CHORD DIAGRAMS WITH DIRECTED CHORDS

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Abstract. Chord diagrams are cubic graphs with two types of edges: the first set of edges comprise a subgraph which is a simple cycle (the frame); the second type of edges (the chords) comprise disconnected 2-vertex subgraphs incident to distinct vertices of the frame. We define associated cubic graphs with directed chords (arcs) while keeping the edges of the frame undirected, and plot all 1, 3, 13, and 121 of them for 2, 4, 6, and 8 vertices.

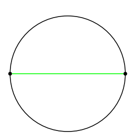
1. Chord Diagrams

The chord diagrams are graphs with two types of edges (i) the frame, a Hamiltonian cycle through the vertices, (ii) chords that connect two distinct vertices of the frame such that the degree of each vertex is 3. The number of vertices, V, is even. The graphs are loopless and connected. Because the degree is 3 at each vertex, the diagrams are *regular*, *cubic* graphs.

Because two distinct types of edges are considered, the diagrams are *signed* graphs [11]. One might plot the two types by attaching a plus or a minus sign to each edge. In this manuscript, the chords are painted green and the edges of the frame black.

Remark 1. *Because one chord and two edges of the frame are incident on each vertex, the graphs are net-regular* [13].

The standard plot of a chord diagram puts the vertices on a circle (or regular V-gon) such that the $V/2$ chords run inside that circle/polygon. Enumeration [3, A054499][15] yields 1 graph on $V = 2$ nodes:



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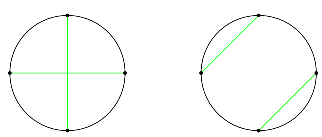
2 graphs on $V = 4$ nodes:



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and 5 graphs on $V = 6$ nodes:

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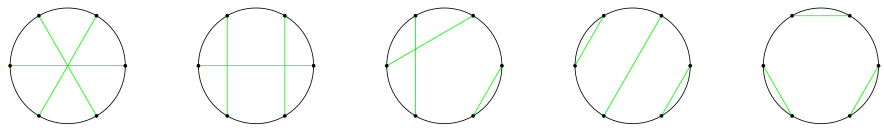
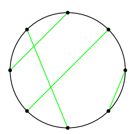


We refer to the basic graph interpretation where rotations or flips of a graph do not create a different graph, i.e., look at the frame as a bracelet. Alternatively one could look at the actual printout/embedding of a graph on/in a sheet of paper and consider diagrams distinct if turning over is not a valid operation to match graphs—necklaces [3, A007769]. For $V \geq 8$ there are more necklaces.

Example 1. *For $V = 8$ exist 17 strung bracelets but 18 strung necklaces because the strung bracelet*



does not have a mirror symmetry and is associated with 2 strung necklaces.

2. Diagrams with directed chords

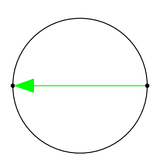
2.1. **Motivation.** The incentive to replace the edges of the chords by directed edges (*arcs*) is a re-interpretation of the chord diagrams as single-electron vacuum polarization diagrams of quantum electrodynamics; the frame is the chain of propagators of the fermion, the edges are the interaction lines. If the interaction is refined to be a *retarded* interaction, the direction of time imposes a distinction of the two vertices. Nevertheless we shall look at the diagrams as purely mathematical/combinatorial objects.

The edges of the frame will be kept undirected.

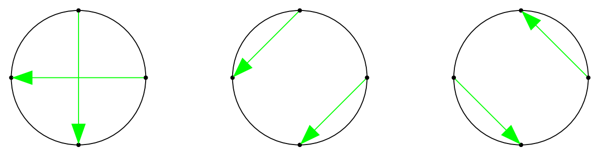
Remark 2. *Because some of the edges are directed and others are not, these are mixed graphs* [6, §5.4].

Remark 3. *Undirected frame edges are equivalent to assigning all frame edges with a uniform clockwise or counter-clockwise orientation as long as the diagrams are considered bracelets—because flipping them over would just map all clockwise circulating frames to counter-clockwise circulating frames and vice versa in a 1-to-1 fashion.*

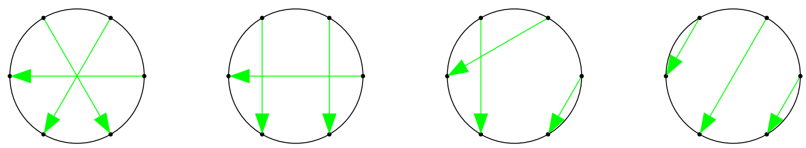
2.2. **Illustrations.** Diagrams with $V = 0, 2, 4, 6, \ldots$ vertices are $1, 1, 3, 13, 121, \ldots$ graphs [3, A260847]—for bracelets, i.e., if flipping is 'allowed' to find matches.



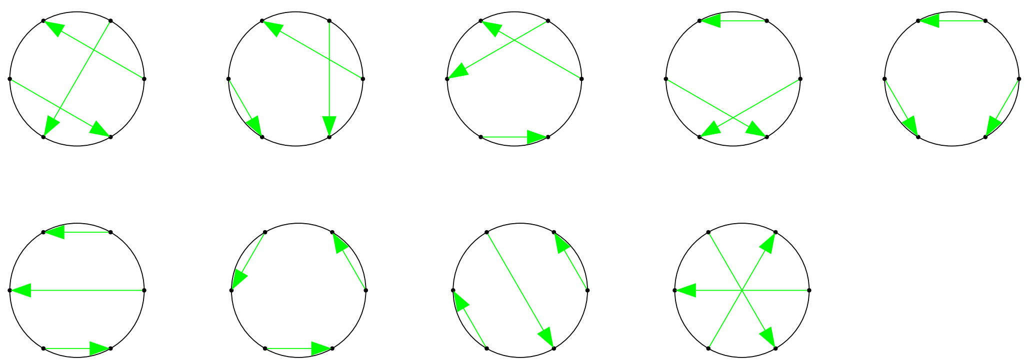
One graph on 2 nodes.

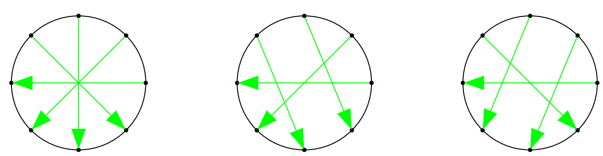


Three graphs on 4 nodes.

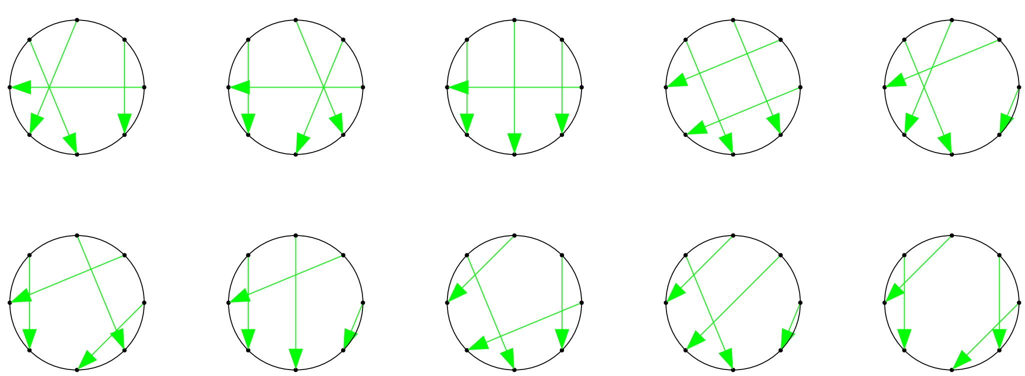


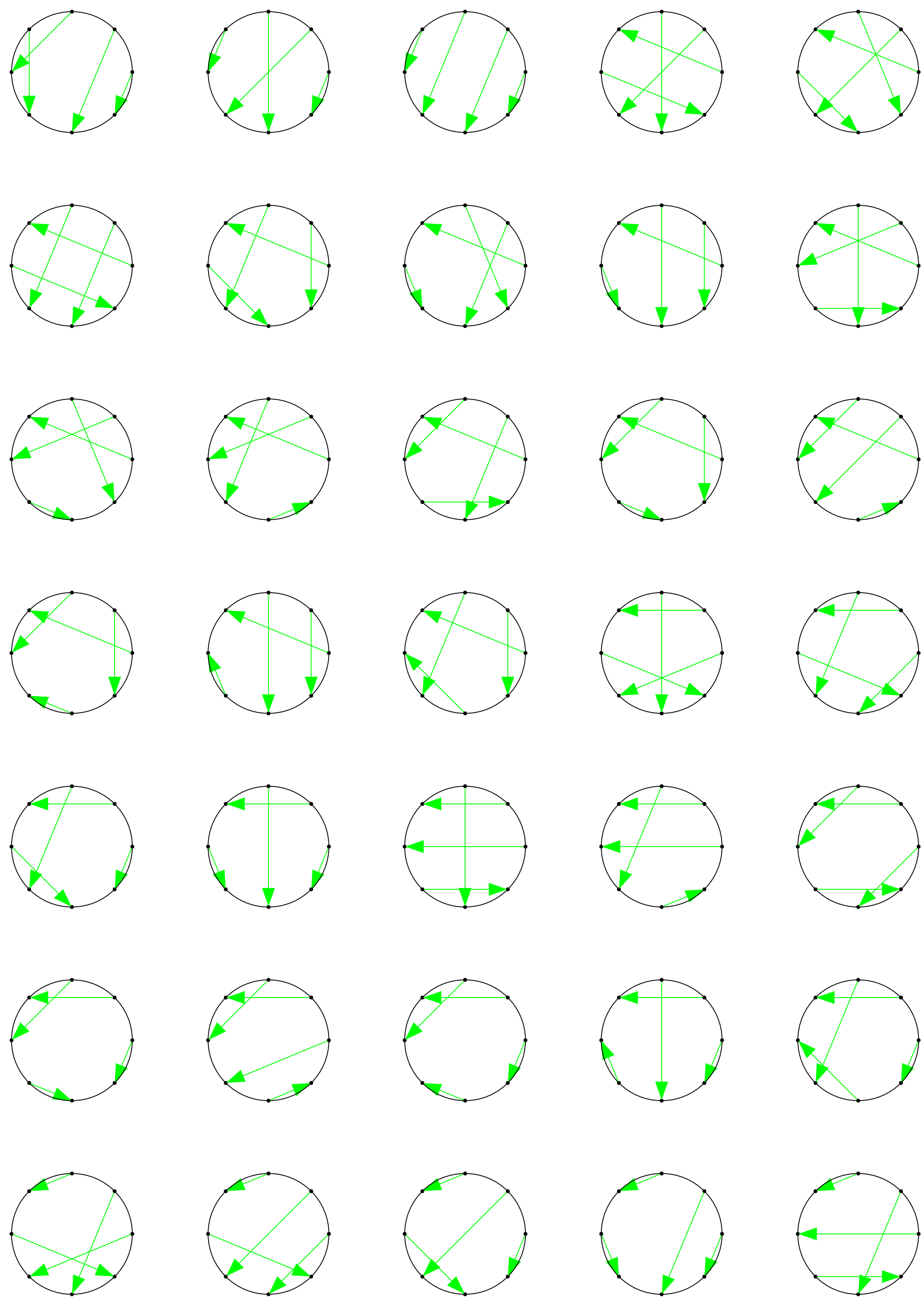
13 graphs on 6 nodes.

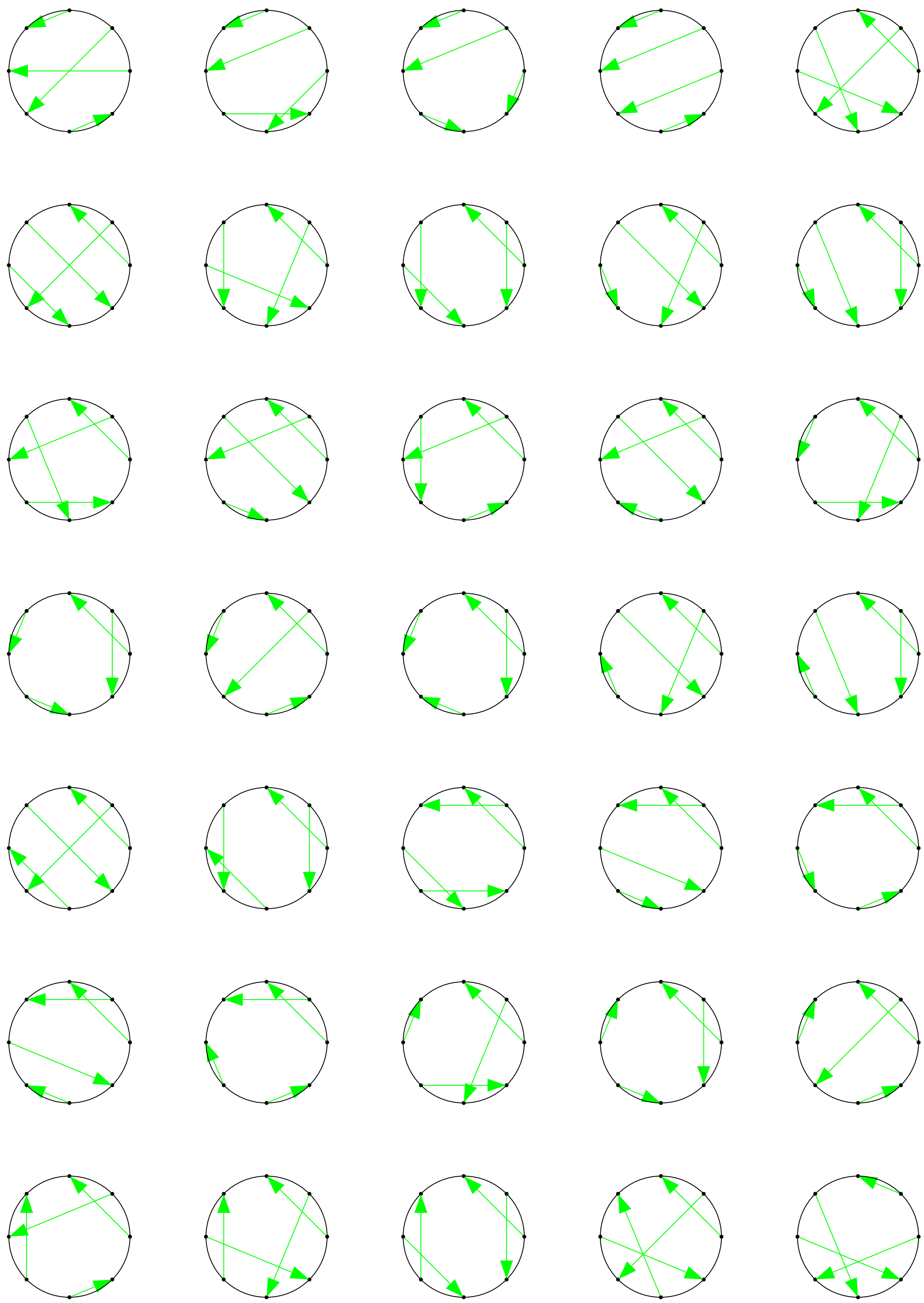


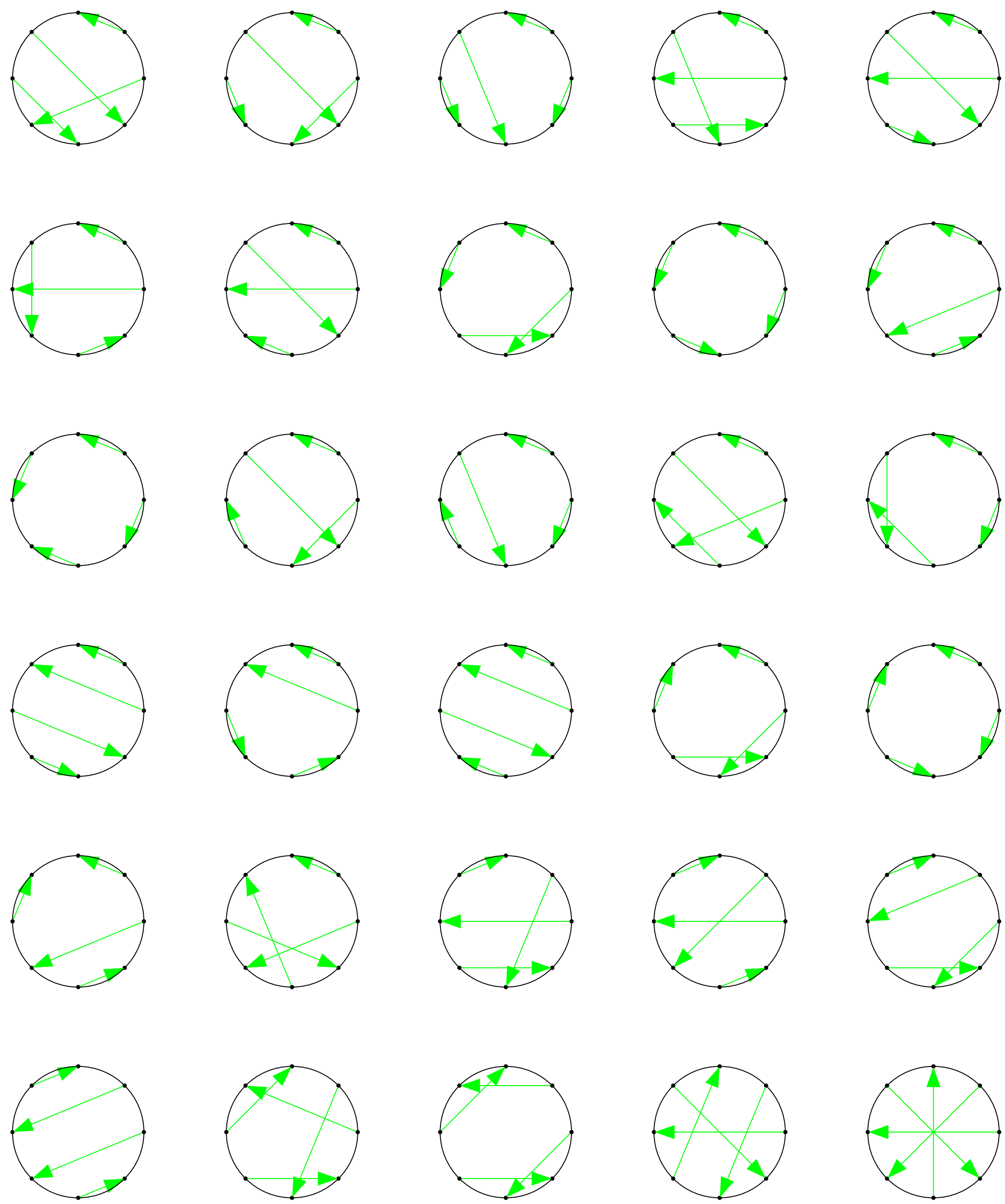


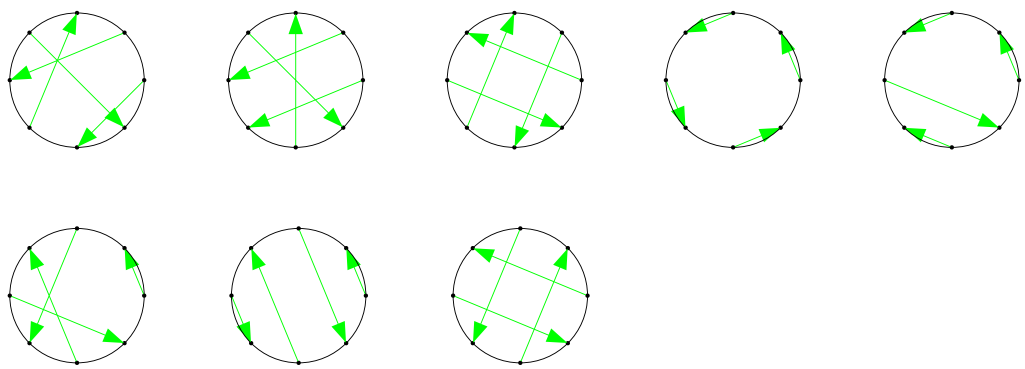
121 graphs on 8 nodes.





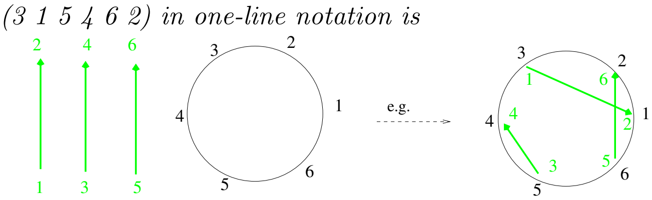






2.3. **Symmetries.** The set of vertex-labeled fixed diagrams with directed chords is based on a set of $V/2$ arcs labeled WLOG with odd numbers at the tails and even numbers at their heads, and a frame with vertices labeled 1 to V (counter clock-wise WLOG) along the Hamiltonian cycle. Glueing the chords into the frame results in a unique representation as a permutation mapping labels 1, 2,... V of the arcs vertices to labels of the frame. Each permutation of $[1, \dots V]$ represents one vertex-labeled fixed diagram.

Example 2. *An example for $V = 6$ and inception of the three chords permuted as (3 1 5 4 6 2) in one-line notation is*



Remark 4. *Other representations are available: (i) The inverse permutation, which collects the edge labels as one moves around the cycle of the frame, serves the same purpose. (ii) One may invent a generalized Lederberg-Coxeter-Frucht LCF notation for the distances between tails and heads along the cycle* [4].

The following symmetry analysis matches broadly the counting of isomers in organic chemistry [14, 10]; our frame is a kind of carbon skeleton of a molecule; the chords are ligands that are attached to the carbons at the skeleton's vertices. The reduction to the *unlabeled* chord diagrams acknowledges symmetries of the frame and of the chords:

- Cyclic permutation of the frame labels 1, 2,... V (rotation of the diagram) does not generate distinct diagrams. The frame is equipped with the symmetry of the cyclic C_V group of V elements. If we consider bracelets, flipping the frame (generating a mirror image) does not generate distinct diagrams either, and the symmetry is the dihedral D_{2V} group—a larger group if $V > 2$. We denote this group isomorphic to C_V or D_{2V} by A. A generator of the group for the necklaces is $(1, 2, 3, \dots, V)$ (cycle notation for the permutations); the additional generator for the bracelets is $(1, V)(2, V-1) \dots (V/2, V/2+1)$ (cycle notation for the permutation, the mirror operation).

- Permutations of the $V/2$ chords create indistinguishable unlabeled diagrams. This is isomorphic to the symmetry group $S_{V/2}$, in our notation any permutation of the odd labels of the arcs' tails; the labels of the heads must follow up accordingly. We denote this group isomorphic to the group $S_{V/2}$ by B. One possible set of $V/2-1$ generators are the star transpositions $\{(1,3)(2,4), (1,5)(2,6), \ldots, (1,V-1)(2,V)\}$ (cycle notation) [12, 7]. The first element in each permutation is the permutation of the odd labels of the tails, the second element the permutation of the even labels of the heads.

Remark 5. *The symmetry group B of order $(V/2)!$ regards relabeled arcs as equivalent as long as the crossing pattern does not change. In the language of ligand substitution, the entire set of (possibly intersecting) chords is considered a single ligand.*

A and B are subgroups of the full symmetric group S_V. Adopting the theory of counting isomers, the number of equivalence classes of the unlabeled diagrams with directed chords is the number of double cosets of $A\backslash S_V/B$ [2, §8.5][8, Th. 11.3][1].

Remark 6. *The diagrams in the necklace interpretation (flipping not allowed to find matches) are counted in* [3, A260296].

Appendix A. GAP implementation

Construction of the double cosets is illustrated with the following GAP program [5]. It provides one way to construct one representative of each coset as a permutation similar to the one-line representation of Example 2.

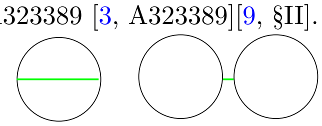
```
#!/usr/bin/env gap

# GAP program: Generate double coset representation of directed chord diagrams
# with V vertices.
# @param V the positive even number, number of vertices
#       The number of chords is V/2.
# @param isBracelet True if flips are allowed to find matches,
#                   false if only the necklace symmetry is considered.
# @param printRep True if a representative of each double coset is
#               printed; false if only the number of cosets is reported
singleV := function(V,isBracelet,printRep)
    local groupB,groupA,S,c,d,auxlist,idx,oneLcyc,oneLflip,oneLswap ;

    if isBracelet then
        Print("bracelet V = ",V,"\n") ;;
    else
        Print("necklace V = ",V,"\n") ;;
    fi;

    # The generator for the frame which is the cyclic group on [V]
    # has the cyclic representation (1 2 3 ,... V)
    # We use the one-line representation (2 3 .... V 1)
    oneLcyc := [2..V] ;;
    Add(oneLcyc,1) ;;

    # The generator for the frame which allows flipping over
    # has the one-line representation (V V-1 V-2 ... 3 2 1)
    oneLflip := [V,V-1..1] ;
    # The group A of the symmetries of the frame has the
    # two generators oneLflip and oneLcyc if considering
    # bracelets, else onlye oneLcyc (necklaces)
    if isBracelet then
        groupA := Group(PermList(oneLflip),PermList(oneLcyc));;
    else
        groupA := Group(PermList(oneLcyc));;
    fi ;;
```

```
    # Print("A ", groupA,"\n") ;

    # this is just 2*V, dihedral group
    Print("# order A ",Size(groupA),"\n") ;;

    auxlist :=[] ;;
    # For the generators of the group of order V/2 swapping
    # any two arcs, the generators have the cyclic representation
    # (1,3)(2,4), (1,5)(2,6), ...(1,V-1)(2,V), as star transpositions
    # Swapping the odd integers means swapping the tails, and swapping
    # the even integers means swapping the attached heads of the arcs.
    if V > 2 then
        for idx in [3,5..V-1] do
            # start with nothing swapped
            oneLswap := [1..V] ;
            # swap 1 <-> idx (odd integer, tail)
            oneLswap[1] := idx ;
            oneLswap[idx] := 1 ;
            # swap 2 <-> idx+1 (even integer, head)
            oneLswap[2] := idx+1 ;
            oneLswap[idx+1] := 2 ;
            # write generator number (idx-1)/2 as a permutation
            auxlist[(idx-1)/2] := PermList(oneLswap) ;
        od ;;
    else
        # if V=2, the group of swapping (the single arc) is the trivial group of order 1
        auxlist[1] := PermList([1]) ;
    fi ;;
    groupB:= CallFuncList(Group,auxlist);;
    # Print("B ", groupB,"\n") ;;

    # this is just (V/2)!, symmetric group of swapping arcs
    Print("# order B ",Size(groupB),"\n") ;




    S:=SymmetricGroup(V);;

    c:= CommutatorSubgroup(groupB,groupA);
    Print("# order of commut subgroup B,A ",Size(c),"\n") ;


    c:= ConjugateSubgroups(S,groupB);
    Print("# order of conj subgroups B ",Size(c),"\n") ;
    c:= ConjugateSubgroups(S,groupA);
    Print("# order of conj subgroups A ",Size(c),"\n") ;

    # Generate the double cosets
    d := DoubleCosetRepsAndSizes(S,groupB,groupA) ;;
    Print("Number of D-cosets ",Size(d),"\n") ;
    if printRep then
        for idx in [1..Length(d)] do
            # Print the coset representativ in reduced cycle notation,
            # number of elements, and in one-line representation
            Print("Cos ",idx," ",d[idx]," ",ListPerm(d[idx][1]),"\n") ;
        od;
    fi ;



    c := Normalizer(S,groupB) ;
    Print("# Order of Normalizer B ",Size(c),"\n") ;
    c := Normalizer(S,groupA) ;
    Print("# Order of Normalizer A ",Size(c),"\n") ;


end;

# main loop over 1,..6 chords, i.e. 2,4,...,12 vertices
```

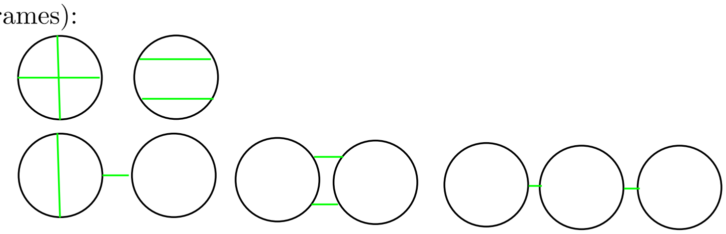
```
for Vhalf in [1..6] do
    V := 2*Vhalf ;
    singleV(V,true,true) ; # bracelet
    singleV(V,false,true) ; # necklace
od;
```

Appendix B. Outlook

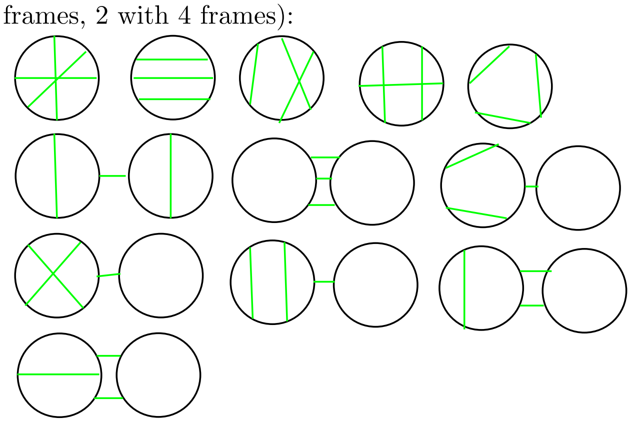
Connected chord diagrams with one or more frames can be defined as in sequence A323389 [3, A323389][9, §II]. Two of these exist with 2 vertices:

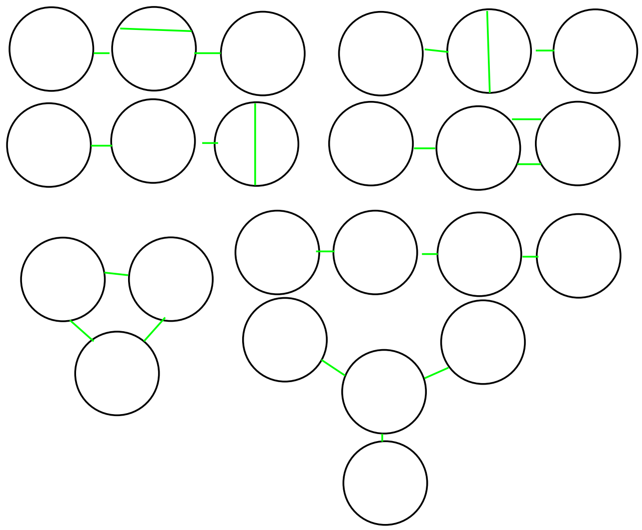


Five of these exist with 4 vertices (2 with 1 frame, 2 with 2 frames, 1 with 3 frames):



Nineteen of these exist with 6 vertices (5 with 1 frame, 7 with 2 frames, 5 with 3 frames, 2 with 4 frames):





Refining chord edges to directed arcs could also be implemented for these.

References

1. Jack Button, Maurice Chiodo, and Mariano Zeron-Medina Laris, *Coset intersection graphs for groups*, Am. Math. Monthly **121** (2014), no. 10, 922–926. MR 3295666
2. Bijan Davvaz, *A first course in group theory*, Springer Verlag, New York, 2021. MR 4387692
3. O. E. I. S. Foundation Inc., *The On-Line Encyclopedia Of Integer Sequences*, (2024), https://oeis.org/. MR 3822822
4. Roberto Frucht, *A canonical representation of trivalent hamiltonian graphs*, J. Graph Theory **1** (1976), no. 1, 45–60. MR 0463029
5. The GAP Group, *GAP – groups, algorithms, and programming, version 4.12.2*, 2023.
6. Frank Harary and Edgar M. Palmer, *Graphical enumeration*, Academic Press, New York, London, 1973. MR 0357214
7. J. Irving and A. Rattan, *Minimal factorizations of permutations into star transpositions*, Disc. Math. **309** (2009), no. 6, 1435–1442. MR 2510551
8. Dinesh Khattar and Neha Agrawal, *Group theory*, Springer Verlag, New York, 2023. MR 4648023
9. Richard J. Mathar, *Feynman diagrams of the QED vacuum polarization*, vixra:1901.0148 (2019).
10. C. Alden Mead, *Table of marks and double cosets in isomer counting*, J. Am. Chem. Soc. **109** (1987), no. 7, 2130–2137.
11. Reza Naserasr, Edita Rollová, and Éric Sopena, *Homomorphisms of signed graphs*, J. Graph Theory **79** (2015), no. 3, 178–212. MR 3346138
12. Igor Pak, *Reduced decompositions of permutations in terms of star transpositions, generalized catalan numbers and k-ary trees*, Disc. Math. **204** (1999), no. 1-3, 329–35. MR 1691876
13. Xiangyu Ren and Xiaowang Li, *Note on a zero net-regular signed graph*, Disc. Appl. Math. **327** (2023), 28–32. MR 4521314
14. Ernst Ruch, Werner Hässelbarth, and Bernd Richter, *Doppelnebenklassen als klassenbegriff und nomenklaturprinzip für isomere und ihre abzählung*, Theoret. Chim. Acta (Berl.) **19** (1970), 288–300.
15. A. Stoimenow, *On the number of chord diagrams*, Disc. Math. **218** (2000), no. 1–3, 209–233. MR 1754336

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