A proof of Riemann hypothesis

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2024-10-17

Abstract

The great German mathematician Bernhard Riemann has predicted that all the non-trivial zeros of the zeta function are on the vertical line which intersects the real line at $\frac{1}{2}$ (the critical line). This prediction was named on his honer as "Riemann hypothesis". In this paper we investigate the distribution of the non-trivial zeros of the zeta function and deduce a closed form of them theoretically. These results are then examined by comparing their prediction with the accepted values and sampling the critical strip.

1 Introduction

Riemann hypothesis has far reaching consequences in analytic number theory (Stopple, J. 2003), Computation (Wojciechowski, J. 2003) and physics (Schumayer & Hutchinson. 2011). In the second section of this paper we will present and prove a lemma and use it to prove the Riemann hypothesis which will lead to a closed form of the non-trivial zeros of the zeta function. In the third section, we will examine the result obtained in the previous section by producing the first 30 non-trivial zeros based on the deduced closed form and compare them with the previously calculated and accepted values and sample the critical strip for further confirmation. Finally we will state the ramifications of the remarkable agreement between the predicted and accepted values along with other considerations to conclude this paper.

2 Theoretical deduction

Lemma. *Let $\zeta(z)$ be any arbitrary complex function, and let $F_1(z), F_2(z), ..., F_n(z)$ be functions with the property $F_i(z) = F_i(1-z)$ for all $i \in \{1, ..., n\}$, also assume that there exist functions $g_1(z), g_2(z), ..., g_n(z)$ such that:*

$$\zeta(z) = g_1(z)F_1(z),\ \zeta(z) = g_2(z)F_2(z), ..., \zeta(z) = g_n(z)F_n(z) \tag{1}$$

then

$$\begin{aligned}\zeta(z) &= g_i(z)\frac{g(z) \pm 1}{g_i(z) \pm g_i(1-z)}\zeta(1-z) \\ &= g_1(z)...g_n(z)\frac{g(z) \pm 1}{g_1(z)...g_n(z) \pm g_1(1-z)...g_n(1-z)}\zeta(1-z)\end{aligned} \tag{2}$$

where $g(z)$ is a function satisfying the following condition:

$$\zeta(z) = g(z)\zeta(1-z) \tag{3}$$

Proof. we will prove by induction, assume that the lemma holds for $n = k$ and let $F_{k+1}(z)$ satisfies the stated conditions, then letting $F^*(z) = F_k(z)F_{k+1}(z)$, we will get:

$$\begin{aligned}\zeta(z) = g^*(z)F^*(z) &\Rightarrow \zeta(z) \pm \zeta(1-z) = g^*(z)F^*(z) \pm g^*(1-z)F^*(1-z) \\ &\Rightarrow g(z)\zeta(1-z) \pm \zeta(1-z) = g^*(z)F^*(z) \pm g^*(1-z)F^*(z) \\ &\Rightarrow [g(z) \pm 1]\zeta(1-z) = [g^*(z) \pm g^*(1-z)]F^*(z) \\ &\Rightarrow F^*(z) = \frac{g(z) \pm 1}{g^*(z) \pm g^*(1-z)}\zeta(1-z)\end{aligned} \tag{4}$$

and hence we get:

$$\zeta(z) = g^*(z)\frac{g(z) \pm 1}{g^*(z) \pm g^*(1-z)}\zeta(1-z) \tag{5}$$
$$= g_1(z)...g_{k+1}(z)\frac{g(z) \pm 1}{g_1(z)...g_{k+1}(z) \pm g_1(1-z)...g_{k+1}(1-z)}\zeta(1-z)$$

so if we take $k = 1$ and use induction the result follows [please note that the $g_1(z), ..., g_n(z)$ in (2) and (5) are different from the $g_1(z), ..., g_n(z)$ in (1) except for $g_i(z)$]. □

Theorem. *Let $\zeta(s)$ be the generalized zeta function such that:*

$$\mathbb{C}\backslash 1 \xrightarrow{\zeta(s)} \mathbb{C} \tag{6}$$

then the only zeros of this function are the trivial zeros on the negative even integers and the non-trivial zeros on the critical line (Conrey, J. B. 2003) which is the vertical line in the complex plane that intersects the real line at $\frac{1}{2}$ (Riemann hypothesis).

Proof. From the definition of the generalized zeta function, we have:

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{(1-s)}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) \Rightarrow \frac{\zeta(s)}{\zeta(1-s)} = \frac{\frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})}}{\frac{\pi^{\frac{1-s}{2}}}{\Gamma(\frac{1-s}{2})}} \tag{7}$$
$$\Rightarrow \zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}F_1(s)$$

however, with a well-known manipulation, we get:

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s) \tag{8}$$

and hence, taking $g_1(s)$ and $g(s)$ to be:

$$g_1(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \quad \text{and} \quad g(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s) \tag{9}$$

we get, from the above lemma, the following result:

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}\frac{\left[2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s) \pm 1\right]}{\frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})} \pm \frac{\pi^{\frac{1-s}{2}}}{\Gamma(\frac{1-s}{2})}}\zeta(1-s) \tag{10}$$

to illustrate the above lemma, we will deduce, from the gamma function properties (Askey & Ranjan, 2010), another form of the description with different functions $g_2(s)$ and $F_2(s)$:

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s) \tag{11}$$
$$= 2^s\pi^{s-1}\frac{\pi}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)}\Gamma(1-s)\zeta(1-s)$$
$$= 2^s\pi^{s-1}\frac{\pi}{\frac{\sqrt{\pi}\Gamma(s)}{2^{s-1}\Gamma\left(\frac{1+s}{2}\right)}\Gamma\left(1-\frac{s}{2}\right)}\Gamma(1-s)\zeta(1-s)$$
$$= 2^s\pi^{\frac{s}{2}}\frac{1}{\frac{\pi^{\frac{s}{2}-s+\frac{1}{2}}\Gamma(s)}{2^{s-1}\Gamma\left(\frac{1+s}{2}\right)}\Gamma\left(1-\frac{s}{2}\right)}\Gamma(1-s)\zeta(1-s)$$
$$= \frac{\frac{2^s\pi^{\frac{s}{2}}\Gamma(\frac{1+s}{2})}{\Gamma(s)}}{\frac{2^{1-s}\pi^{\frac{1-s}{2}}\Gamma(1-\frac{s}{2})}{\Gamma(1-s)}}\zeta(1-s) \Rightarrow \frac{\zeta(s)}{\zeta(1-s)} = \frac{\frac{2^s\pi^{\frac{s}{2}}\Gamma(\frac{1+s}{2})}{\Gamma(s)}}{\frac{2^{1-s}\pi^{\frac{1-s}{2}}\Gamma(1-\frac{s}{2})}{\Gamma(1-s)}}$$

so, according to the established lemma, there must exist a function $F_2(s)$ with the property $F_2(s) = F_2(1-s)$, otherwise, it should not be canceled when $\zeta(s)$ is divided by $\zeta(1-s)$, and hence we get:

$$\zeta(s) = \frac{2^s\pi^{\frac{s}{2}}\Gamma(\frac{1+s}{2})}{\Gamma(s)}F_2(s) \quad \text{where} \quad g_2(s) = \frac{2^s\pi^{\frac{s}{2}}\Gamma(\frac{1+s}{2})}{\Gamma(s)} \tag{12}$$

so taking $g(s)$ as before and using the lemma again, we get:

$$F_2(s) = \frac{\left[2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s) \pm 1\right]}{\frac{2^s\pi^{\frac{s}{2}}\Gamma(\frac{1+s}{2})}{\Gamma(s)} \pm \frac{2^{1-s}\pi^{\frac{1-s}{2}}\Gamma(1-\frac{s}{2})}{\Gamma(1-s)}}\zeta(1-s) \tag{13}$$

therefore, the complete description of the zeta function without any cancellation is given by:

$$\zeta(s) = \frac{2^s\pi^{\frac{s}{2}}\Gamma(\frac{1+s}{2})}{\Gamma(s)}\frac{\left[2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s) \pm 1\right]}{\frac{2^s\pi^{\frac{s}{2}}\Gamma(\frac{1+s}{2})}{\Gamma(s)} \pm \frac{2^{1-s}\pi^{\frac{1-s}{2}}\Gamma(1-\frac{s}{2})}{\Gamma(1-s)}}\zeta(1-s) \tag{14}$$

this final description of the zeta function [along with the description given in (10) or any generalized description satisfying the conditions of the above lemma] has all the missing information in the previous known description and it is complete, for if we assume that there are other missing functions $F_1(s), ..., F_n(s)$ that were omitted by cancellation then they must all have the property $F_1(s) = F_1(1-s), ..., F_n(s) = F_n(1-s)$ and hence we can take $F^*(s) = F_1(s) \times ... \times F_n(s)$ and we will get the same description. Now from the preceding argument, we have:

$$\begin{aligned}\zeta(s) &= \frac{2^s\pi^{\frac{s}{2}}\Gamma(\frac{1+s}{2})}{\Gamma(s)}\frac{\left[2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s) \pm 1\right]}{\frac{2^s\pi^{\frac{s}{2}}\Gamma(\frac{1+s}{2})}{\Gamma(s)} \pm \frac{2^{1-s}\pi^{\frac{1-s}{2}}\Gamma(1-\frac{s}{2})}{\Gamma(1-s)}}\zeta(1-s) \\ &= \left[\frac{2^s\pi^{\frac{s}{2}}\Gamma(\frac{1+s}{2})}{\Gamma(s)}\frac{2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s) + 1}{\frac{2^s\pi^{\frac{s}{2}}\Gamma(\frac{1+s}{2})}{\Gamma(s)} + \frac{2^{1-s}\pi^{\frac{1-s}{2}}\Gamma(1-\frac{s}{2})}{\Gamma(1-s)}}\right]^n \\ &\quad \left[\frac{2^{1-s}\pi^{\frac{1-s}{2}}\Gamma(1-\frac{s}{2})}{\Gamma(1-s)}\frac{2^{1-s}\pi^{-s}\sin\left(\frac{\pi(1-s)}{2}\right)\Gamma(s) - 1}{\frac{2^{1-s}\pi^{\frac{1-s}{2}}\Gamma(1-\frac{s}{2})}{\Gamma(1-s)} - \frac{2^s\pi^{\frac{s}{2}}\Gamma(\frac{1+s}{2})}{\Gamma(s)}}\right]^{n-1} \zeta(1-s)\end{aligned} \tag{15}$$

and we also have

$$\begin{aligned}\zeta(s) &= \frac{2^s\pi^{\frac{s}{2}}\Gamma(\frac{1+s}{2})}{\Gamma(s)}\frac{\left[2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s) \pm 1\right]}{\frac{2^s\pi^{\frac{s}{2}}\Gamma(\frac{1+s}{2})}{\Gamma(s)} \pm \frac{2^{1-s}\pi^{\frac{1-s}{2}}\Gamma(1-\frac{s}{2})}{\Gamma(1-s)}}\zeta(1-s) \\ &= \left[\frac{2^s\pi^{\frac{s}{2}}\Gamma(\frac{1+s}{2})}{\Gamma(s)}\frac{2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s) - 1}{\frac{2^s\pi^{\frac{s}{2}}\Gamma(\frac{1+s}{2})}{\Gamma(s)} - \frac{2^{1-s}\pi^{\frac{1-s}{2}}\Gamma(1-\frac{s}{2})}{\Gamma(1-s)}}\right]^n \\ &\quad \left[\frac{2^{1-s}\pi^{\frac{1-s}{2}}\Gamma(1-\frac{s}{2})}{\Gamma(1-s)}\frac{2^{1-s}\pi^{-s}\sin\left(\frac{\pi(1-s)}{2}\right)\Gamma(s) - 1}{\frac{2^{1-s}\pi^{\frac{1-s}{2}}\Gamma(1-\frac{s}{2})}{\Gamma(1-s)} - \frac{2^s\pi^{\frac{s}{2}}\Gamma(\frac{1+s}{2})}{\Gamma(s)}}\right]^{n-1} \zeta(1-s)\end{aligned} \tag{16}$$

and hence, by equating these two equations and canceling the common factors, we get:

$$\left[\frac{2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s) + 1}{\frac{2^s\pi^{\frac{s}{2}}\Gamma(\frac{1+s}{2})}{\Gamma(s)} + \frac{2^{1-s}\pi^{\frac{1-s}{2}}\Gamma(1-\frac{s}{2})}{\Gamma(1-s)}}\right]^n = \left[\frac{2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s) - 1}{\frac{2^s\pi^{\frac{s}{2}}\Gamma(\frac{1+s}{2})}{\Gamma(s)} - \frac{2^{1-s}\pi^{\frac{1-s}{2}}\Gamma(1-\frac{s}{2})}{\Gamma(1-s)}}\right]^n \tag{17}$$

according to the previously established results, equation (17) ought to hold for any power n and while the non-trivial zeros produced by the left-hand side of (17) leads to the indeterminate form $\frac{0}{0}$, the right-hand side which have a definite value should reveals the ambiguity of the left-hand side and vice versa. So if the absolute value of the denominator of the related unambiguous side is greater than 2 (which is true for all the solutions of the closed form describing the non-trivial zeros except $s = \frac{1}{2}$, as we will demonstrate below) then the indeterminate form will approach zero as n approaches infinity which entails that the zeta function assumes its non-trivial zeros at those values (there are other values in which (17) approaches zero as the power n increases to infinity, however, the non-trivial zeros will be closer to zero than any other values for any given power n). It is clear that the value of the power n is irrelevant when deducing the form describing those zeros. So from (14) and (17), the zeros of the zeta function are:

$$\frac{2^s \pi^{\frac{s}{2}} \Gamma(\frac{1+s}{2})}{\Gamma(s)} = 0 \Rightarrow s = -2n \quad \text{where} \quad n \in \mathbb{Z}^+ \tag{18}$$

and

$$\begin{aligned} 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \pm 1 = 0 \Rightarrow \sin^2\left(\frac{\pi s}{2}\right) &= \frac{\pi^{2-2s}}{2^{2s}\Gamma(1-s)^2} \\ \Rightarrow \Re\left[\sin^2\left(\frac{\pi s}{2}\right)\right] &= \Re\left[\frac{\pi^{2-2s}}{2^{2s}\Gamma(1-s)^2}\right] \\ \Rightarrow \Re(s) &= \frac{1}{2} \end{aligned} \tag{19}$$

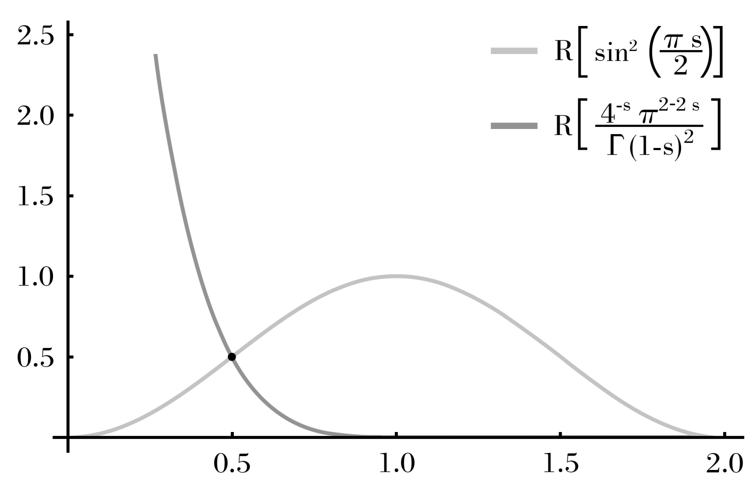
□

The trivial zeros in (18) occur at the negative even integer poles of the gamma function since at the negative odd integers we get the indeterminate form $\frac{\infty}{\infty}$. On the other hand, the non-trivial zeros occur only on the vertical line which intersects the real line at $\frac{1}{2}$ (the critical line, see the figure below) and their distribution is given by the following equation:

$$\sin^2\left(\frac{\pi s}{2}\right) = \frac{\pi^{2-2s}}{2^{2s}\Gamma(1-s)^2} \tag{20}$$

or if we take $g(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})} \frac{\Gamma(\frac{1-s}{2})}{\pi^{\frac{1-s}{2}}}$, we would get the following more simple and elegant form of the distribution of the non-trivial zeros:

$$\frac{\Gamma\left(\frac{s}{2}\right)^2}{\pi^s} = \frac{\Gamma\left(\frac{1-s}{2}\right)^2}{\pi^{1-s}} \tag{21}$$



it is worth to mention again that (20) or (21) has a real solution at $s = \frac{1}{2}$, however, while at this point we get the indeterminate form $\frac{0}{0}$, it is not a zero of the zeta function since the unambiguous side will tend to infinity as the power n goes to infinity (its denominator is less than 2). Also, the previously calculated non-trivial zeros (Odlyzko, A. 2024) are described by the formula $g(s)+1 = 0$ while the non-trivial zeros associated with the formula $g(s)-1 = 0$ are not captured by the assumptions on which those zeros are derived and hence they constitute a new set of non-trivial zeros on the critical line which were never calculated before.

3 Examination

As stated before the assumptions, on which the accepted values of the non-trivial zeros (Odlyzko, A. 2024) were calculated, capture only the non-trivial zeros described by the closed form $g(s) + 1 = 0$. To examine that, we produced the 30 non-trivial zeros based on this form and compared them with the corresponding accepted values (Table 1). As it is obvious from this table the absolute value of the divergence of each corresponding pair is less than 1 which confirms the theoretically deduced relation. The decimal divergence between them strongly support previous result (Mei, X. 2020) questioning the validity of the assumptions used to calculate the accepted values and claiming that $\zeta_1(a, b) \neq 0$ and $\zeta_2(a, b) \neq 0$. The plausible justification of this discrepancy is that while the proposed assumptions [$\zeta_1(a, b) = 0$ and $\zeta_2(a, b) = 0$] to derive the accepted values are not true, they are very close to zero such that they lead to a good approximations to the non-trivial zeros of zeta function.

Table 1: The first 30 predicted non-trivial zeros of zeta function based on the form $g(s) + 1 = 0$ and their corresponding accepted values along with their divergences.

Predicted value	**Accepted value**	**Divergence**	**Predicted value**	**Accepted value**	**Divergence**
14.5179	14.1347	0.3832	67.1327	67.0798	0.0529
20.6540	21.0220	-0.3680	69.7638	69.5464	0.2174
25.4915	25.0109	0.4806	72.3542	72.0672	0.2870
29.7385	30.4249	-0.6864	74.9071	75.7047	-0.7976
33.6238	32.9351	0.6887	77.4254	77.1448	0.2806
37.2567	37.5862	-0.3295	79.9116	79.3374	0.5742
40.7000	40.9187	-0.2187	82.3676	82.9104	-0.5428
43.9935	43.3271	0.6664	84.7955	84.7355	0.0600
47.1647	48.0052	-0.8405	87.1970	87.4253	-0.2283
50.2333	49.7738	0.4595	91.9266	92.4919	-0.5653
53.2141	52.9703	0.2438	94.2575	94.6513	-0.3938
56.1186	56.4462	-0.3276	96.5672	95.8706	0.6966
58.9560	59.3470	-0.3910	98.8569	98.8312	0.0257
61.7335	60.8318	0.9017	101.1275	101.3179	-0.1904
64.4574	65.1125	-0.6551	103.3799	103.7255	-0.3456

As a strong further computational support of the deduced form, we sampled the complex numbers in the critical strip throughout the interval $[0, 1]$ by taking 20000 complex numbers and their complex conjugates in 1000 equally spaced vertical lines in that interval and applied the derived form on them which confirms that all the predicted non-trivial zeros (based on the deduced closed form) are on the critical line and they are all complex conjugates.

4 Conclusion

In this paper we have established theoretically the closed form of the distribution of the non-trivial zeros of zeta function and it turns out that the accepted values of these zeros are reproduced almost exactly by a branch of this form represented by the equation $g(s) + 1 = 0$, and there are a new set of non-trivial zeros described by the equation $g(s)-1 = 0$ that is completely missed by the assumptions used to derive the accepted values. Also this result reconcile the discrepancy between the result obtained by the compassion method of infinite series (Mei, X.

2020) which predicts that $\zeta_1(a,b) \neq 0$ and $\zeta_2(a,b) \neq 0$ and the non-trivial zeros obtained by the negation of the latter assumptions since the deduced closed form predicts that the currently derived non-trivial zeros (e.g. those derived by Andrew Odlyzko) are biased by about less than 1 from the exact values.

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