Optimization of response curve by Chebyshev polynomials

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ABSTRACT

In control system synthesis, it is interesting to use orthogonal basis functions such as various polynomials and series.

However, there are still no examples of applying spectral methods to closed-loop control systems and poor applications to nonlinear plants.

In this paper, we proposed a control method of allowing the state of the plant to pass through the desired points that the user suggests subjectively in case a model of the control plant is given and the boundary values at a given time point are given.

In other words, to generate the optimal response curve of the Volza problem, we used the standard Chebyshev pseudo-spectral (PS) method, which deals with the state and control of the plant with Chebyshev polynomial approximation, based on which the optimization problem is considered to be a nonlinear programming problem. At that time we obtained the desired control quantity u with the polynomial coefficients and thus implemented the control.

In this paper, we demonstrated the practical applicability of this method by showing not only examples of linear plants but also applications of nonlinear plants.

keywords: Chebyshev polynomials , pseudo-spectral, Response curve optimization, Optimal control problem.

1. Introduction

In this paper we propose an easier method for solving the constrained optimization problem using Chebyshev polynomial approximation than the preceding method.

To date, a number of papers related to Chebyshev polynomial approximation of optimal control problems have been presented [1,3, 5,8].

In 2012, a paper presented the penalized local quadratic interpolation approach as a method for solving constrained optimization problems using Legendre approximation and the pseudo-spectral integral-differential matrix [6~10].

Since then, several papers have discussed the approximation process using pseudo-spectral integral-differential matrices, which are difficult to consider and difficult to understand due to the use of complex formulas [11~16].

Hence, we have considered a simpler approach from a practical point of view.

We chose the Clenshaw-Curtis quadrature method as an easy-to-realistic way to satisfy the accuracy in the integral calculation and used Chebyshev pseudo-spectral method whereby the accuracy of the boundary points in the approximation is very high.

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Several methods, including robust and Chebyshev approximations, have been introduced in connection with quadrature selection. Clenshaw-Curtis quadrature method is easy to reduce the computation and the process.

As an example of the application of this paper, we have compared the results with those presented in the previous non-linear plants like airplanes.

We believe that the approach presented here has a simple and easy-to-implement merit compared to the previous methods, and thus is an advance in Legendre approximation theory.

Here is the outline of this paper.

In Section 2, we set the optimal control problem with linear terminal constraints.

In Section 3, we consider the Chebyshev pseudo-spectral method and the formulation of Chebyshev approximation for nonlinear optimal control problem.

Section 4 presents numerical calculations and applications.

2. Problem Statement

Direct numerical solutions are widely used in practical optimization problems to achieve better performance of the system and to realize computer-based programming.

The pseudo-spectral (PS) method has the meaning that the designer places control and state variables across a given point, in this paper, based on the Chebyshev method, transforms the nonlinear optimal control problem into an algebraic system represented by Chebyshev coefficients, and describes the method of simultaneous determination of the linear equations consisting of Chebyshev coefficients by applying Pontryagin's maximum principle.

Suppose there is the following nonlinear system:

$$\dot{x}(t) = H(t, x(t), u(t)) \tag{1}$$

Here, the initial and final conditions are $x(0) = x_0$, $x(h) = x_h$ and $x(t)$ and $u(t)$ are states and control vectors of order $n \times 1$, $q \times 1$.

$H(t, u(t))$ is assumed to be smooth or non-smooth continuous functions in $[0, h] \times U$.

There exists the state satisfying (1) and a two-point boundary value condition $x(0) = x_0$, $x(h) = x_h$ and control variable pair $(x(t), u(t))$.

The problem is to obtain an optimal control $u(t)$ that minimizes the following objective function, satisfying Eq.(1) in $0 \le t \le h$.

$$J(\mathrm{t}, \mathrm{x(t)}, \mathrm{u(t)}) = \int_0^h f(t, x(t), u(t)) dt \tag{2}$$

The $f(t, x(t))$ in (2) is approximated by $f(t, x(t)) = c^T(t) x(t)$.

In addition, assuming a sufficiently smooth function, the following inequality state constraints can be considered.

$$\Phi(t, x(t)) \le 0 \tag{3}$$

3. Chebyshev pseudo-spectral method

The pseudo-spectral method uses orthogonal functions as basic functions to approximate any functions including periodicity and non-periodicity.
In this method, the weight of Chebyshev polynomials is great at the end of the interval and small in the middle, so the accuracy of the boundary points in the approximation is very high.
This is suitable for response curve optimization that allows us to pass a given boundary point.
Polynomial interpolation of various functions based on Chebyshev polynomials provides approximations with almost consistent accuracy in the interval [-1, 1] and its convergence has also been demonstrated in previous works [3].

3.1 Differential Equation Matrix

The first derivative of the control variable u and the state variable x in formula (1) is approximated as follows.

$$\begin{cases} \dot{x}^N(t) = \sum_{k=0}^{N} \dot{T}_k(t) X_k \\ \dot{u}^N(t) = \sum_{k=0}^{N} \dot{T}_k(t) U_k \end{cases} \tag{4}$$

, where $T_k(t)$ is the Chebyshev polynomial and X_k, U_k are the expansion coefficients when the state variables and control variables are evolved.
As mentioned above, all differential equations are approximated by Chebyshev polynomials.

3.2 Integral scheme

For the calculation of integral part, quadrature method is used.
What is important in quadrature is how to weigh the point under consideration and its point.
We used the Clenshaw-Curtis quadrature method, which is expressed as follows.

$$\int_{-1}^{1} p(t)dt = \sum_{k=0}^{N} p(t_k)\varpi_k \tag{5}$$

, where p is a given function for variable t and ϖ_k is the weight at each point.
When N is even, the weight is as follows.

$$\varpi_0 = \varpi_N = 1/(N^2 - 1) \tag{6}$$

$$\varpi_s = \varpi_{N-s} = \frac{4}{N} \sum_{j=0}^{N/2''} \frac{1}{1-4j^2} \cos\frac{2\pi js}{N} \qquad \left(s = 1,2,\ldots,\frac{N}{2} \right) \tag{7}$$

When N is odd, the weight is as follows.

$$\varpi_0 = \varpi_N = 1/N^2 \tag{8}$$

$$\varpi_s = \varpi_{N-s} = \frac{4}{N} \sum_{j=0}^{N-1/2''} \frac{1}{1-4j^2} \cos\frac{2\pi js}{N} \qquad \left(s = 1,2,\ldots,\frac{N-1}{2} \right) \tag{9}$$

3.3 Nonlinear Planning Issues

Following the above procedure, the response curve optimization problem is transformed into the following nonlinear programming problem.
The objective function is as follows.

$$J(\mathrm{t}, \mathrm{x}(\mathrm{t}), \ \mathrm{u}(\mathrm{t})) = \int_0^{\tau_f} f(t, x(t), u(t))dt \tag{10}$$

Requirements to be met are as follows.

$$f_1 \le f(\dot{x}(\tau), x(\tau), u(\tau), \tau) \le f_u \tag{11}$$

$$\psi_{01} \le \psi(x(\tau_0), u(\tau_0), \tau_0) \le \psi_{0u} \tag{12}$$

$$\psi_{f1} \le \psi(x(\tau_f), u(\tau_f), \tau_f) \le \psi_{fu} \tag{13}$$

$$g_1 \le g(x(\tau), u(\tau), \tau) \le g_u \tag{14}$$

$$\begin{bmatrix} x_1 \\ u_1 \end{bmatrix} \le \begin{bmatrix} x(\tau) \\ u(\tau) \end{bmatrix} \le \begin{bmatrix} x_u \\ u_u \end{bmatrix} \tag{15}$$

4. Numerical Solution Example

All calculations in the following numerical solutions were performed by the MATLAB program.

We also used the function “fminco n” of MATLAB optimization toolbox with SQP algorithm for nonlinear programming solutions.

Example 1: As a nonlinear plant, the following plant is selected.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1^{\ 2} + x_1 x_2 + x_2 u \\ y &= x_1 \end{aligned} \tag{16}$$

Let y(0)=0, y(0.8)=0.8, y(0.9)=0.9, y(1)=1 be the points that the response curve should pass.

Then the objective function is chosen as follows.

$$J = \int_0^{0.5} e^2(t)dt \tag{17}$$

For this subject, the simulation results by the above algorithm are as follows.

$$y = -0.00153t^4 + 0.0202t^3 - 0.00491t^2 + 0.996t - 0.005375$$
$$x = -0.0061t^3 + 0.0607t^2 - 0.0098t + 0.996$$
$$u = 0.0272t^4 - 0.0282t^3 - 1.0073t^2 - 0.8634t - 0.0023$$
$$c = -0.028t^4 - 0.137t^3 - 0.172t^2 + 1.01t - 0.863$$
$$fval = 8.381e^{-6} \tag{18}$$

When performing the above simulation, we consider the process of sine waves, where the sine function is treated by a polynomial approximation using Fourier series expansion.

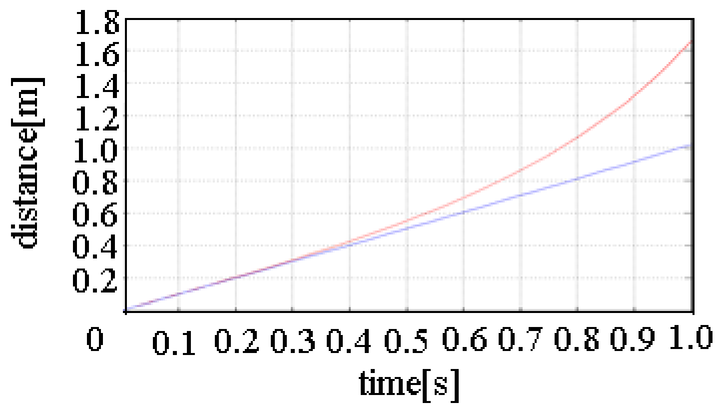

Fig.1 Output with and without controller

(Red: Output without controller, Blue: Output with controller)

Fig.2 shows the error of the sine polynomial approximation with the order up to 4. The higher the order of the approximation gets, the higher the accuracy of the approximation gets, and the lower the error gets. However, in such cases, the computational cost is high and the computation time is long, so the computational power and real-time of the computer must be taken into account.

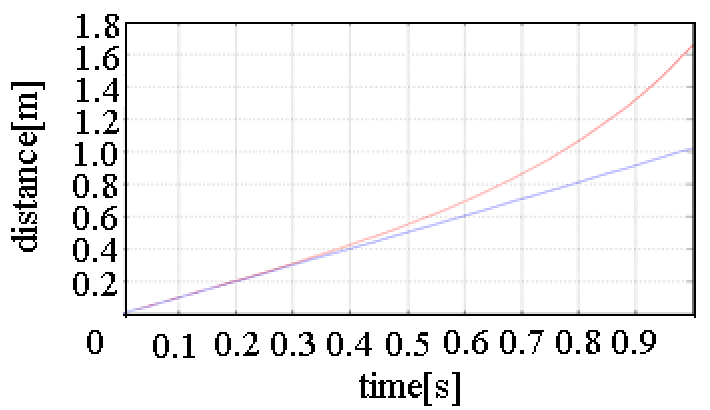

Fig 2. Error of sine polynomial approximation of order 4
(red-Real Trajectory , blue-Approximate Trajectories)

Example 2: Let the following plant be given.

$$\dot{y}(t) = x(t)$$
$$\dot{x}(t) = -\sin(y(t)) + u(t) \quad (19)$$

$$J = \int_0^{0.5} e^2(t)dt \quad (20)$$

As above, we assume that the objective and objective functions are given and the response curve that the output of the object must follow is given as follows.

$$g(t) = t^2 + t + 1 \quad (21)$$

The simulation results are as follows.

$$y = 12.767t^6 - 42.47t^5 + 54.032t^4 - 32.545t^3 + 10.2t^2 + 0.0019t + 1$$
$$u = 4.5982t^6 - 14.9t^5 + 402.44t^4 - 863.44t^3 + 652.53t^2 - 195.21t + 21.25$$
$$fval = 5.7194e^{-5} \quad (22)$$

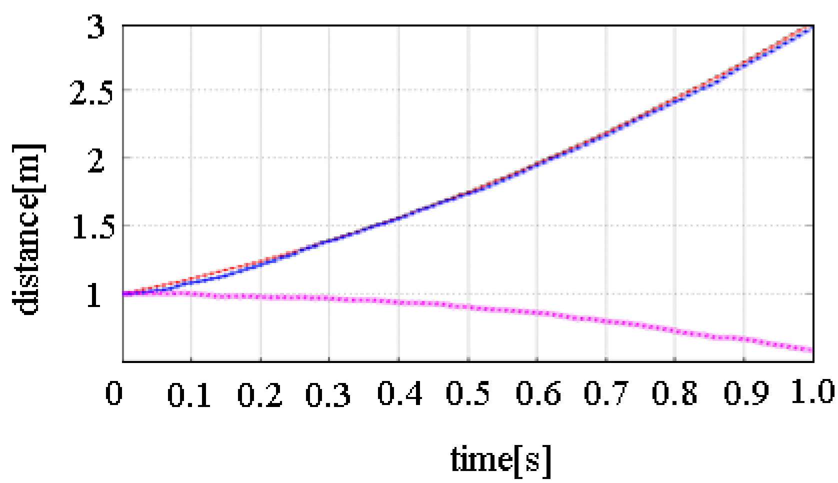

Fig 3. Output response curve

(Red- Desired response curve, Blue-Response curve with controller
Pink- Response curve without controller)

As shown in the figure, it can be seen that the output response of the object follows exactly the desired response curve.

Example 3: Consider the problem of trajectory of an airplane. The vehicle's motion model is as follows.

$$\dot{y}(t) = V(t)\sin(\alpha(t))$$
$$\dot{x}(t) = V(t)\cos(\alpha(t))$$
$$m\dot{V} = P_0 - X(t) - mg\sin(\alpha(t)) \tag{23}$$

The parameters in this equation are as follows.

$$m = 15kg, \quad \alpha = 100mrad, \quad V_0 = 920\,m/s$$
$$X(t) = \frac{1}{2}\rho_0 V(t)^2 \frac{\pi}{4} d^2 \cos(\alpha(t))$$
$$P_0 = \frac{mV_0}{t_0} = \frac{15\times 920}{0.002}, \quad t_0 = 0.002s$$

The objective function is then defined as follows to minimize the distance traveled by the aircraft.

$$J = \int_0^{0.5} D(t)dt \tag{24}$$

We consider the problem of finding the law of change of angle α such that the spacecraft passes $(x_0, y_0) = (0,0)$ and $(x_f, y_f) = (1000,10)$.

Solving the above problem, the result is as follows.

$$\alpha(\mathrm{t}) = 2.301\,e^{-8}t^4 + 1.97e^{-6}t^3 - 1.678\,e^{-3}t^2 - 1.073\,e^{-2}t + 1.708\,e^{-2} \tag{25}$$

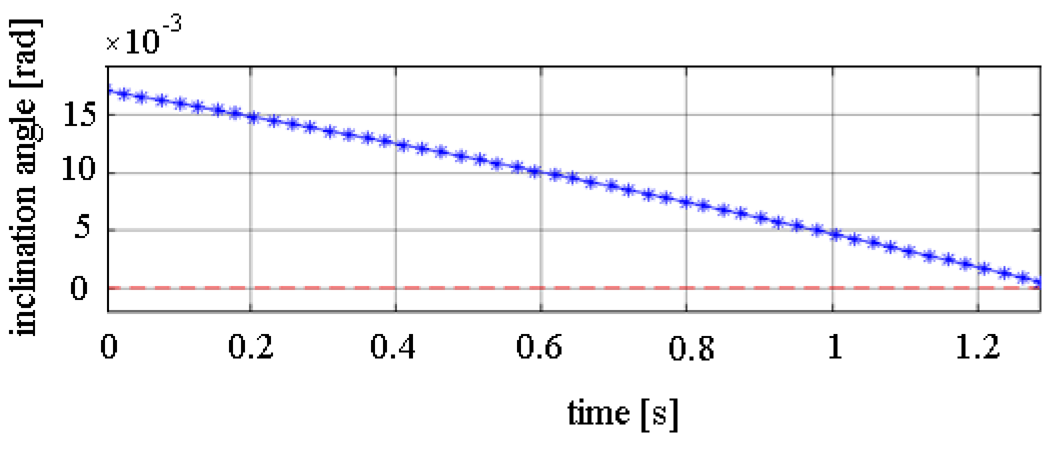

Fig 4. Orbit inclination angle curve

Fig 4. compares the trajectory of the spacecraft obtained by the proposed method with the trajectory obtained by the previous optimization method.

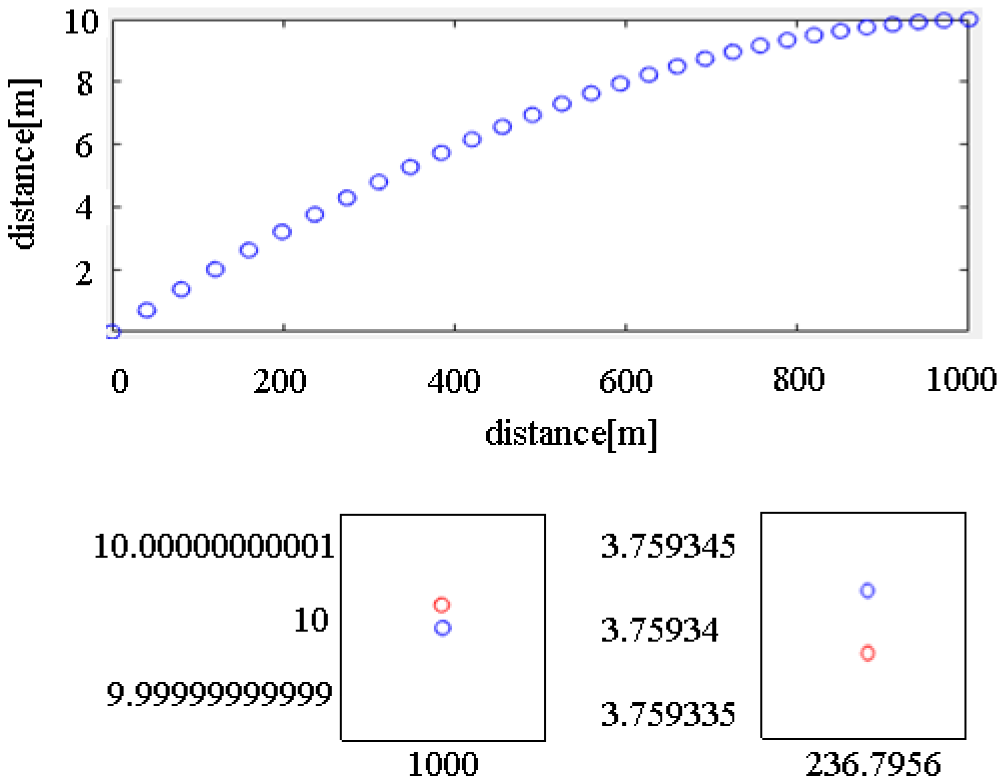

Fig 5. Trajectories of the spacecraft obtained by the proposed method
and trajectories obtained by previous optimization method.
(red-previous optimization method , blue- proposed method)

5. Conclusions

In this paper, we described a method for solving optimal control problems with constraint and boundary conditions as nonlinear programming problems using approximations by Chebyshev polynomials.

The method presented in this paper overcomes the drawback that if the object is complicated for nonlinear objects, the controller must be complicated by complexity, and approximates the object with known polynomials, making the modularization of controller design feasible and construction of controllers simple even if the object is complex.

In particular, it can be simply applied to control if the uncertainty has been or analytical solution is difficult.

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