Proof of the Collatz Conjecture for the Natural Numbers

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Abstract

This document proves that the Collatz Conjecture is true for the Natural numbers excluding zero. Use is made of the probability distribution of even and odd numbers in supposed diverging Collatz sequences to establish that Collatz sequences do not diverge, have a finite number of terms and are bounded. Finally proof by contradiction, the pigeon hole principle and proof by induction are used to prove that the Collatz Conjecture is true via two theorems.

Keywords Collatz Conjecture, Syracuse Conjecture, $3x + 1$ Conjecture, Ulam conjecture, Hailstone sequence or Hailstone numbers.

1 Introduction

The Collatz Conjecture is named after the mathematician Lothar Collatz who introduced it in 1937. Current and past research is presented in [1], [2], [3]. The solution has proved elusive and the famous mathematician P. Erdos remarked that "Mathematics may not be ready for such problems."[4] By 2020, the conjecture had been verified by computer for all starting values up to 2^{68}.[5] However a mathematical proof that would prove that the conjecture is true for all Natural numbers greater than zero has yet to be proven.

2 Definitions

Definition 2.1 (Natural Numbers) *The symbol N^+ denotes the Natural numbers not including 0.*

Definition 2.2 (Collatz function) *The Collatz function, $C : \mathbf{N}^+ \to \mathbf{N}^+, n \in \mathbf{N}^+$ is shown in Equation 2.1.*

$$C(n) = \begin{cases} n/2, & \text{if } 2 \mid n. \\ 3n+1, & \text{otherwise.} \end{cases} \tag{2.1}$$

Definition 2.3 (generates) *The phrase n generates denotes the Collatz sequence of numbers iteratively calculated with a starting value of n where the next term in the sequence is obtained by using the Collatz function on the proceeding one.*

Definition 2.4 (Collatz Conjecture) *The Collatz Conjecture asserts that Collatz sequences generated from the set of Natural numbers has a term* *equal to 1 or phrased differently reaches 1. A Collatz sequence as defined herein is deemed terminated upon reaching 1.*

Definition 2.5 (The k^{th} generated term of a Collatz sequence) *$C^k(n)$ where $k, n \in \mathbf{N}^+$ is defined as the k^{th} generated term of a sequence starting from n. Note it is the $(k+1)^{th}$ term of that sequence.*

Definition 2.6 (cycling) *A Collatz sequence which repeats one of the numbers in the sequence will cycle through numbers already in the sequence and is defined as cycling.*

As an example Equation 2.2 shows that $C^2(7) = 11$ which is the 3rd term of that sequence and the number 7 generates a Collatz sequence that reaches 1. Because it reaches 1 then 7 is a number that is in accord with the Collatz Conjecture.

$$(7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1) \tag{2.2}$$

3 Probability that the Parity of $\mathbf{C}^{\infty}(\mathbf{n})$ is Even

Lemma 3.1 *If $n \in 2\mathbf{N}^+$ is randomly chosen then $P\{n/2 \in 2\mathbf{N}^+\}$ is $\frac{1}{2}$.*

Proof

Consider the set of even numbers, $\{2, 4, 6, 8..\}$. Upon dividing the elements of this set by 2 the result is $\{1, 2, 3, 4, ...\}$. Note that half of the terms are even and half are odd. Thus for a randomly selected even number n, this implies that the probability that $\frac{n}{2}$is even or odd is $\frac{1}{2}$. $\square$

Theorem 3.2 is developed from [6] and thus the author entirely deserves credit for it.

Theorem 3.2 *$P\{C^{\infty}(n) \ \in 2\mathbf{N}^{+}\}$ is $\frac{2}{3}$.*

Proof

The top portion of Figure 1 shows a probability tree for even and odd terms that are generated from a random starting number n in a Collatz sequence. $P\{C(n) \in 2\mathbf{N}\}$ is the probability that C(n) is even. This will be written simply as $P\{C(n)\}$. This implies the probability of an odd C(n) is $1 - P\{C(n)\}$. Also note that $P\{C(n)\} = \frac{1}{2}$ by lemma 3.1.

The next level of the tree represents $C^2(n)$.

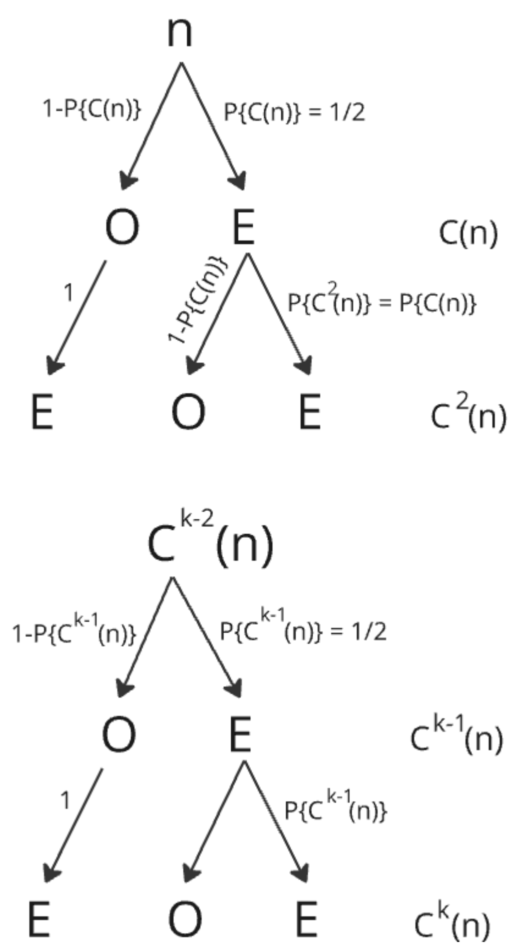


Fig 1. Probability Tree of the Parity of terms in a Collatz sequence.

$P\{C^2(n) \in 2\mathbf{N}^+\}$ which will be written simply as $P\{C^2(n)\}$ is calculated as

$$\begin{aligned} P\{C^2(n)\} &= (1 - P\{C(n)\}).1 + \frac{1}{2}.P\{C(n)\} \\ &= 1 - \frac{1}{2}P\{C(n)\} \end{aligned} \tag{3.1}$$

Or in general, from observation of the lower portion of Figure 1,

$$P\{C^k(n)\} = 1 - \frac{1}{2}P\{C^{k-1}(n)\} \tag{3.2}$$

where again it is understood that the probability being calculated is for an even number. Applying Equation 3.2 in a recursive manner,

$$\begin{aligned} P\{C^k(n)\} &= 1 - \frac{1}{2}(1 - \frac{1}{2}P\{C^{k-2}(n)\}) \\ &= 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + ... + (-1)^k \frac{P\{C(n)\}}{2^{k-1}} \\ &= 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + ... + (-1)^k \frac{\frac{1}{2}}{2^{k-1}} \\ &= 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + ... + (-\frac{1}{2})^k \end{aligned} \tag{3.3}$$

Taking the limit as k approaches infinity,

$$\lim_{k\to\infty} P\{C^k(n)\} = \lim_{k\to\infty} \left[1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + ... + (-\frac{1}{2})^k\right] \tag{3.4}$$

Noting that the last term disappears and that the remaining rhs terms form a geometric series with a = 1 and r = $-\frac{1}{2}$.

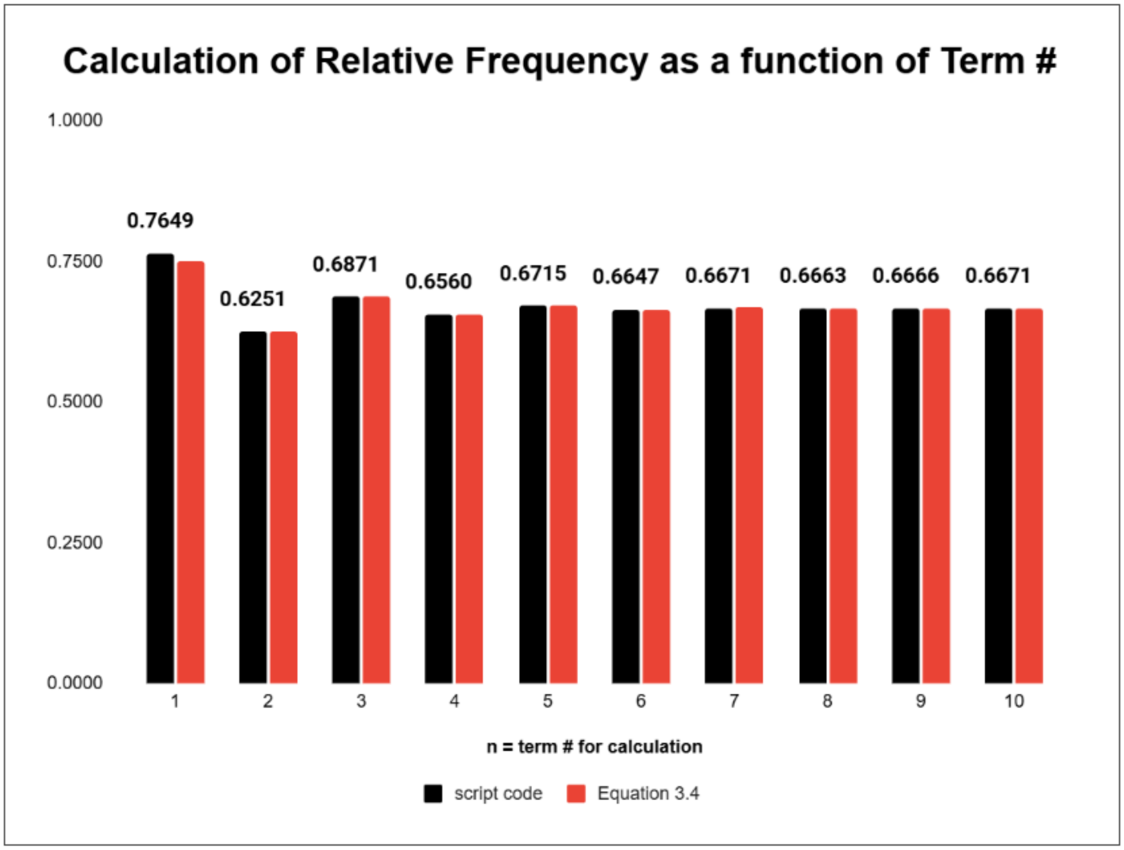
$$
\begin{aligned}
P\{C^{\infty}(n)\} &= \frac{a}{1-r} \\
&= \frac{1}{1-\{-\frac{1}{2}\}} \\
&= \frac{1}{1+\frac{1}{2}} \\
&= \frac{1}{\frac{3}{2}} \\
&= \frac{2}{3}
\end{aligned} \tag{3.5}
$$

Therefore, $P\{C^{\infty}(n) \ \in 2\mathbf{N}^{+}\}$ is $\frac{2}{3}$. $\square$

The program listed in Figure 3 was written in Google Sheets scripting language which generated 100,000 different Collatz sequences from distinct random starting numbers. The probability of each of the first 10 terms being even was calculated, and the results are shown in Figure 4. The same calculations are also plotted using Equation 3.3. The script code-calculated values are in agreement with the theoretical values as
$.6671 = P\{C^{10}(n)\} \approx P\{C^{\infty}(n)\} = \frac{2}{3}$.

```
function collatzProb(nseqs=1000000,termnum=10) {
// nseqs is the number of Collatz sequences
// termnum is the number of the term being considered
let n = 0;
let even = 0;
let total=nseqs;
while (nseqs !== 0) {
 n = n + Math.floor(Math.random()*1000)
 j=termnum;
 while (j !== 0) {
  if (n % 2 === 0) {
   n /= 2;
  } else {
   n = n * 3 + 1;
  }
j--;}
if (n % 2 === 0) {
 even++;   }
 nseqs--;
}
return even/total;
     }
```

Fig 3. Code to calculate that each of $C^1(n)...C^{10}(n)$ is even.



term #	Script Code rel freq	Eqn 3.4 probability	Error
1	0.7649	0.7500	1.99%
2	0.6251	0.6250	0.02%
3	0.6871	0.6875	0.06%
4	0.6560	0.6563	0.04%
5	0.6715	0.6719	0.06%
6	0.6647	0.6641	0.10%
7	0.6671	0.6680	0.13%
8	0.6663	0.6660	0.04%
9	0.6666	0.6670	0.05%
10	0.6671	0.6665	0.09%

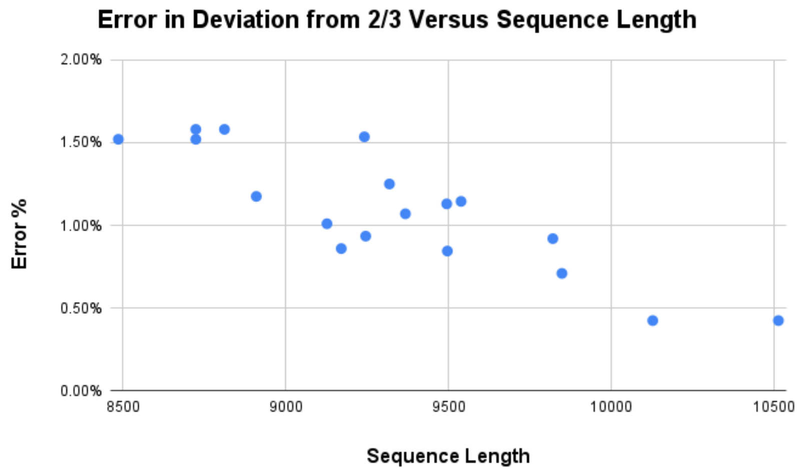
Fig 4. Probability that a term is even.

Corollary 3.3 *In an infinitely long Collatz sequence, assuming it exists, $\frac{2}{3}$ of the terms are even and $\frac{1}{3}$ are odd. The probability of a term being even far from n is 2/3 and thus 2/3 of the sample space must be even in an infinitely long sequence and approximately so in a finite one. This presumes that there are no powers of 2 encountered; otherwise, the sequence does not have an infinite number of terms. This distinction must be noted as any powers of 2 result in all successive terms being even and thus the probabilities would not be applicable.*

To numerically confirm this result for long sequences, a program was written in the Pari/GP programming language [7]. This program is shown below.

```
ColEven(n) = {
 my(even=0,total=1,pof2=0);
 if(n%2==0, even=1,0);
 while (n != 1,pof2=log(n)/log(2);
   if (truncate(pof2)==pof2,if(total==1,return(1),
                 return(even/total*1.0000001,total)));
   if (n%2 == 0,n /= 2;even=even+1,n = n * 3 + 1);
 total=total+1);}
```

Fig 5. Code to calculate the ratio of even terms to the total number of terms in a long sequence before a power of 2 is encountered.



Start Number	% Even Terms	Sequence Length	Deviation from 2/3 as %
220!	0.6768	8485	1.52%
221!	0.6724	9170	0.86%
222!	0.6745	8909	1.18%
223!	0.6734	9126	1.01%
224!	0.6729	9245	0.94%
225!	0.6768	8723	1.52%
226!	0.6772	8723	1.58%
227!	0.6723	9496	0.85%
228!	0.6772	8811	1.58%
229!	0.6738	9367	1.07%
230!	0.6695	10127	0.43%
231!	0.6714	9848	0.71%
232!	0.675	9318	1.25%
233!	0.6742	9494	1.13%
234!	0.6743	9538	1.15%
235!	0.6728	9820	0.92%
236!	0.6769	9241	1.54%
237!	0.6695	10513	0.43%
238!	0.6744	9722	1.16%
239!	0.6713	10291	0.70%
240!	0.6717	10228	0.76%

Fig 6. Percent of even terms before a power of 2 is encountered as a function of sequence length.

The data in Figure 6 was generated using the single Pari/GP command

```
for (i=220,240,ColEven(i!))
```

Figure 6 agrees with the theoretical result. The long sequences examined have a percentage of even terms close to 2/3 and the trend shows that the result becomes closer to 2/3 for larger sequences. This check has not invalidated what was proven and provides some confidence that a mistake was not made in the theoretical calculations above.

4 Proof that the Collatz Conjecture is True $\forall\ n \in \mathbf{N}^+$

Theorem 4.1 *If the numbers from 1 to n generate Collatz sequences that reach 1, this implies that n+1 generates a sequence that reaches 1.*

Proof

Assumption 4.1 *All of the numbers from 1 to n generate Collatz sequences that reach 1.*

Remark 4.1 *If the numbers from 1 to n can be shown to generate Collatz sequences that reach 1 then if any number larger than n in the course of the generation of the sequence results in a term that is between 1 to n then it will continue along an already established Collatz sequence and will reach 1.*

There are 4 possibilities with regard to a Collatz sequence generated from n+1:

(1) The sequence generated from n+1 diverges.

(2) The sequence generated from n+1 does not diverge but cycles at a number greater than n such that no term in the sequence is below n+1.

(3) The sequence generated from n+1 does not diverge and does not cycle above n but does not reach 1.

(4) The sequence reaches 1.

Remark 4.2 *If (1) , (2) and (3) are false then the only option left is (d), and thus Theorem 4.1 is true.*

Proposition 4.2 *The sequence generated from n+1 diverges.*

Proof

If a Collatz sequence diverges then it is infinitely long. This implies in such a sequence, if it exists, there are no numbers that are a power of 2, otherwise on encountering this term the sequence would proceed directly to 1 and thus not be divergent. Encountering a power of 2 would upset the probabilities that were calcuated in Theorem 3.2 therefore it is important to point out that this situation does not exist in the case of an infinite sequence.

Consider a Collatz sequence where α is the fraction of the terms that are odd and 1-α is the fraction of those that are even. Each odd term increases the first number in the sequence by approximately 3 and each even term decreases the initial term by 2. Assume that the odd terms instead of increasing the first term by approximately 3, instead increases it by 3.9 without the addition of 1. This simplifies the algebra and allows an upper bound to be calculated. Note that for $n > 1$ (trivial Collatz sequence) $3.9n > 3n + 1$. An upper bound for the kth term can then be formulated. As there are k terms, (n+1) is multiplied by 3.9 αk times and divided by two $(1 - \alpha)k$ times. This can be written as

$$C^k(n+1) < (n+1) \times \frac{3.9^{\alpha \cdot k}}{2^{(1-\alpha)\cdot k}} \tag{4.1}$$

Taking limits as k approaches infinity and noting by Theorem 3.2 that α the proportion of odd terms approaches $1 - \frac{2}{3} = \frac{1}{3}$.

$$\begin{aligned} \lim_{k\to\infty} C^k\left(n+1\right) &< \lim_{k\to\infty}\left(n+1\right) \times \frac{3.9^{\frac{1}{3}\cdot k}}{2^{\frac{2}{3}\cdot k}} \\ &= (n+1) \times \lim_{k\to\infty} \left(\frac{3.9}{4}\right)^{\frac{1}{3}\cdot k} \\ &= (n+1) \times 0 = 0 \text{ where } k, n \in \mathbf{N}^+ \end{aligned} \tag{4.2}$$

This is a contradiction. The term at infinity is $\notin \mathbf{N}^+$. Therefore the reverse of what was assumed is true. A Collatz sequence has a finite number of terms. It also implies that Collatz sequences are bounded. Without loss of generality assume the number of terms are even. Odd terms are always followed by even terms so at most k/2 terms can be odd. Each odd even

pair has the net effect of increasing the starting number n+1 by 3/2 approximately. Therefore certainly $(n+1)2^{k/2}$ exceeds any term in the sequence and is therefore an upper bound. If the number of terms are odd then $(n+1)2^{(k+1)/2}$ can be used as an upper bound.
We can conclude that a Collatz sequence does not diverge, has a finite number of terms, and is bounded.

Therefore proposition 4.2 is false. □

Proposition 4.3 *The sequence generated from n+1 does not diverge but cycles at a number greater than n such that no term in the sequence is below n+1.*

Proof

If n+1 is even then upon dividing by 2 the next term in the sequence is less than n+1 which implies the sequence reaches 1 proving for even n+1 the sequence does not cycle above n.

Consider odd n+1. Note Figure 2 where it is seen that C(n+1) = 3n+4 as n+1 is odd. The next term is (3n+4)/2 as the preceeding term is even. Then there are two possibilities: (3n+4)/4 and the term (9n+14)/2. This implies that odd n+1 generates terms of the form

$$\frac{\alpha n + \beta}{2^\delta} \text{ where } \alpha, \beta, \delta, n \in \mathbf{N}^+ \tag{4.3}$$

For proposition 4.2 to be true

$$C^k(n+1) = C^j(n+1) \text{ where } j, k, n \in \mathbf{N}^+ \text{ and } j > k > 2 \qquad (4.4)$$

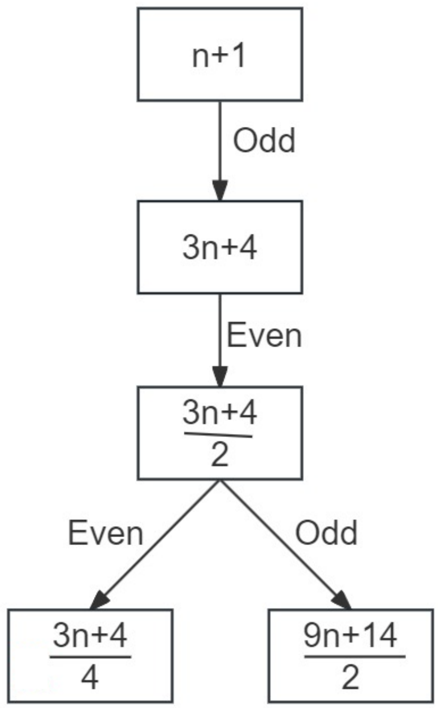


Fig 2. Tree structure showing terms that n+1 may generate

Say $C^k(n+1) = p$ and s cycles later produces p again.

$$C^k(n+1) = p = C^{k+s} \text{ where } k, n, p, s \in \mathbf{N} \qquad (4.5)$$

From observation of equations 5,6 and 7

$$p = \frac{\alpha p + \beta}{2^\delta} \text{ where } p, \alpha, \beta, \delta \in \mathbf{N}^+ \tag{4.6}$$

Comparing terms from Equation 8

$$1 = \frac{\alpha}{2^\delta} \tag{4.7}$$

$$0 = \frac{\beta}{2^\delta} \tag{4.8}$$

The only solution which satisfies equations 9 and 10 is when $\beta = 0$ and α is a power of 2, which implies p is a power of 2 which generates subsequent terms that monotonically decrease to 1 contradicting proposition 4.3. Therefore proposition 4.3 is false. □

Proposition 4.4 *The sequence does not diverge and does not cycle above n but does not reach 1.*

Proof It has been proven that Collatz sequences are bounded. Assume t equals the number of Natural numbers greater than n but less than an upper bound M and let any number in this range be equal to q. Then generate q to term t+1. Because no number repeats above n, as there is no cycling, in a worst case scenario where all the t numbers in the range were exhausted then by the pigeonhole principle term t+1 has to be in the range from 1 to n which, from remark 4.1 implies that the sequence reaches 1. This is in contradiction to Proposition 4.4 which is then false. $\square$

As propositions 4.2 ,4.3 and 4.4 are false then the only conclusion that can be reached is that Theorem 4.1 is true. $\square$

Theorem 4.5 *Collatz sequences which are generated from the Natural numbers reach 1.*

Proof

The Collatz sequence for a starting value of 1 is as follows

$$(1) \tag{4.9}$$

As n = 1 generates a Collatz sequences that reaches 1, by Theorem 4.2 this implies that n+1 = 2 also generates a Collatz sequence that reaches 1. Continuing in this manner then by induction n generates a Collatz sequence that reaches 1 $\forall\ n \in \mathbf{N}^+$. Therefore Theorem 4.5 is true. $\square$

Corollary 4.6 *There are no repeated numbers in a Collatz sequence otherwise there would be a cycle and the sequence would never reach 1.*

Corollary 4.7 *At some point in a Collatz sequence a term is encountered for the first time such that*

$$C^k(n) = 2^j \ \textit{where}\ j, n, k \in \mathbf{N}^+ \tag{4.10}$$

In other words eventually a term in a generated sequence is a power of 2. This is the only way that the sequence could eventually reach 1.

5 Conclusion

(a) Collatz sequences reach 1 for all starting values $n \in \mathbf{N}^+$.

(b) Collatz sequences do not diverge, have a finite number of terms and are bounded.

(c) Collatz sequences have distinct terms.

(d) Collatz sequences eventually reach a term that is a power of 2.

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