THE G-DRAZIN INVERTIBILITY OF SUM WITH ORTHOGONAL CONDITIONS

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ABSTRACT. In this article, we present novel additive properties for the g-Drazin inverse within the context of Banach algebras. Subsequently, we provide representations for the g-Drazin inverse of block operator matrices defined over Banach spaces. These findings build upon and extend various well-known results, such as those by Bu, Feng, and Bai (J. Comput. Appl. Math., 218(2012), 10226–10237) and Dopazo and Martnez-Serrano (Linear Algebra Appl., 432(2010), 1896–1904).

1. INTRODUCTION

Let $\mathcal{A}$ be a complex Banach algebra with an identity 1. An element $a \in \mathcal{A}$ has g-Drazin inverse (i.e., generalized Drazin inverse) if there exists $b \in \mathcal{A}$ such that $b = bab, ab = ba, a - a^2b \in \mathcal{A}^{qnil}$. Here, $\mathcal{A}^{qnil} = \{a \in \mathcal{A} \mid 1 + ax \in \mathcal{A}^{-1}$ whenever $ax = xa\}$. Such b is unique, if it exists, and denote it by a^d. The Drazin inverse a^D of a is defined by replacing the preceding $\mathcal{A}^{qnil}$ into the set of all nilpotents in $\mathcal{A}$. The Drazin and g-Drazin inverses are valuable tools in the realms of matrix and operator theory. They have found applications across various disciplines, including ordinary differential equations, statistics, probability, Markov chains, among others.

The Drazin and g-Drazin invertiblity of the sum in a Banach algebra was is extensively studied by many authors from different views, e.g., [1, 2, 3, 4, 6] and [11]. The representation of $(P+Q)^D$ was obtained when $PQP = 0$ and $PQ^2 = 0$ for two complex matrices P and Q (see [10, Theorem 2.1]). After that, this conditions were extended to $P^3Q = 0, QPQ = 0$ and $QP^2Q = 0$ for complex matrices P and Q by Bu, Feng and Bai(see [3, Theorem 3.1]). In [**?**, Theorem 2.2], the representation of the g-Drazin inverse $(P+Q)^d$ of the sum of P, Q was obtained under the assumption $PQ = 0$ for bounded linear operators P and Q. The g-Drazin invertibility of $a + b$ was also concerned

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with in the case of $aba = 0$ and $ab^2 = 0$ for two elements a and b in a Banach algebra (see [1, Theorem 2.4]). The motivation of this paper is to extend the preceding conditions to the general setting.

In Section 2, we derive new additive properties for the sum of g-Drazin invertible elements in a Banach algebra. These results provide a generalization of Theorem 2.2 from [10, Theorem 2.2].

Let X be a complex Banach space and $\mathcal{B}(X)$ denote the Banach algebra of all bounded linear operators on X. Let

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \ (*)$$

be a block operator matrix, where $A \in \mathcal{B}(X), B \in \mathcal{B}(Y,X), C \in \mathcal{B}(X,Y)$ and $D \in \mathcal{B}(Y)$ are g-Drazin invertible and X, Y are complex Banach spaces. In Section 3, we present new expressions for the g-Drazin inverse of matrix M, which extend Theorem 4.1 and Theorem 4.2 from [3, Theorem 4.1 and Theorem 4.2] as well as Theorem 2.2 from [5, Theorem 2.2].

Throughout the paper, all Banach algebras are complex with an identity. We use $\mathcal{A}^d$ to denote the set of all g-Drazin invertible elements in $\mathcal{A}$. Let $a^\pi = 1 - aa^d$ for any $a \in \mathcal{A}^d$. $\mathbb{N}$ stands for the set of all natural numbers.

2. ADDITIVE RESULTS

The purpose of this section is to establish the new formulas for the g-Drazin of the sum in a Banach algebra. We start by extending [3, Theorem 3.1] as follows.

Lemma 2.1. *Let $a, b, ab \in \mathcal{A}^d$. If $a^2b = 0, b^2a = 0$, then $a + b \in \mathcal{A}^d$. In this case,*

$$\begin{array}{rcl} (a+b)^d & = & a^d + b^d + a(ba)^d + b(ab)^d + a(b^d)^2 + b(a^d)^2 \\ & + & (aba + bab)z(a^2 + b^2), \\ \alpha & = & ab + ba, \beta = a^2 + b^2, \\ \alpha^d & = & (ab)^d + (ba)^d, \beta^d = (a^d)^2 + (b^d)^2, \\ z & = & \sum\limits_{i=0}^{\infty} (\alpha^d)^{i+3}\beta^i\beta^\pi + \sum\limits_{i=0}^{\infty} \alpha^i\alpha^\pi(\beta^d)^{i+3} \\ & - & (\alpha^d)^2\beta^d - \alpha^d(\beta^d)^2. \end{array}$$

Proof. Let $\alpha = ab + ba, \beta = a^2 + b^2$. Since $(ab)(ba) = (ba)(ab) = 0$, we have $\alpha^d = (ab)^d + (ba)^d$. Likewise, $\beta^d = (a^d)^2 + (b^d)^2$. Set

$$M = \begin{pmatrix} ab + ba & 1 \\ 0 & a^2 + b^2 \end{pmatrix}.$$

Then

$$M^d = \begin{pmatrix} \alpha^d & z_1 \\ 0 & \beta^d \end{pmatrix},$$

where

$$z_1 = \sum_{i=0}^{\infty} (\alpha^d)^{i+2} \beta^i \beta^{\pi} + \sum_{i=0}^{\infty} \alpha^i \alpha^{\pi} (\beta^d)^{i+2} - \alpha^d \beta^d.$$

Let $z = \alpha^d z_1 + z_1 \beta^d$. Then

$$(M^d)^2 = \begin{pmatrix} (\alpha^d)^2 & z \\ 0 & (\beta^d)^2 \end{pmatrix}.$$

Clearly, $M = \left(\begin{pmatrix} 1 \\ a^2 + b^2 \end{pmatrix} (ab + ba, 1) \right)^2$. By Cline's formula (see [7, Theorem 2.1]), we have $(a+b)^2 = (ab+ba, 1) \begin{pmatrix} 1 \\ a^2 + b^2 \end{pmatrix}$ has g-Drazin inverse. In light of [9, Corollary 2.2], $a+b$ has g-Drazin inverse. Furthermore, we have

$$\begin{array}{l} (a+b)^d \\ = (a+b)[(a+b)^2]^d \\ = (a+b)(ab+ba, 1)(M^d)^2 \begin{pmatrix} 1 \\ a^2 + b^2 \end{pmatrix} \\ = (aba + bab, a+b) \begin{pmatrix} (\alpha^d)^2 & z \\ 0 & (\beta^d)^2 \end{pmatrix} \begin{pmatrix} 1 \\ a^2 + b^2 \end{pmatrix} \\ = (aba + bab)(\alpha^d)^2 + (aba + bab) z (a^2 + b^2) + (a+b)(\beta^d)^2)(a^2 + b^2) \\ = a^d + b^d + a(ba)^d + b(ab)^d + a(b^d)^2 + b(a^d)^2 + (aba + bab) z (a^2 + b^2). \end{array}$$

Moreover, we have

$$z = \sum_{i=0}^{\infty} (\alpha^d)^{i+3} \beta^i \beta^{\pi} + \sum_{i=0}^{\infty} \alpha^i \alpha^{\pi} (\beta^d)^{i+3} - (\alpha^d)^2 \beta^d - \alpha^d (\beta^d)^2.$$

as asserted. □

Lemma 2.2. *Let $a, b \in \mathcal{A}^d$. If $a^2 b = 0, b^2 = 0$ and $(ab)^2 = 0$, then $a + b \in \mathcal{A}^d$. In this case,*

$$(a+b)^d = a^d + b(a^d)^2 + ab(a^d)^3 + bab(a^d)^4.$$

Proof. In view of Lemma 2.3, we have

$$\begin{array}{l}(a+b)^d = (aba+bab)z(a^2+b^2)+a^d+b(a^d)^2,\\ \alpha = ab+ba, \beta = a^2,\\ \alpha^d = 0, \beta^d = (a^d)^2,\\ z = \sum\limits_{i=0}^{\infty}\alpha^i(\beta^d)^{i+3} = (\beta^d)^3+\alpha(\beta^d)^4+\alpha^2(\beta^d)^5.\end{array}$$

Hence,

$$\begin{array}{rcl} z &=& (a^d)^6+\alpha(a^d)^8+\alpha^2(a^d)^{10}\\ &=& (a^d)^6+ab(a^d)^8+b(a^d)^7+bab(a^d)^9.\end{array}$$

Therefore

$$\begin{array}{rl} & (a+b)^d\\ = & (aba+bab)[(a^d)^4+ab(a^d)^6+b(a^d)^5+bab(a^d)^7]+a^d+b(a^d)^2\\ = & a^d+b(a^d)^2+ab(a^d)^3+bab(a^d)^4,\end{array}$$

as required. □

We now generalize [3, Theorem 3.1] from the Drazin inverse of complex matrices to the g-Drazin inverse in Banach algebras.

Theorem 2.3. *Let $a, b \in \mathcal{A}^d$. If $a^3b = 0, bab = 0$ and $ba^2b = 0$, then $a+b \in \mathcal{A}^d$. In this case,*

$$\begin{array}{rcl} (a+b)^d &=& \alpha^d a + b^d + \alpha b z_2 a + \alpha^2 b z_2 \alpha^d a + \alpha^2 b z_2 (\alpha^d)^2 a\\ &+& \alpha^2 b(\beta^d)^2 z_1 a + \alpha^2 b(\beta^d)^2 z_1 \alpha^d a + \alpha^2 b(\beta^d)^3 z_1 a\\ &+& b z_1 a + \alpha b(\beta^d)^2 + \alpha^2 b(\beta^d)^3 + \alpha^2 b(\beta^d)^4,\\ \alpha &=& a^2+ab, \beta = b^2+ab,\\ \alpha^d &=& (a^d)^2+ab(a^d)^4, \beta^d = (b^d)^2+a(b^d)^3,\end{array}$$

$$\begin{array}{rcl} z_1 &=& \sum\limits_{i=0}^{\infty}(\beta^d)^{i+2}(a+b)\alpha^i\alpha^\pi + \sum\limits_{i=0}^{\infty}\beta^i\beta^\pi(a+b)(\alpha^d)^{i+2}\\ &-& \beta^d(a+b)\alpha^d,\\ z_2 &=& \sum\limits_{i=0}^{\infty}\beta^i\beta^\pi(a+b)(\alpha^d)^{i+3} + \sum\limits_{i=0}^{\infty}(\beta^d)^{i+3}(a+b)\alpha^i\alpha^\pi\\ &-& \beta^d(a+b)(\alpha^d)^2 - (\beta^d)^2(a+b)\alpha^d.\end{array}$$

Proof. Let $M = \begin{pmatrix} a^2+ab & a^2b+ab^2\\ a+b & b^2+ab\end{pmatrix}$. Then $M = A+B$, where

$$A = \begin{pmatrix} a^2+ab & 0\\ a+b & b^2+ab\end{pmatrix}, B = \begin{pmatrix} 0 & a^2b+ab^2\\ 0 & 0\end{pmatrix}.$$

Let $\alpha = a^2 + ab, \beta = b^2 + ab$. Clearly, we have $(ab)^2 = 0, a^2(ab) = a^3b = 0$ and $b^2(ab) = 0$. In light of [?, Theorem 2.3], we have

$$\alpha^d = (a^d)^2 + ab(a^d)^4, \beta^d = (b^d)^2 + a(b^d)^3.$$

By virtue of [8, Theorem 2.1],

$$A^d = \begin{pmatrix} \alpha^d & 0 \\ z_1 & \beta^d \end{pmatrix},$$

where

$$z_1 = \sum_{i=0}^{\infty} (\beta^d)^{i+2}(a+b)\alpha^i\alpha^\pi + \sum_{i=0}^{\infty} \beta^i\beta^\pi(a+b)(\alpha^d)^{i+2} - \beta^d(a+b)\alpha^d.$$

Let $z_2 = z_1\alpha^d + \beta^d z_1$. Then

$$(A^d)^2 = \begin{pmatrix} (\alpha^d)^2 & 0 \\ z_2 & (\beta^d)^2 \end{pmatrix}.$$

Clearly, $B^2 = 0$, and so $B^d = 0$. We compute that

$$\begin{array}{l} B(A^d)^2 = \begin{pmatrix} \alpha b z_2 & \alpha b(\beta^d)^2 \\ 0 & 0 \end{pmatrix}, \\ AB(A^d)^3 = \begin{pmatrix} \alpha^2 b z_2 \alpha^d + \alpha^2 b(\beta^d)^2 z_1 & \alpha^2 b(\beta^d)^3 \\ (a+b)\alpha b z_2 \alpha^d + (a+b)\alpha b(\beta^d)^2 z_1 & (a+b)\alpha b(\beta^d)^3 \end{pmatrix}, \\ BAB(A^d)^4 = \\ \begin{pmatrix} \alpha^2 b z_2(\alpha^d)^2 + \alpha^2 b(\beta^d)^2 z_1\alpha^d + \alpha^2 b(\beta^d)^3 z_1 & \alpha^2 b(\beta^d)^4 \\ (a+b)\alpha b z_2(\alpha^d)^2 + (a+b)\alpha b(\beta^d)^2 z_1\alpha^d + (a+b)\alpha b(\beta^d)^3 z_1 & (a+b)\alpha b(\beta^d)^4 \end{pmatrix}. \end{array}$$

Since $ba^2b = 0$, we see that $A^2B = 0$ and $(AB)^2 = 0$. According to Lemma 2.4, we have

$$\begin{array}{rcl} M^d & = & A^d + B(A^d)^2 + AB(A^d)^3 + BAB(A^d)^4 \\ & = & \begin{pmatrix} \Gamma & \Delta \\ \Lambda & \Xi \end{pmatrix}, \end{array}$$

where

$$\begin{array}{rcl} \Gamma & = & \alpha^d + \alpha b z_2 + \alpha^2 b z_2 \alpha^d + \alpha^2 b(\beta^d)^2 z_1 + \alpha^2 b z_2(\alpha^d)^2 \\ & + & \alpha^2 b(\beta^d)^2 z_1 \alpha^d + \alpha^2 b(\beta^d)^3 z_1, \\ \Delta & = & \alpha b(\beta^d)^2 + \alpha^2 b(\beta^d)^3 + \alpha^2 b(\beta^d)^4, \\ \Lambda & = & z_1 + (a+b)\alpha b z_2 \alpha^d + (a+b)\alpha b(\beta^d)^2 z_1 + (a+b)\alpha b z_2(\alpha^d)^2 \\ & + & (a+b)\alpha b(\beta^d)^2 z_1\alpha^d + (a+b)\alpha b(\beta^d)^3 z_1, \\ \Xi & = & \beta^d + (a+b)\alpha b(\beta^d)^3 + (a+b)\alpha b(\beta^d)^4, \end{array}$$

Since $M=(\begin{pmatrix} a\\ 1\end{pmatrix}(1,b))^2$, it follows from [9, Corollary 2.2] that $N:=\begin{pmatrix} a\\ 1\end{pmatrix}(1,b)$ has g-Drazin inverse and $(N^d)^2=(N^2)^d=M^d$. By using Cline's formula again,

$$\begin{array}{rcl}(a+b)^d&=&(1,b)(N^d)^2\begin{pmatrix} a\\ 1\end{pmatrix}=(1,b)M^d\begin{pmatrix} a\\ 1\end{pmatrix}\\ &=&(1,b)\begin{pmatrix} \Gamma&\Delta\\ \Lambda&\Xi\end{pmatrix}\begin{pmatrix} a\\ 1\end{pmatrix}=\Gamma a+b\Lambda a+\Delta+b\Xi.\end{array}$$

Since $b(a+b)\alpha b=0$, by the computation, we complete the proof. □

Lemma 2.4. *Let $\mathcal{A}$ be a Banach algebra, and let $a,b\in\mathcal{A}^d$. If $ab^2=0,ba^2=0$ and $(ba)^3=0$, then $a+b\in\mathcal{A}^d$. In this case,*

$$\begin{array}{rcl}(a+b)^d&=&(a+b,ab+b^2)M^d\begin{pmatrix} a\\ 1\end{pmatrix},\\ M^d&=&F^d+(F^d)^2G+(F^d)^3G^2+(F^d)^3GF+(F^d)^4G^2F,\\ G&=&\begin{pmatrix} a^2b+aba&a^3b+abab\\ 0&a^2b+bab\end{pmatrix},\\ F&=&\begin{pmatrix} a^3&0\\ a^2+ab+ba+b^2&b^3\end{pmatrix},\\ F^d&=&\begin{pmatrix} (a^d)^3&0\\ (a^d)^4+(b^d)^4+(b^d)^5aa^{\pi}&(b^d)^3\end{pmatrix}.\end{array}$$

Proof. Set

$$M=\begin{pmatrix} a^3+a^2b+aba&a^3b+abab\\ a^2+ab+ba+b^2&a^2b+bab+b^3\end{pmatrix}.$$

Then

$$\begin{array}{rcl}M&=&\begin{pmatrix} a^2b+aba&a^3b+abab\\ 0&a^2b+bab\end{pmatrix}+\begin{pmatrix} a^3&0\\ a^2+ab+ba+b^2&b^3\end{pmatrix}\\ &:=&G+F.\end{array}$$

Moreover, we have

$$\begin{array}{rcl}F&=&\begin{pmatrix} a^3&0\\ a^2+ba&0\end{pmatrix}+\begin{pmatrix} 0&0\\ b^2+ab&b^3\end{pmatrix}\\ &:=&H+K.\end{array}$$

Then

$$H^i=\begin{pmatrix} a^{3i}&0\\ (a^2+ba)a^{3(i-1)}&0\end{pmatrix},K^i=\begin{pmatrix} 0&0\\ b^{3(i-1)}(b^2+ab)&b^{3i}\end{pmatrix}.$$

In view of [8, Theorem 2.1], we have

$$(H^d)^i = \begin{pmatrix} (a^d)^{3i} & 0 \\ (a^d)^{3i+1} & 0 \end{pmatrix},$$
$$(K^d)^i = \begin{pmatrix} 0 & 0 \\ (b^d)^{3(i+1)}(b^2+ab) & (b^d)^{3i} \end{pmatrix} \ (i \in \mathbb{N}).$$

Then

$$H^\pi = \begin{pmatrix} a^\pi & 0 \\ -a^d & 1 \end{pmatrix}, K^\pi = \begin{pmatrix} 1 & 0 \\ -(b^d)^3(b^2+ab) & b^\pi \end{pmatrix}.$$

Since $HK = 0$, it follows by [**?**, Theorem 2.3] that

$$\begin{array}{rcl} F^d & = & \sum\limits_{i=0}^{\infty} K^i K^\pi (H^d)^{i+1} + \sum\limits_{i=0}^{\infty} (K^d)^{i+1} H^i H^\pi \\ & = & K^\pi H^d + K^d H^\pi \\ & + & \sum\limits_{i=1}^{\infty} K^i K^\pi (H^d)^{i+1} + \sum\limits_{i=1}^{\infty} (K^d)^{i+1} H^i H^\pi. \end{array}$$

We compute that

$$\begin{array}{rcl} K^\pi H^d & = & \begin{pmatrix} 1 & 0 \\ -(b^d)^3(b^2+ab) & b^\pi \end{pmatrix} \begin{pmatrix} (a^d)^3 & 0 \\ (a^d)^4 & 0 \end{pmatrix} \\ & = & \begin{pmatrix} (a^d)^3 & 0 \\ (a^d)^4 & 0 \end{pmatrix}, \\ K^d H^\pi & = & \begin{pmatrix} 0 & 0 \\ (b^d)^6(b^2+ab) & (b^d)^3 \end{pmatrix} \begin{pmatrix} a^\pi & 0 \\ -a^d & 1 \end{pmatrix} \\ & = & \begin{pmatrix} (0 & 0 \\ (b^d)^4 & (b^d)^3 \end{pmatrix}, \\ K^i K^\pi (H^d)^{i+1} & = & 0, \\ (K^d)^{i+1} H^i H^\pi & = & \{ \begin{array}{cc} \begin{pmatrix} 0 & 0 \\ (b^d)^5 a a^\pi & 0 \end{pmatrix} & i = 1 \\ 0 & i \geq 2. \end{array} \end{array}$$

Therefore

$$F^d = \begin{pmatrix} (a^d)^3 & 0 \\ (a^d)^4 + (b^d)^4 + (b^d)^5 a a^\pi & (b^d)^3 \end{pmatrix}.$$

We see that

$$G^2 = \begin{pmatrix} a^2 baba & a^2 babab \\ 0 & 0 \end{pmatrix}, G^3 = 0, GFG = 0 \text{ and } GF^2 = 0.$$

Then $G^d = 0$ and $G^\pi = I$. Then

$$M^d = F^d + (F^d)^2 G + (F^d)^3 G^2 + (F^d)^3 GF + (F^d)^4 G^2 F.$$

Obviously, $M = N^3$, where $N = \begin{pmatrix} a \\ 1 \end{pmatrix}(1, b)$. Therefore

$$(a+b)^d = (a+b, ab+b^2)M^d \begin{pmatrix} a \\ 1 \end{pmatrix},$$

as asserted. □

Lemma 2.5. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}^d$. If*

$$ba^2 = 0, b^2 = 0 \text{ and } (ba)^2 = 0,$$

then $a + b \in \mathcal{A}^d$. In this case,

$$(a+b)^d = a^d + (a^d)^2 b + (a^d)^3 ba + (a^d)^4 bab.$$

Proof. Using the notations in Lemma 2.6, we have $G^2 = 0$ and

$$F^d = \begin{pmatrix} (a^d)^3 & 0 \\ (a^d)^4 & 0 \end{pmatrix}, (F^d)^3 GF = \begin{pmatrix} (a^d)^6 bab & 0 \\ (a^d)^7 bab & 0 \end{pmatrix},$$
$$(F^d)^2 G = \begin{pmatrix} (a^d)^4 b + (a^d)^5 ba & (a^d)^3 b + (a^d)^5 bab \\ (a^d)^5 b + (a^d)^6 ba & (a^d)^4 b + (a^d)^6 bab \end{pmatrix}.$$

Hence,

$$M^d = \begin{pmatrix} (a^d)^3 + (a^d)^4 b + (a^d)^5 ba + (a^d)^6 bab & (a^d)^3 b + (a^d)^5 bab \\ (a^d)^4 + (a^d)^5 b + (a^d)^6 ba + (a^d)^7 bab & (a^d)^4 b + (a^d)^6 bab \end{pmatrix}.$$

Therefore we compute that

$$\begin{aligned} (a+b)^d &= (a+b, ab)M^d \begin{pmatrix} a \\ 1 \end{pmatrix} \\ &= a^d + (a^d)^2 b + (a^d)^3 ba + (a^d)^4 bab, \end{aligned}$$

as asserted. □

We have at our disposal all the information necessary to prove the following result.

Theorem 2.6. *Let $a, b \in \mathcal{A}^d$. If $ab^3 = 0, aba = 0$ and $ab^2 a = 0$, then $a + b \in \mathcal{A}^d$. In this case,*

$$\begin{aligned} & (a+b)^d \\ = \; & a^d + (a^d)^2 b + (a^d)^3 b^2 + b^d + (b^d)^3 ab + \sum_{i=0}^{\infty} b(\beta^d)^{i+2}(a+b)\alpha^i \alpha^\pi a \\ + \; & \sum_{i=0}^{\infty} b\beta^i \beta^\pi (a+b)(\alpha^d)^{i+2} a - b^d (a+b) a^d, \\ & \alpha = a^2 + ab, \beta = b^2 + ab, \alpha^d = (a^d)^2 + (a^d)^3 b, \beta^d = (b^d)^2 + (b^d)^4 ab. \end{aligned}$$

Proof. Let

$$M=\begin{pmatrix} a^2+ab & a^2b+ab^2\\ a+b & b^2+ab\end{pmatrix}.$$

Then $M=A+B$, where

$$A=\begin{pmatrix} a^2+ab & 0\\ a+b & b^2+ab\end{pmatrix}, B=\begin{pmatrix} 0 & a^2b+ab^2\\ 0 & 0\end{pmatrix}.$$

Let $\alpha=a^2+ab, \beta=b^2+ab$. Clearly, we have $(ab)^2=0, (ab)a^2=(aba)a=0$ and $ab(b^2)=ab^3=0$. In light of [**?**, Theorem 2.3], we have

$$\alpha^d=(a^d)^2+(a^d)^3b, \beta^d=(b^d)^2+(b^d)^4ab.$$

By virtue of [8, Theorem 2.1],

$$A^d=\begin{pmatrix} \alpha^d & 0\\ z_1 & \beta^d\end{pmatrix},$$

where

$$z_1=\sum_{i=0}^{\infty}(\beta^d)^{i+2}(a+b)\alpha^i\alpha^{\pi}+\sum_{i=0}^{\infty}\beta^i\beta^{\pi}(a+b)(\alpha^d)^{i+2}-\beta^d(a+b)\alpha^d.$$

Let $z_{i+1}=z_i(\alpha^d)^i+(\beta^d)^iz_i (i\in\mathbb{N})$. Then

$$(A^d)^{i+1}=\begin{pmatrix} (\alpha^d)^{i+1} & 0\\ z_{i+1} & (\beta^d)^{i+1}\end{pmatrix}.$$

Obviously, $B^2=0$, and then $B^d=0$. We compute that

$$(A^d)^2B=\begin{pmatrix} 0 & (a^d)^2b+(a^d)^3b^2\\ 0 & z_2(a^2b+ab^2)\end{pmatrix}, BA=\begin{pmatrix} a^2b^2 & 0\\ 0 & 0\end{pmatrix},$$
$$(A^d)^3BA=\begin{pmatrix} 0 & 0\\ z_3a^2b^2 & 0\end{pmatrix}, (A^d)^4BAB=0, BA^2=0, (BA)^2=0.$$

In light of Lemma 2.7, we have

$$\begin{aligned} M^d &= A^d+(A^d)^2B+(A^d)^3BA+(A^d)^4BAB\\ &= \begin{pmatrix} \alpha^d & (a^d)^2b+(a^d)^3b^2\\ z_1+z_3a^2b^2 & \beta^d\end{pmatrix}.\end{aligned}$$

Since $M = (\begin{pmatrix} a \\ 1 \end{pmatrix}(1,b))^2$, it follows from [9, Corollary 2.2] that $N := \begin{pmatrix} a \\ 1 \end{pmatrix}(1,b)$ has g-Drazin inverse and $(N^d)^2 = (N^2)^d = M^d$. By using Cline's formula again,

$$\begin{array}{rcl} (a+b)^d & = & (1,b)(N^d)^2 \begin{pmatrix} a \\ 1 \end{pmatrix} = (1,b)M^d \begin{pmatrix} a \\ 1 \end{pmatrix} \\ & = & (1,b) \begin{pmatrix} \alpha^d & (a^d)^2 b + (a^d)^3 b^2 \\ z_1 + z_3 a^2 b^2 & \beta^d \end{pmatrix} \begin{pmatrix} a \\ 1 \end{pmatrix} \\ & = & \alpha^d a + b z_1 a + (a^d)^2 b + (a^d)^3 b^2 + b\beta^d \\ & = & a^d + (a^d)^2 b + (a^d)^3 b^2 + b^d + (b^d)^3 ab + b z_1 a, \end{array}$$

as required. □

3. Block operator matrices

The aim of this section is to investigate the g-Drazin inverse for the block matrix M given by $(*)$. We now generalized [3, Theorem 4.1] and derive

Theorem 3.1. *Let $A \in \mathcal{B}(X)$ have g-Drazin inverse, $D \in \mathcal{B}(Y)$ and M be given by $(*)$. Let $W = AA^d + A^dBCA^d$. If AW has g-Drazin inverse, $ABCA^\pi = 0, A^\pi ABC = 0$ and $D = CA^dB$, then M has g-Drazin inverse. In this case,*

$$\begin{array}{rcl} M^d & = & P^d + P(QP)^d + Q(PQ)^d + Q(P^d)^2 \\ & + & (PQP + QPQ)Z(P^2 + Q^2), \\ Z & = & \sum\limits_{i=0}^{\infty} (\alpha^d)^{i+3} \beta^i \beta^\pi + \sum\limits_{i=0}^{\infty} \alpha^i \alpha^\pi (\beta^d)^{i+3} - (\alpha^d)^2 \beta^d - \alpha^d (\beta^d)^2, \\ \alpha & = & PQ + QP, \beta = P^2 + Q^2, \\ \alpha^d & = & (PQ)^d + (QP)^d, \beta^d = (P^d)^2, \\ P & = & \begin{pmatrix} A^2A^d & AA^dB \\ C & CA^dB \end{pmatrix}, Q = \begin{pmatrix} AA^\pi & A^\pi B \\ 0 & 0 \end{pmatrix}, \\ P^d & = & \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} [(AW)^d]^2 \begin{pmatrix} A, AA^dB \end{pmatrix} \\ & + & \sum\limits_{i=1}^{\infty} \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} [(AW)^d]^{i+2} \begin{pmatrix} A^{i+1}A^\pi + AA^dBCA^{i-1}A^\pi, 0 \end{pmatrix}. \end{array}$$

Proof. One easily checks that

$$M = \begin{pmatrix} A & B \\ C & CA^dB \end{pmatrix} = P + Q,$$

where

$$P = \begin{pmatrix} A^2A^d & AA^dB \\ C & CA^dB \end{pmatrix}, Q = \begin{pmatrix} AA^\pi & A^\pi B \\ 0 & 0 \end{pmatrix}.$$

We check that $P^2Q=0, Q^2P=0$ and PQ has g-Drazin inverse. Clearly, $Q^d=0$. In view of Lemma 2.3, we have

$$\begin{array}{rcl} M^d & = & P^d+P(QP)^d+Q(PQ)^d+Q(P^d)^2\\ & + & (PQP+QPQ)Z(P^2+Q^2),\\ \alpha & = & PQ+QP, \beta=P^2+Q^2,\\ \alpha^d & = & (PQ)^d+(QP)^d, \beta^d=(P^d)^2,\\ Z & = & \sum\limits_{i=0}^{\infty}(\alpha^d)^{i+3}\beta^i\beta^{\pi}+\sum\limits_{i=0}^{\infty}\alpha^i\alpha^{\pi}(\beta^d)^{i+3}\\ & - & (\alpha^d)^2\beta^d-\alpha^d(\beta^d)^2. \end{array}$$

Also we have

$$P=P_1+P_2, P_1=\begin{pmatrix} A^2A^d & AA^dB\\ CAA^d & CA^dB \end{pmatrix}, P_2=\begin{pmatrix} 0 & 0\\ CA^{\pi} & 0 \end{pmatrix},$$

$P_2P_1=0$ and $P_2^2=0$. Hence $P_2^d=0$. Obviously, we have

$$P_1=\begin{pmatrix} AA^d\\ CA^d \end{pmatrix}\begin{pmatrix} A, AA^dB \end{pmatrix}.$$

By hypothesis, $\begin{pmatrix} A, AA^dB \end{pmatrix}\begin{pmatrix} AA^d\\ CA^d \end{pmatrix}=AW$ has g-Drazin inverse. By using Cline's formula, we get

$$P_1^d=\begin{pmatrix} AA^d\\ CA^d \end{pmatrix}[(AW)^d]^2\begin{pmatrix} A, AA^dB \end{pmatrix}.$$

For any $n\in\mathbb{N}$, we compute that

$$\begin{array}{l} (P_1^d)^i=\begin{pmatrix} AA^d\\ CA^d \end{pmatrix}[(AW)^d]^{i+1}\begin{pmatrix} A, AA^dB \end{pmatrix},\\ P_2^i=\begin{pmatrix} A^iA^{\pi} & 0\\ CA^{i-1}A^{\pi} & 0 \end{pmatrix}. \end{array}$$

According to [?, Theorem 2.3], P has g-Drazin inverse and

$$\begin{array}{rcl} P^d & = & \sum\limits_{i=0}^{\infty}P_1^iP_1^{\pi}(P_2^d)^{i+1}+\sum\limits_{i=0}^{\infty}(P_1^d)^{i+1}P_2^iP_2^{\pi}\\ & = & \sum\limits_{i=0}^{\infty}(P_1^d)^{i+1}P_2^i\\ & = & \begin{pmatrix} AA^d\\ CA^d \end{pmatrix}[(AW)^d]^2\begin{pmatrix} A, AA^dB \end{pmatrix}\\ & + & \sum\limits_{i=1}^{\infty}\begin{pmatrix} AA^d\\ CA^d \end{pmatrix}[(AW)^d]^{i+2}\begin{pmatrix} A^{i+1}A^{\pi}+AA^dBCA^{i-1}A^{\pi}, 0 \end{pmatrix}. \end{array}$$

This completes the proof. □

Corollary 3.2. *Let $A \in \mathcal{B}(X)$ have g-Drazin inverse, $D \in \mathcal{B}(Y)$ and M be given by $(*)$. Let $W = AA^d + A^dBCA^d$. If AW has g-Drazin inverse, $ABC = 0$ and $D = CA^dB$, then M has g-Drazin inverse. In this case,*

$$\begin{array}{l} M^d = P^d + (P^d)^2Q, \\ P = \begin{pmatrix} A & AA^dB \\ C & CA^dB \end{pmatrix}, Q = \begin{pmatrix} 0 & A^{\pi}B \\ 0 & 0 \end{pmatrix}, \\ P^d = \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} [(AW)^d]^2 \begin{pmatrix} A, AA^dB \end{pmatrix} \\ + \sum\limits_{i=1}^{\infty} \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} [(AW)^d]^{i+2} \begin{pmatrix} A^{i+1}A^{\pi} + BCA^{i-1}A^{\pi}, 0 \end{pmatrix}. \end{array}$$

Proof. Since $QP = 0$, we easily obtain the result by Theorem 3.1. □

Lemma 3.3. *Let $A \in \mathcal{B}(X)$ have g-Drazin inverse and $M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$. Let $W = AA^d + A^dBCA^d$. If AW has g-Drazin inverse, $CA^dBC = 0, ABCA^{\pi} = 0$ and $A^{\pi}ABC = 0$, then M has g-Drazin inverse. In this case,*

$$\begin{array}{rcl} M^d & = & G^d + (G^d)^2H, \\ G^d & = & P^d + P(QP)^d + Q(PQ)^d + Q(P^d)^2 \\ & + & (PQP + QPQ)Z(P^2 + Q^2), \\ H & = & \begin{pmatrix} 0 & 0 \\ 0 & -CA^dB \end{pmatrix}, \end{array}$$

where P, Q, Z is defined as in Theorem 3.1.

Proof. Clearly, we have

$$M = \begin{pmatrix} A & B \\ C & CA^dB \end{pmatrix} = G + H,$$

where

$$G = \begin{pmatrix} A & B \\ C & CA^dB \end{pmatrix}, H = \begin{pmatrix} 0 & 0 \\ 0 & -CA^dB \end{pmatrix}.$$

Then we check that $GH = 0, H^2 = 0$. By virtue of [**?**, Theorem 2.3], $M^d = G^d + (G^d)^2H$. Applying Theorem 3.1 to G, we complete the proof. □

Theorem 3.4. *Let $A \in \mathcal{B}(X)$ have g-Drazin inverse, $D \in \mathcal{B}(Y)$ have g-Drazin inverse and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Let $W = AA^d + A^dBCA^d$. If AW has g-Drazin* *inverse,* $CA^dBC = 0, ABCA^\pi = 0, A^\pi ABC = 0, BDC = 0$ *and* $BD^2 = 0$, *then* M *has g-Drazin inverse. In this case,*

$$\begin{array}{rcl} M^d & = & \sum_{i=0}^{\infty} L^iL^\pi(K^d)^{i+1} + \sum_{i=0}^{\infty}(L^d)^{i+1}K^iK^\pi \\ & + & \sum_{i=0}^{\infty} L^iL^\pi(K^d)^{i+2}L + \sum_{i=0}^{\infty}(L^d)^{i+3}K^iK^\pi L \\ & - & L^dK^dL - (L^d)^2KK^dL, \end{array}$$

$$K = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}, L = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}, H = \begin{pmatrix} 0 & 0 \\ 0 & -CA^dB \end{pmatrix},$$

$$\begin{array}{rcl} K^d & = & G^d + (G^d)^2H, \\ G^d & = & P^d + P(QP)^d + Q(PQ)^d + Q(P^d)^2 \\ & + & (PQP + QPQ)Z(P^2 + Q^2), \end{array}$$

where P, Q, Z *is defined as in Theorem 3.1.*

Proof. Write $M = K + L$, where

$$K = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}, L = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}.$$

Then we have $K^d = G^d + (G^d)^2H$, where G, H as defined in Theorem 3.3. Since $BDC = 0$ and $BD^2 = 0$, we check that $KLK = 0, KL^2 = 0$. Then we have

$$\begin{array}{rcl} M^d & = & \sum_{i=0}^{\infty} L^iL^\pi(K^d)^{i+1} + \sum_{i=0}^{\infty}(L^d)^{i+1}K^iK^\pi \\ & + & \sum_{i=0}^{\infty} L^iL^\pi(K^d)^{i+2}L + \sum_{i=0}^{\infty}(L^d)^{i+3}K^iK^\pi L \\ & - & L^dK^dL - (L^d)^2KK^dL. \end{array}$$

The theorem is therefore established. □

The following example illustrates that Theorem 3.4 is a nontrivial generalization of [5, Theorem 2.2].

Example 3.5. *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, *where*

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

We easily check that $CA^dBC = 0, ABCA^\pi = 0, A^\pi ABC = 0,$ $BDC = 0$ *and* $BD^2 = 0$. *Then applying Theorem 3.4, we compute that*

$$\begin{array}{rcl} M^d & = & \sum_{i=0}^{\infty} L^i L^\pi (K^d)^{i+1} + \sum_{i=0}^{\infty} (L^d)^{i+1} K^i K^\pi \\ & + & \sum_{i=0}^{\infty} L^i L^\pi (K^d)^{i+2} L + \sum_{i=0}^{\infty} (L^d)^{i+3} K^i K^\pi L \\ & - & L^d K^d L - (L^d)^2 K K^d L. \end{array}$$

Since L *is nilpotent, we have* $L^d = 0$, *and so* $M^d = \sum_{i=0}^{\infty} L^i L^\pi (K^d)^{i+1} + \sum_{i=0}^{\infty} L^i L^\pi (K^d)^{i+2} L$, *where* $K^d = G^d + (G^d)^2 H$. *As* $H = 0$, *we get* $K^d = G^d = P^d + Q(P^d)^2$. *Moreover, we have* $P^d = (P_1)^d + (P_1^d)^2 P_2 = (P_1)^d = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ *which implies that* $LK^d = 0$ *and* $K^d L = 0$. *Then* $M^d = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$. *But* $BC \neq 0$.

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