that the event that (3k+1) is a perfect square number is inevitable, which proves the Collatz conjecture.

4. Conjecture

When (3k+1) is a perfect square number, the cumulative probability of stopping the operation $\Sigma 1/2^t < 1$.

5. References

[1] http://en.wikipedia.org/wiki/Collatz_conjecture
[2] Lagarias, J. C., The 3x+1 problem: an overview. The ultimate challenge: the 3x+1 problem, 3-29, Amer. Math. Soc., Providence, RI, 2010.