Understanding Universal Disjunction Better

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October 24, 2024

In Quantificational Logic (QL), the direction of entailment that distinguishes the two forms of universal disjunction is a well known but poorly understood asymmetry. We focus on formulae with just two predicates.

$$\forall x(Bx \lor Rx) \not\dashv\vDash \forall x(Bx) \lor \forall x(Rx). \tag{1}$$

Eq 1 reminds us universal disjunction in prenex form does not entail the disjunction of universally quantified formulae. Entailment in the other direction is on the other hand valid. The asymmetry is a basic tenet of QL. However, attempts to illuminate the inequality tend to be cursory. Lemmon, for instance, considers existential conjunction. [1]

$$\exists x(Bx) \land \exists x(Rx) \not\dashv\vDash \exists x(Bx \land Rx). \tag{2}$$

When the domain is the set of positive integers and B is even and R is odd, the fact some integers are even and some are odd does not mean some integers are both even and odd. In the case of universal quantification, from the fact all the integers are either even or odd it does not follow all the integers are even or all the integers are odd. As we shall see, Lemmon's example leaves much out. Beyond Lemmon, elucidations are thin on the ground. This paper explores a semantics to plug the lacuna.

Our investigation is limited to the following notation with the usual rules for well formed formulae.

$$\forall, \exists, B, R, x, a, \neg, \land, \lor, \rightarrow, (,).$$

This limited set of symbols is able to express both forms of universal disjunction. The same fragment also allows an infinite number of well formed formulae of which the majority are logically equivalent. This leaves a finite number of logically distinct propositions. A semantics is introduced to account for this finite set. To that end we introduce the idea of a semantic tile. A tile is a semantic atom. The term 'tile' avoids confusion with another well used term 'atomic sentence'. Unlike an atomic sentence a semantic tile may be syntactically complex. It is also an apt name given the mosaic system we are about to introduce and we shall also refer to the shapes that form the elements of a grid as tiles. Hopefully without introducing too much confusion. The essential idea is that a semantic tile is a possibly true proposition only entailed by logically equivalent sentences or those sentences that express contradiction. Two or more possibly true and logically equivalent sentences only entailed by contradiction are the same tile. No two tiles may be true together. Every complex proposition constructed from a set of tiles is truth functional.

We first introduce a 4×4 grid of tiles. A Boolean 1 is a white (ivory) tile and a Boolean Zero is a black tile. The grid is able to express 2^{16} possible meanings as syntactic combinations of the four propositions whose grid patterns are as Figure 1. The four propositions are rudimentary and the reader should be able to quickly determine the grids preserve the correct set of logical relationships.

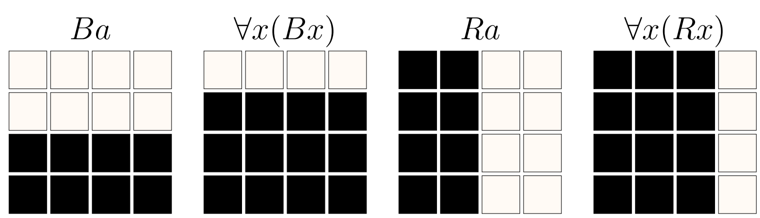

Figure 1:

As we continue to make use of the basic 4×4 outline it may seem odd that propositions about a specific element of the domain help to define a categorical proposition, but this is a reality of QL.

An immediate problem is that the basic 4×4 grid is insufficient to express universal disjunction in the prenex form; for this we need 32 tiles. The pattern of tiles shown in Figure 2 is the the grid form we shall use.

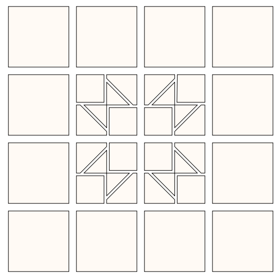

Figure 2:

The reader may prefer the grid to be laid out differently but this particular pattern was settled upon sometime ago for reasons now partly forgotten. Admittedly Figure 2 is something of a mosaic and we adopt the habit of referring to grid patterns as mosaics. The center four squares of the original 4×4 grid are now a cell that consists of an additional five tiles. Take note that the double triangle or hour glass shape is a single tile. The 32 tiles make possible 2^{32} different meanings (a number approaching 4.3 billion). However, we are focused on the two forms of universal disjunction when both predicates are positive. The mosaics for these are as Figure 3.

$\forall x(Bx) \vee \forall x(Rx)$ $\quad$ $\forall x(Bx \vee Rx)$

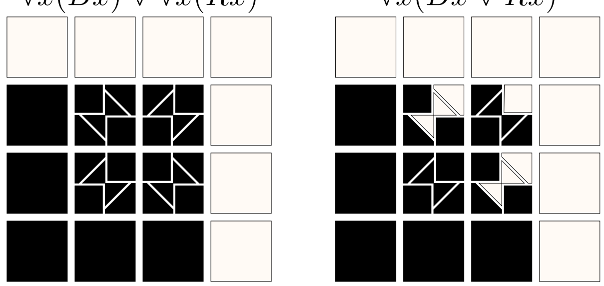

Figure 3:

We take a closer look at the set of 12 tiles that form the pattern for universal disjunction seen on the right hand image of Figure 3. The formula with the simplest syntax is given for each tile, and its location on the grid is noted. A natural language example helps illuminate the different meanings. We consider the elements of the domain are flower stalks with red or blue petals. The same point is also made with an illustration and a Venn diagram.

In the more complicated propositions an arrow points out stalk a. The full suite of illustrations and Venn diagrams make better sense once it is recognised the arrow makes a negative claim when stalk a has petals of one colour. The Venn diagrams also help keep track of the logical possibilities.

P1. $\forall x(Bx) \wedge \forall x(Rx)$

Every stalk has a blue petal and a red petal.

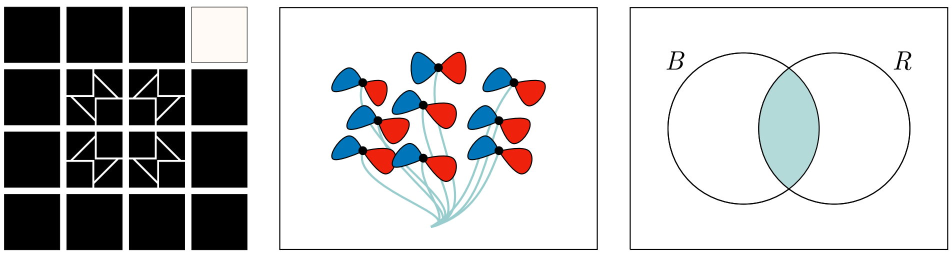

Figure 4:

P2. $\forall x(Bx) \wedge \forall x(\neg Rx)$

Every stalk has a blue petal, none has a red petal.

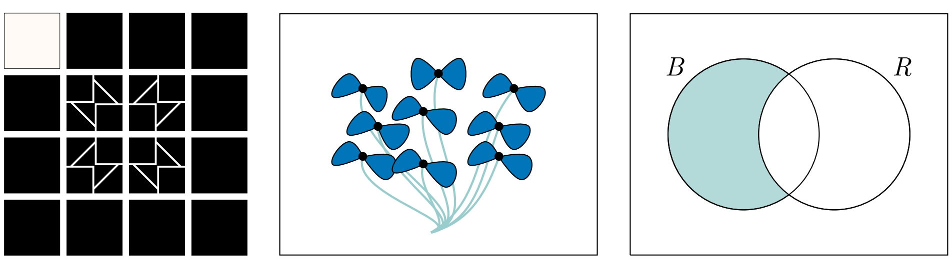

Figure 5:

P3. $\forall x(Rx) \wedge \forall x(\neg Bx)$

Every stalk has a red petal, none has a blue petal.

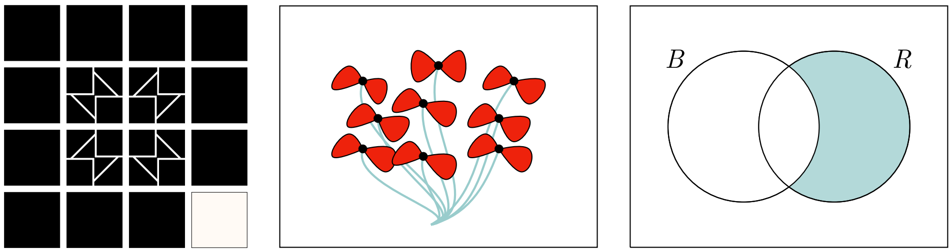

Figure 6:

P4. $\forall x(Bx) \wedge \exists x(\neg Rx) \wedge Ra$

Every stalk has a blue petal.
Some stalks do not have a red petal, but stalk *a* does.

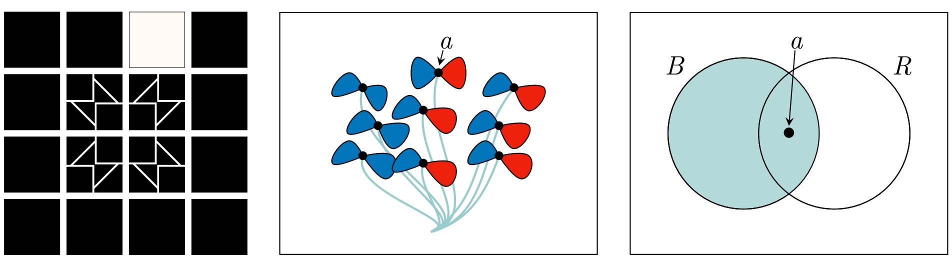

Figure 7:

P5. $\forall x(Rx) \land \exists x(\neg Bx) \land Ba$

Every stalk has a red petal.
Some stalks do not have a blue petal, but stalk a does.

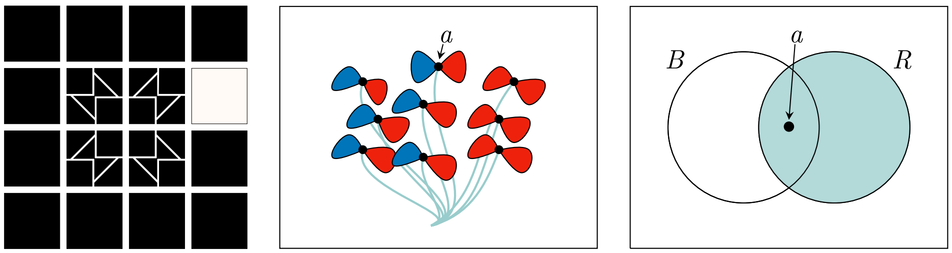

Figure 8:

P6. $\forall x(Bx) \land \exists x(Rx) \land \neg Ra$

Every stalk has a blue petal.
Some stalks have a red petal, but stalk a does not.

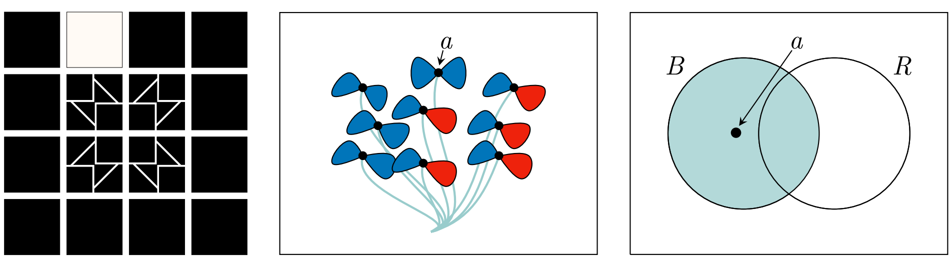

Figure 9:

P7. $\forall x(Rx) \wedge \exists x(Bx) \wedge \neg Ba$

Every stalk has a red petal.
Some stalks have a blue petal, but stalk a does not.

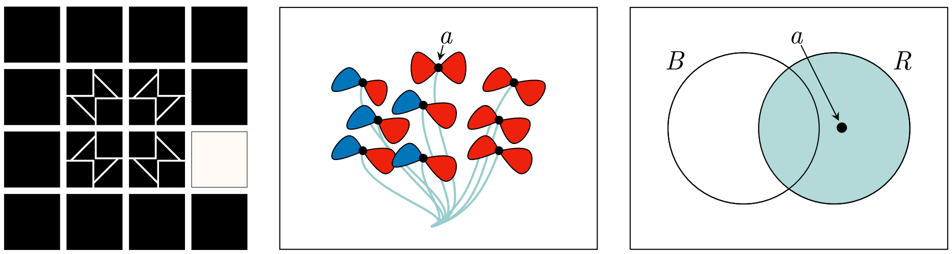

Figure 10:

P8. $\forall x(Bx \vee Rx) \wedge \exists x(\neg Bx) \wedge \exists x(\neg Rx) \wedge Ba \wedge Ra$

Every stalk has a blue or red petal.
Some stalks do not have a blue petal
and some do not have red,
but stalk a has both.

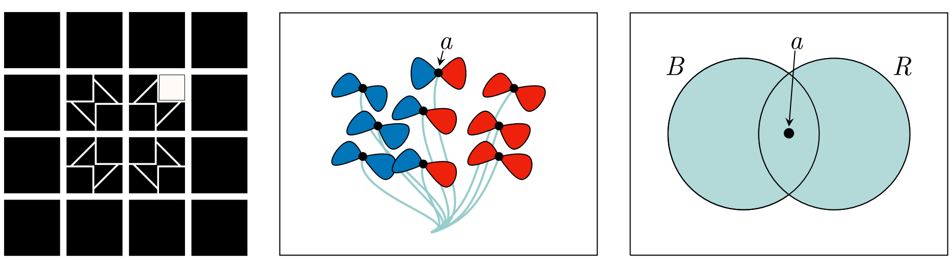

Figure 11:

P9. $\forall x(Bx \vee Rx) \wedge \exists x(Bx \wedge Rx) \wedge \exists x(\neg Bx) \wedge \neg Ra$

Every stalk has a blue petal or red petal.
Some stalks have both a blue petal and a red petal.
Some stalks do not have a blue petal.
Stalk a does not have a red petal.

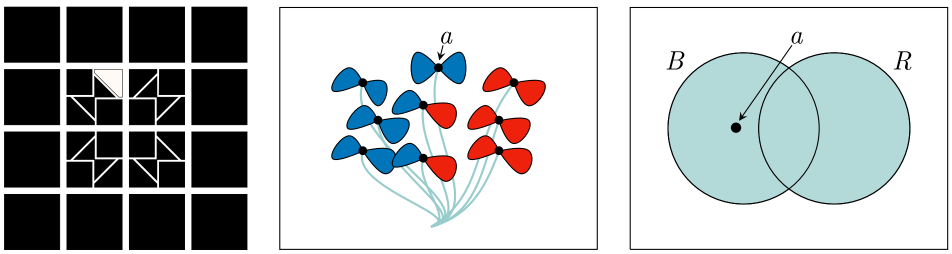

Figure 12:

P10. $\forall x(Bx \vee Rx) \wedge \exists x(Bx \wedge Rx) \wedge \exists x(\neg Rx) \wedge \neg Ba$

Every stalk has a blue or red petal.
Some stalks have both a blue petal and a red petal.
Some stalks do not have a red petal.
Stalk a does not have a blue petal.

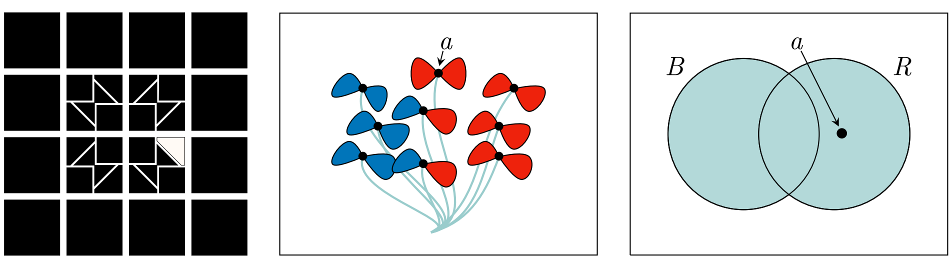

Figure 13:

N.B. It is easier to make sense of the next two propositions if the prenex form is rendered as the logically equivalent universal implication.

P11. $\forall x(Bx \rightarrow \neg Rx) \wedge \forall x(\neg Bx \rightarrow Rx) \wedge \exists x(\neg Bx) \wedge \exists x(Rx) \wedge \neg Ra$

If stalks have a blue petal then they do not have a red petal
and if stalks do not have a blue petal they have a red petal.
Some stalks do not have a blue petal.
Some stalks have a red petal, but stalk a does not.

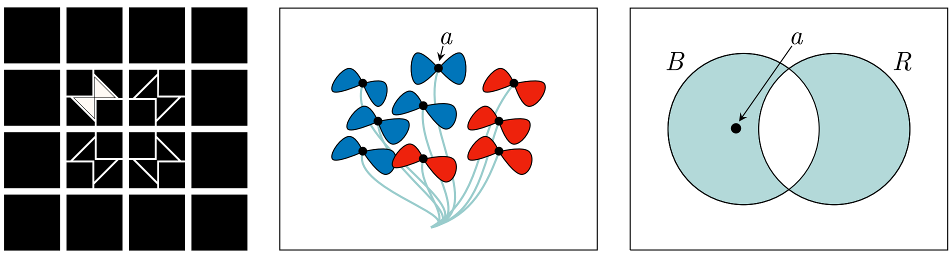

Figure 14:

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P12. $\forall x(Bx \rightarrow \neg Rx) \wedge \forall x(\neg Bx \rightarrow Rx) \wedge \exists x(\neg Rx) \wedge \exists x(Bx) \wedge \neg Ba$

If stalks have a blue petal then they do not have a red petal
and if they do not have a blue petal they have a red petal.
Some stalks do not have a red petal.
Some stalks have a blue petal, but stalk a does not.

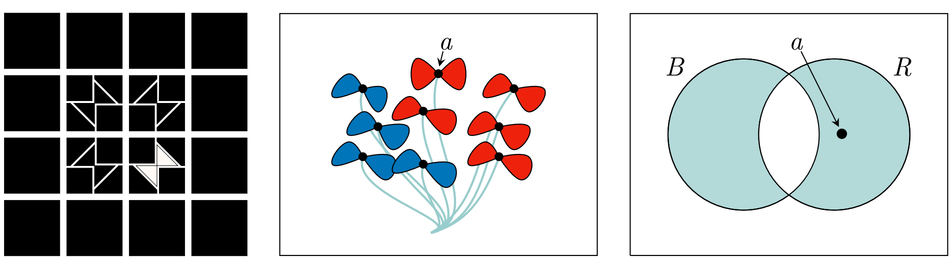

Figure 15:

If we compare the pair P9 and P10 with P11 and P12 it is possible to see just how much is left out by the original example of even and odd integers due to Lemmon. Admittedly the full set of 12 propositions are presented dogmatically. However, for this account of universal disjunction to be correct the following three statements hold true.

$$\vdash \neg(Pn \wedge Pm), \tag{3}$$
where Pn and Pm are any two propositions taken from P1 to P12.

$$\forall x(Bx) \vee \forall x(Rx) \dashv\vdash P1 \vee ... \vee P7. \tag{4}$$

$$\forall x(Bx \vee Rx) \dashv\vdash P1 \vee ... \vee P12. \tag{5}$$

Given the chosen layout for the grid, the mosaics for the alternative universal disjunctions, where one or both predicates is negated, point to their respective corners of the grid as shown by Figure 16.

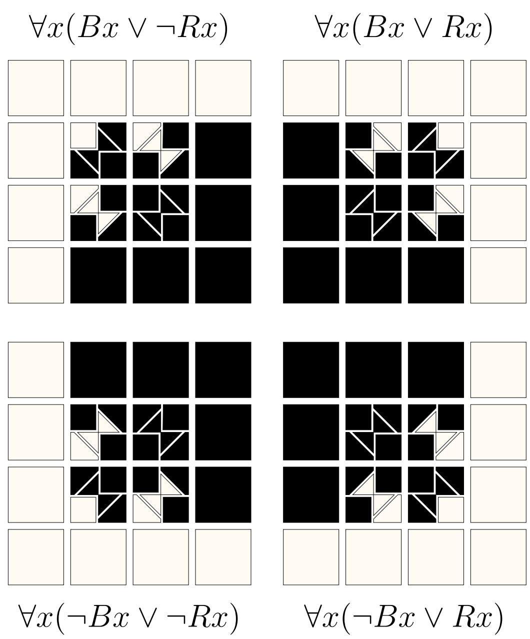

Figure 16:

Through experimentation the reader will find the Figure 16 mosaics preserve the correct set of logical relationships. Keeping to the monadic fragment it is possible to introduce further apparatus such as additional names, variables, and predicates but things get very complicated very quickly. Any initial advantage gained from being able to visualise fragments of monadic predicate logic as a spatial pattern diminish rapidly. Nonetheless, in the simpler case of two predicates the system of mosaics affords a deeper understanding of universal disjunction than usually attempted.

References

[1] Edward John Lemmon. *Beginning logic*. CRC Press, 1971.