Different Approaches for Proving the Pythagorean Theorem using Trigonometry

Tathagata Biswas

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Abstract

Contrary to the claims by Elisha S Loomis in his famous book and popular belief, several approaches towards proving the Pythagorean theorem using trigonometry exists. These approaches essentially use trigonometric identities and concepts that can be derived independent of the identity $sin^2x + cos^2x = 1$, to avoid any circular reasoning. Crucial to the trigonometric approaches are the law of sines, trigonometric angle sum and difference identities and modern definitions of trigonometric functions using the power series and Euler's formula. This article describes these trigonometric proofs of the theorem.

1. Introduction

Elisha Scott Loomis (1852-1940) had claimed in his book, *The Pythagorean Proposition*, containing over 250 proofs of the Pythagorean theorem, that no trigonometric proof of the theorem could exist. His argument was simple: since all fundamental trigonometric formulas are based on the Pythagorean Theorem ($sin^2\theta + cos^2\theta = 1$), using trigonometry to prove the theorem would be circular reasoning.[1] Lately, Calcea Johnson and Ne'Kiya Jackson, high school students from St. Mary's Academy, New Orleans challenged the long held belief by providing a proof of the theorem that used the Brahmagupta's law of sines. Although, details of their proof could not be found in the public domain, a brief outline is provided here.[2]

This recent discovery, has opened new possibilities of proving the Pythagorean theorem using tools from trigonometry, without any circular reasoning. It is thus, reasonable to believe that Johnson and Jackson's proof is not the last of its kind, and other trigonometric proofs of the theorem can exist. Such a proof must essentially use trigonometric results, formulas or identities that can be derived independent of the Pythagorean relation ($sin^2x + cos^2x = 1$). These include the trigonometric functions (defined as ratios of the different sides of a right triangle, or using power series or exponential functions), the angle sum and difference identities (as proven by Jason

Zimba),[3] derivatives of the trigonometric functions (can be derived from taking limits of the functions for angle $(x + dx)$, the Brahmagupta law of sines (proven using triangle similarities).[4] The following sections present different trigonometric proofs of the Pythagorean theorem.

1. **Proof 1**: Using angle subtraction identity (*method 1*)

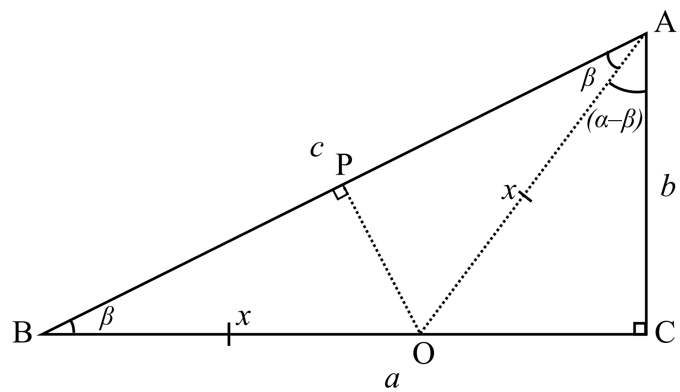

Figure 1

Consider ΔABC with $\angle ACB = 90^\circ, \angle BAC = \alpha, \angle ABC = \beta$ (for $\alpha \geq \beta$) and sides $BC = a$, $AC = b$ and $AB = c$. Construct line from AO (where O is a point on line BC), such that $\angle BAO = \beta$. Also, construct $OP \perp AB\ at\ P$, as in *figure 1*. Thus, it is required to prove: $c^2 = a^2 + b^2$.

We have, ΔAOB an isosceles triangle with ∠ABO = ∠BAO =β and OA = OB (= x).

In ΔOAC, $\angle OAC = (\alpha - \beta)$ and $\frac{\mathrm{OC}}{OA} = \sin(\alpha - \beta)$, or $\mathrm{OC} = x.\sin(\alpha - \beta)$

Now, BC $= \mathrm{OB} + \mathrm{OC} \Rightarrow \mathrm{a} = \mathrm{x} + \mathrm{x}.\ \sin(\alpha - \beta)$

$$\Rightarrow x = \frac{a}{1 + \sin(\alpha - \beta)} \tag{1}$$

In the isosceles triangle ΔAOB:

$$AB = AP + BP \Rightarrow AB = 2BP = 2(\mathrm{OB}) \cdot \cos\beta \qquad (by\ trigonometric\ definitions)$$

$$\Rightarrow c = \frac{2a \cdot \cos\beta}{1 + \sin(\alpha - \beta)} \qquad (from\ equation\ 1)$$

By using the angle subtraction identity: $\sin(\alpha - \beta) = \sin\alpha \cdot \cos\beta - \cos\alpha \cdot \sin\beta$, we get:

$$c = \frac{2a \cdot \cos\beta}{1 + \sin\alpha \cdot \cos\beta - \cos\alpha \cdot \sin\beta}$$

$$\Rightarrow c \cdot (1 + \sin\alpha \cdot \cos\beta - \cos\alpha \cdot \sin\beta) = 2a \cdot \cos\beta$$

$$\Rightarrow c\left(1+\frac{a^2}{c^2}-\frac{b^2}{c^2}\right)=\frac{2a^2}{c} \qquad (by\ trigonometric\ definitions)$$

$$\Rightarrow c^2 = a^2 + b^2 \qquad (hence, proved)$$

2. **Proof 2**: Using angle subtraction identity (*method 2*)

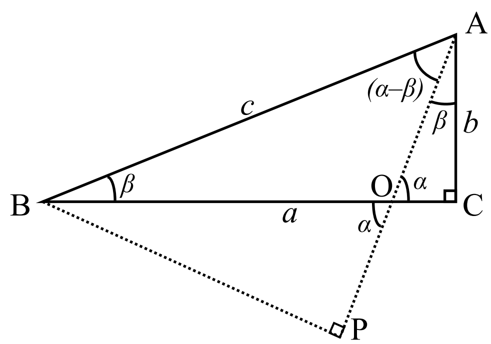

Figure 2

Consider ΔABC with $\angle ACB = 90^{\circ}, \angle BAC = \alpha, \angle ABC = \beta$ (for $\alpha \geq \beta$) and sides $BC = a$, $AC = b$ and $AB = c$. Construct line from AOP (where O is a point on line AC), such that $\angle OAC = \beta$. Also, construct $BP \perp AP\ at\ P$, as in *figure 2*. Thus, it is required to prove: $c^2 = a^2 + b^2$.

In ΔOAC,

$$OA = \frac{b}{\cos\beta}\ and\ OC = (OA)\cdot\sin\beta = \frac{b\cdot\sin\beta}{\cos\beta} \qquad (by\ trigonometric\ definitions)$$

Now,

$$OB = BC - OC = a - \frac{b\cdot\sin\beta}{\cos\beta} \tag{2}$$

In ΔAOB, $\angle \mathrm{OBP} = (90^{\circ} - \alpha) = \beta$

Therefore, by law of sines:

$$\frac{BP}{\sin\alpha} = \frac{OP}{\sin\beta} = \frac{OB}{\sin 90^{\circ}} \tag{3}$$

$$\therefore\ BP = (OB)\cdot\sin\alpha = (\,a - b\tan\beta)\cdot\sin\alpha \qquad (from\ 2)$$

$$\Rightarrow BP = \frac{a^2 - b^2}{c} \tag{4}$$

Also,

$$OP = (OB)\cdot\sin\beta = (\,a - b\tan\beta)\cdot\sin\beta = \frac{b(a^2 - b^2)}{ac} \tag{5}$$

In ΔABP, by law of sines:

$$\frac{BP}{\sin(\alpha-\beta)} = \frac{AP}{\sin 2\beta}$$

$$\Rightarrow\ (BP)\cdot\sin 2\beta =\ (OA+OP)\cdot\sin(\alpha-\beta)$$

$$\Rightarrow (BP)\cdot 2\sin\beta\cos\beta = (OA+OP)\cdot(\sin\alpha\cdot\cos\beta - \cos\alpha\cdot\sin\beta) \qquad (identities)$$

$$\Rightarrow\ 2\left(\frac{a^2-b^2}{c}\right)\left(\frac{ab}{c^2}\right) = \left(\frac{bc}{a}+\frac{b(a^2-b^2)}{ac}\right)\left(\frac{a^2-b^2}{c^2}\right)$$

$$\Rightarrow\ 2\left(\frac{ab}{c}\right) = \left(\frac{bc^2+b(a^2-b^2)}{ac}\right)$$

$$\Rightarrow c^2 = a^2+b^2 \qquad (hence, proved)$$

3. **Proof 3**: Johnson and Jackson's proof (simpler version)

Consider ΔABC with $\angle ACB = 90^{\circ}, \angle ABC = \alpha, \angle BAC = \beta$ and sides $BC = a$, $AC = b$ and $AB = c$. Construct ΔAB′C and rays $\underset{\text{AO}}{\longrightarrow}$ and $\underset{BO}{\longrightarrow}$ such that $\angle BB'O = \alpha$, as in *figure 3.* Thus, it is required to prove: $c^2 =\ a^2+b^2$.

(The construction is like the one Johnson and Jackson had originally used, but omits the infinite chain of similar triangles.)

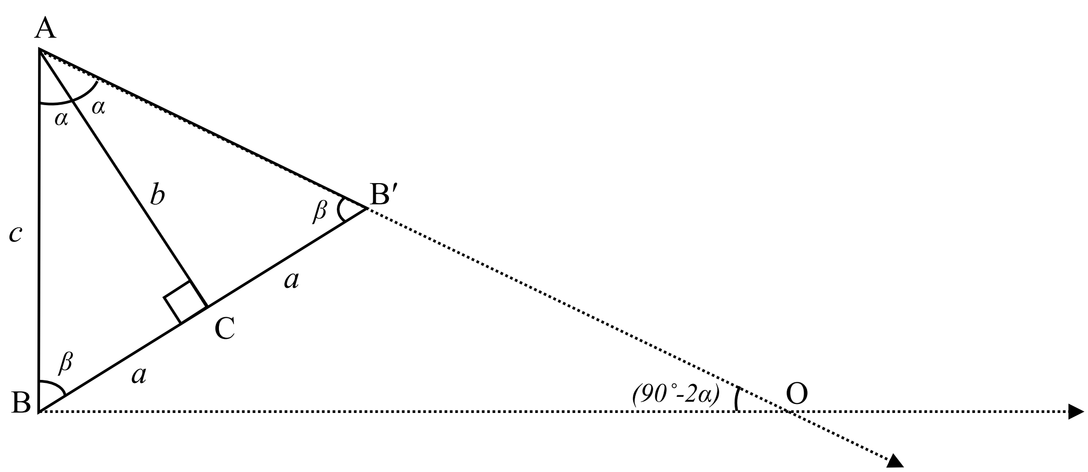

Figure 3

We have, using law of sines:

In ΔBOB′,

$$\frac{\sin(180° - \beta)}{BO} = \frac{\sin(90° - 2\alpha)}{BB'} = \frac{\sin\alpha}{OB'}$$

$$\Rightarrow BO = \frac{2a \cdot \sin\beta}{\cos 2\alpha} \ and\ OB' = \frac{2a \cdot \sin\alpha}{\cos 2\alpha} \tag{6}$$

In ΔABO,

$$\frac{\sin 90°}{AO} = \frac{\sin(90° - 2\alpha)}{AB}$$

$$\Rightarrow (AB' + OB') \cdot \cos 2\alpha = c$$

$$\Rightarrow \left(c + \frac{2a \cdot \sin\alpha}{\cos 2\alpha}\right) \cdot \cos 2\alpha = c \qquad (from\ 6)$$

$$\Rightarrow c \cdot \cos 2\alpha + 2a \cdot \sin\alpha = c$$

$$\Rightarrow c \cdot (\cos^2\alpha - \sin^2\alpha) + 2a \cdot \sin\alpha = c \qquad (expanding\ \cos 2\alpha)$$

$$\Rightarrow c \cdot \left(\frac{b^2}{c^2} - \frac{a^2}{c^2}\right) + 2a \cdot \frac{a}{c} = c$$

$$\Rightarrow c^2 = a^2 + b^2 \qquad (hence, proved)$$

4. **Proof 4**: Simplest proof (without any need for additional construction)

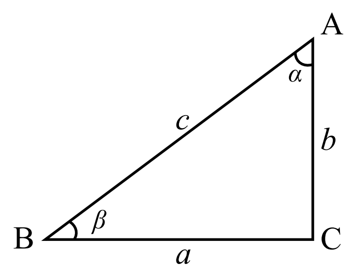

Figure 4

Let, in ΔABC with $\angle ACB = 90^{\circ}, \angle ABC = \alpha, \angle BAC = \beta$ and sides $BC = a$, $AC = b$ and $AB = c$. To prove: $c^2 = a^2 + b^2$.

ΔABC, using the law of sines we get:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin 90°} \tag{7}$$

$$\Rightarrow \frac{a}{\sin(90° - \beta)} = \frac{b}{\sin(90° - \alpha)} = \frac{c}{\sin 90°}$$

$$\Rightarrow \frac{a}{\cos\beta} = \frac{b}{\cos\alpha} = c \tag{8}$$

From equations 7 and 8:

$$a^2 = c^2 \cdot \sin\alpha\cos\beta \tag{9}$$

$$b^2 = c^2 \cdot \cos\alpha\sin\beta \tag{10}$$

Adding 9 and 10:

$$a^2 + b^2 = c^2 \cdot (\sin\alpha\cos\beta + \cos\alpha\sin\beta)$$

$$= c^2 \cdot \sin(\alpha + \beta) \qquad (\alpha + \beta) = 90°$$

$$\Rightarrow a^2 + b^2 = c^2 \qquad (hence, proved)$$

5. **Proof 5**: Using Euler's formula for trigonometric functions

We have,

$$e^{ix} = \cos x + i\sin x \tag{11}$$

$$e^{-ix} = \cos x - i\sin x \tag{12}$$

Solving the equations 11 and 12 we get,

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \tag{13}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \tag{14}$$

Squaring and adding equations 13 and 14 we get,

$$sin^2x + cos^2x = \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^2 + \left(\frac{e^{ix} + e^{-ix}}{2}\right)^2$$

$$= -\frac{1}{4}\left(e^{2ix} + e^{-2ix} - 2e^{ix}\frac{1}{e^{ix}}\right) + \frac{1}{4}\left(e^{2ix} + e^{-2ix} + 2e^{ix}\frac{1}{e^{ix}}\right) = 1 \tag{15}$$

This is true for all values of x, real or complex. In ΔABC (Figure 4) for $0 \leq \alpha \leq 90^\circ$ equation 15 can be rewritten using basic trigonometric definitions:

$$sin^2\alpha + cos^2\alpha = 1 \Rightarrow \frac{a^2}{c^2} + \frac{b^2}{c^2} = 1 \Rightarrow a^2 + b^2 = c^2 \qquad (hence, proved)$$

6. Discussion

Proof 1 and 2 used the angle-subtraction trigonometric identity and the law of sines at important steps. Proof 3, shortened the original version of Johnson and Jackson's proof by bypassing the need for computing infinite geometric series for finding the sides of the larger right triangle. Use of the law of sines and angle sum identity for $\cos 2\alpha$ was instrumental in this process. Proof 4 showed that the law of sines can directly be related with the Pythagorean theorem. Jason Zimba in 2016 had provided a derivation of the angle sum and subtraction identities using triangle similarity (without using $sin^2x + cos^2x = 1$).[5] He himself showed the Pythagorean theorem as a corollary to his discovery.[5] Similarly, the law of sines for any triangle can be derived using triangle similarity,[4] thus avoiding any risk of circular reasoning.

Proof 5 differs from the rest as it uses Euler's formula for defining trigonometric functions, and in a way expands the Pythagorean theorem to the complex plane. Euler's formula can in turn be derived from the power series expansions of e^{ix}, $\cos x$, and $\sin x$ and comparing the real and the imaginary terms. The Maclaurin series expansions of $\cos x$, and $\sin x$ uses serial derivatives of the terms, which do not depend on the Pythagorean theorem. Previously, De Villier in the conlusion to his article in 2023 had contemplated the possibility of a proof using the many different definitions of trigonometric functions.

7. Conclusion

Contrary to the popular belief, trigonometric proofs of the Pythagorean theorem had always existed. Proofs using newer constructions and solutions have been shown.

References

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