Understanding Universal Disjunction

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Abstract

We explore the different meanings of the two versions of universal disjunction. A small but badly neglected aspect of quantificational logic. A semantics is provided to distinguish the alternative meanings; there are twelve for two predicate sentences.

In Quantificational Logic (QL) there are two versions of universal disjunction. (We only consider the simple case of formulae with two predicates). There are individually quantified sentences combined in a disjunction, e.g. $\forall x(Bx) \vee \forall x(Rx)$. A simple English sentence in this form might be: 'everything is blue or everything is red.' This compares to sentences in which the universal quantifier distributes over a disjunction, e.g. $\forall x(Bx \vee Rx)$. The English sentence might be: 'everything is blue or red'. The sentence forms are not equivalent and their asymmetric entailment is a basic tenet of QL.

$$\forall x(Bx \vee Rx) \dashv\nvdash \forall x(Bx) \vee \forall x(Rx). \tag{1}$$

Eq. (1) reminds us distributed universal disjunction does not entail individually quantified sentences, whilst entailment in the other direction is valid. Lemmon provides an example to illustrate the difference.[1]. Consider the domain whose elements are the positive integers. If all the numbers are even

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or odd, it does not mean they are all even or that they are all odd. As cogent as the example is, we shall see it also leaves much out. Beyond Lemmon, elucidations that delve deeper are thin on the ground; so room is left for a fuller account.

Our investigation is limited to the following notation with the usual rules for well formed formulae.

$$\forall, \exists, B, R, x, a, \neg, \wedge, \vee, \rightarrow, (,).$$

This limited set of symbols allows an infinite number of well formed formulae, but as the majority are logically equivalent redundancy is also infinite. However, the set of logically distinct meanings expressible in this fragment is finite. A semantics is introduced to account for this finite set. It will then be helpful to introduce the concept of a semantic tile. Unlike an atomic sentence a semantic tile is syntactically complex whilst also a semantic atom. This means it is a proposition only entailed by logically equivalent sentences or those that express contradiction. Where there are many logically equivalent sentences that are semantic atoms, they are the same tile.

Meanings expressible in our limited fragment may be distinguished using a truth function which we can present on a 4×4 array. A white (ivory) tile represents a Boolean 1 (true) and a black tile is a Boolean 0 (false). Each tile on the array also represents a semantic tile. The 16 tiles account for 2^{16} possible meanings that are combinations of the four propositions shown in Figure 1.

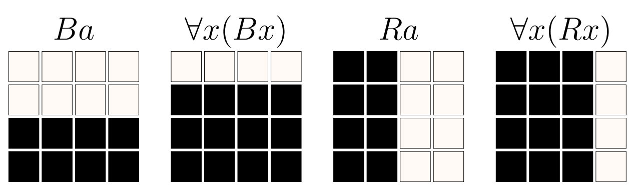


Figure 1

The basic 4×4 array is insufficient to express distributed universal disjunction; for this we need the 32 tile array of Figure 2.

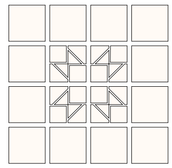


Figure 2

The double triangle (hour glass) is a single tile. The actual pattern is negotiable but the additional five tiles that form each of the four central square cells is not. As Figure 2 resembles a mosaic we refer to grid patterns as 'mosaics'. The 32 tiles make possible 2^{32} different meanings (a number approaching 4.3 billion) but we focus on the two forms of universal disjunction when both predicates are positive. The mosaics for these are as Figure 3.

$\forall x(Bx) \vee \forall x(Rx)$ $\quad$ $\forall x(Bx \vee Rx)$

Figure 3

Where one or both predicates is negated the mosaic *points* to the respective corner of the array as shown at Figure 4.

$\forall x(Bx \vee \neg Rx)$ $\quad$ $\forall x(\neg Bx \vee \neg Rx)$ $\quad$ $\forall x(\neg Bx \vee Rx)$

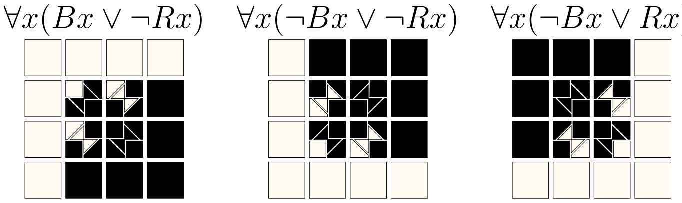
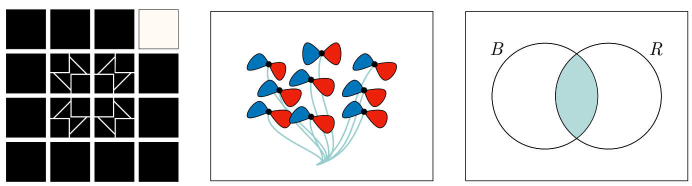


Figure 4

Mosaics help us unpack the alternative meanings that distinguish the two versions of universal disjunction. To illustrate the point we restrict ourselves to Figure 3. Each tile represents an infinite number of well formed formula but to keep things simple we only consider the formula with the simplest syntax. The domain is flower stalks. The stalks have blue (B) or red (R) petals. The meaning of the sentence in the form of the formula is illustrated with an English sentence, a picture and a Venn diagram. On the Venn diagram the additional arrow makes more sense once recognised it makes a negative claim when stalk a has petals of one colour. Twelve propositions (P1 to P12) fully account for Figure 3 and provide a deeper delve into the difference between the two forms of universal disjunction.

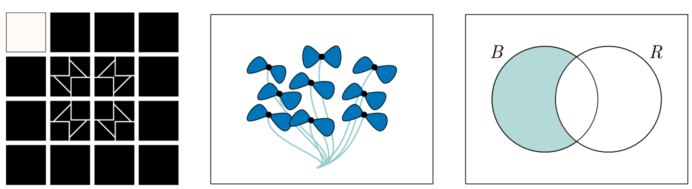
P1. $\forall x(Bx) \wedge \forall x(Rx)$

Every stalk has a blue petal and a red petal.



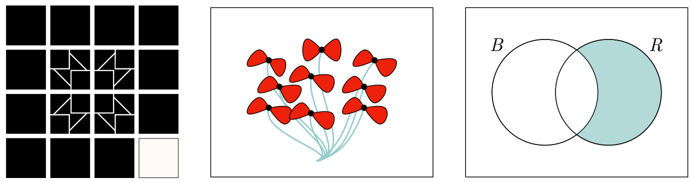
P2. $\forall x(Bx) \wedge \forall x(\neg Rx)$

Every stalk has a blue petal, none has a red petal.



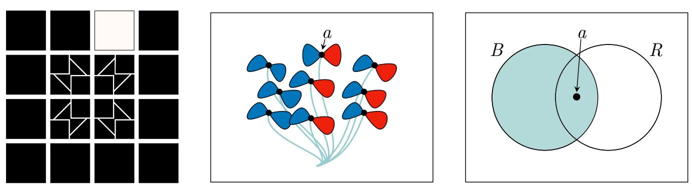
P3. $\forall x(Rx) \wedge \forall x(\neg Bx)$

Every stalk has a red petal, none has a blue petal.



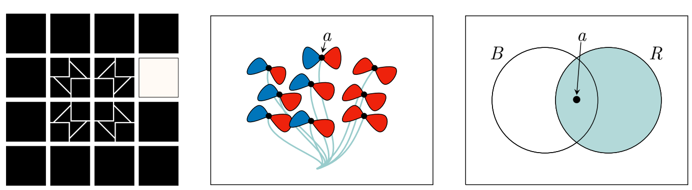
P4. $\forall x(Bx) \wedge \exists x(\neg Rx) \wedge Ra$

Every stalk has a blue petal.
Some stalks do not have a red petal, but stalk a does.



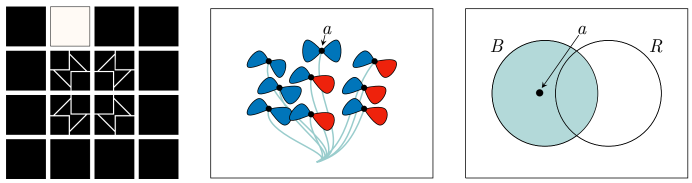
P5. $\forall x(Rx) \wedge \exists x(\neg Bx) \wedge Ba$

Every stalk has a red petal.
Some stalks do not have a blue petal, but stalk a does.



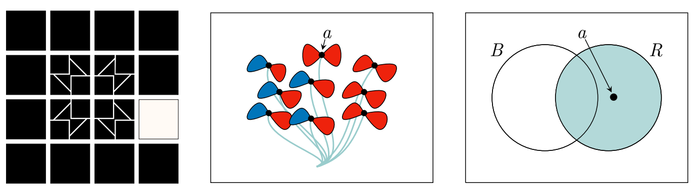
P6. $\forall x(Bx) \wedge \exists x(Rx) \wedge \neg Ra$

Every stalk has a blue petal.
Some stalks have a red petal, but stalk a does not.



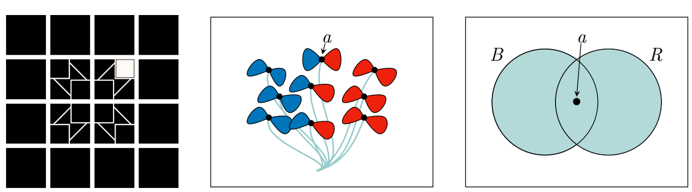
P7. $\forall x(Rx) \wedge \exists x(Bx) \wedge \neg Ba$

Every stalk has a red petal.
Some stalks have a blue petal, but stalk a does not.



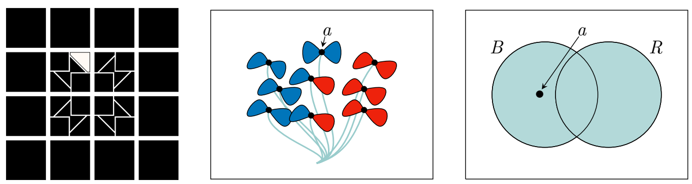
P8. $\forall x(Bx \vee Rx) \wedge \exists x(\neg Bx) \wedge \exists x(\neg Rx) \wedge Ba \wedge Ra$

Every stalk has a blue or red petal.
Some stalks do not have a blue petal
and some do not have red,
but stalk a has both.



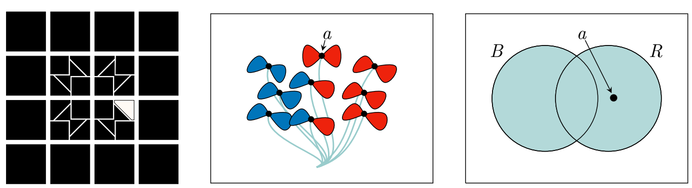
P9. $\forall x(Bx \vee Rx) \wedge \exists x(Bx \wedge Rx) \wedge \exists x(\neg Bx) \wedge \neg Ra$

Every stalk has a blue petal or red petal.
Some stalks have both a blue petal and a red petal.
Some stalks do not have a blue petal.
Stalk a does not have a red petal.



P10. $\forall x(Bx \vee Rx) \wedge \exists x(Bx \wedge Rx) \wedge \exists x(\neg Rx) \wedge \neg Ba$

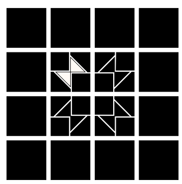
Every stalk has a blue or red petal.
Some stalks have both a blue petal and a red petal.
Some stalks do not have a red petal.
Stalk a does not have a blue petal.

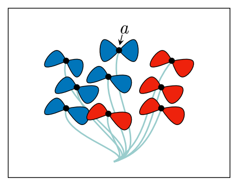


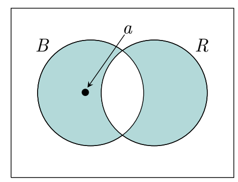
It is easier to make sense of the next two propositions if the the distributed quantifier is rendered as the logically equivalent universal implication.

P11. $\forall x(Bx \rightarrow \neg Rx) \wedge \forall x(\neg Bx \rightarrow Rx) \wedge \exists x(\neg Bx) \wedge \exists x(Rx) \wedge \neg Ra$

If stalks have a blue petal then they do not have a red petal
and if stalks do not have a blue petal they have a red petal.
Some stalks do not have a blue petal.
Some stalks have a red petal, but stalk *a* does not.

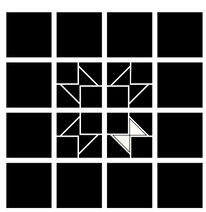


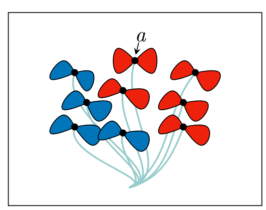


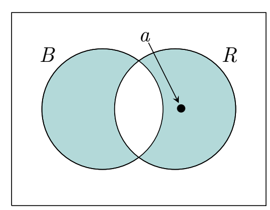


P12. $\forall x(Bx \rightarrow \neg Rx) \wedge \forall x(\neg Bx \rightarrow Rx) \wedge \exists x(\neg Rx) \wedge \exists x(Bx) \wedge \neg Ba$

If stalks have a blue petal then they do not have a red petal
and if they do not have a blue petal they have a red petal.
Some stalks do not have a red petal.
Some stalks have a blue petal, but stalk *a* does not.







If propositions P1 to P7 are compared with the larger set P1 to P12 it is obvious just how much information Lemmon's elucidation passes over. Admittedly the full set of 12 propositions have been introduced dogmatically. If there is any doubt, attention to the Venn diagrams should give confidence all the logical possibilities have been counted. Moreover, this approach is founded on three valid arguments.

$$\vdash \neg(Pn \wedge Pm),$$
$$\text{where Pn and Pm are any two propositions taken from P1 to P12.} \tag{2}$$

Eq. (2) is the contrary clause that insists no two propositions P1 to P12 may be true together.

$$\forall x(Bx) \vee \forall x(Rx) \dashv\vdash P1 \vee ... \vee P7. \tag{3}$$

Eq. (3) confirms the disjunction of two universally quantified sentences is equal to the disjunction P1 to P7.

$$\forall x(Bx \vee Rx) \dashv\vdash P1 \vee ... \vee P12. \tag{4}$$

Eq. (4) confirms distributed universal quantification is equal to the disjunction P1 to P12.

There is also a theorem for the 32 tile mosaic.

$$\vdash P1 \vee ... \vee P32 \tag{5}$$

Eq. (5) confirms Figure 2 represents a truth table tautology. To prove Eq. (5) it is only necessary to prove the top left quartile of the mosaic is a disjunction of eight propositions equivalent to $Ba \wedge Ra$. A principle of symmetry applies to the remaining quartiles.

In conclusion: the advantages of the semantic approach advocated here make them self felt if and when Figure 3 serves as a quick reminder why Eq. (1) holds true.

References

[1] Edward John Lemmon. *Beginning logic*. CRC Press, 1971.