Goldbach Prime Pairs and its Distribution for Integers of Form $2(n+1)^2$

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Abstract

In this article, detailed study on the distribution of the Goldbach prime pairs for the even integers of form $2(n+1)^2$ were carried out. The experimental proof of the formulated conjectures were given using the algorithm. The results suggest that, there will always be Goldbach prime pair for expression, $(n)(n-1) < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - (n)(n-1)$, were $n = 1, 2, 3, ...$ Additionally, the gap between Goldbach first prime pair, that is, $(p'_f - p_f)$ was found to be always less than its corresponding n value after $n = 2538$, hence, $Gap(p'_f - p_f) << n$, where, $n = 2539, 2540, 2541, ...$

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1. Introduction

Goldbach's well-known conjecture states that *"Every even integer greater than 2 can be written as the sum of two prime numbers"* [1]. Goldbach's conjecture deals with the possibility of writing every even integer greater than 2 as sum of two prime numbers (p, p'), however, the conjecture does not deal with the distribution of such two prime numbers (p, p') or the interval in which it will certainly be falling for all the positive integers or for some selective positive integers. The Legendre's conjecture states that, *"There is a prime number between n^2 and $(n+1)^2$ for every positive integer n"* [2]. The Bertrand's postulate, which has been proven and states that *"for any $n > 1$, there is always at least one prime p such that $n < p < 2n$"* [3]. The Legendre's conjecture and Bertrand's postulate deals with the distribution of the prime numbers. Hence, with the idea of finding the possibility of ascertaining the interval in which there will be certainty

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that Goldbach prime numbers (p, p' pairs) would be present, in this article, systemic approaches have been shown to obtained the mathematical expressions which shows the possibility of the presence of the Goldbach prime pairs for the even integers of form $2(n+1)^2$, where, $n = 1, 2, 3, ...$

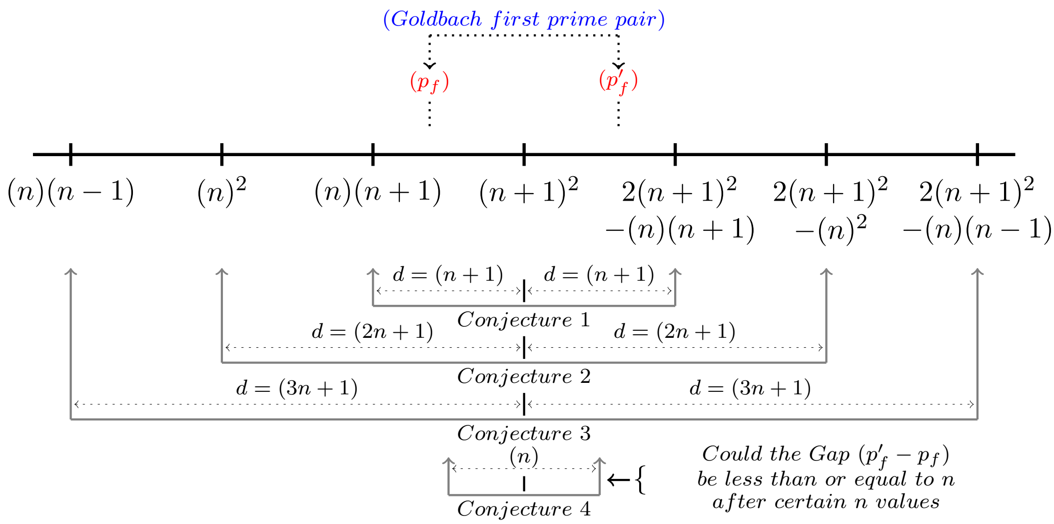

FIGURE 1. Schematic illustration of the proposed conjectures.

Conjecture 1: There is always existing a Goldbach first prime pair (p_f, p'_f), such that, $(n)(n+1) < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - (n)(n+1)$ and $p_f + p'_f = 2(n+1)^2$ and $(n+1)^2 - p_f = p'_f - (n+1)^2$, where, $n = 1, 2, 3, ... - \{4, 6, 10, 16, 19, 21, 22, 23, 30, 33, 36, 43, 48, 49, 56, 57, 61, 66, 72, 76, 81, 106, 117, 127, 130, 132, 141, 210, 276, 289, 333, 418\}$.

Conjecture 2: There is always existing a Goldbach first prime pair (p_f, p'_f), such that, $(n)^2 < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - (n)^2$ and $p_f + p'_f = 2(n+1)^2$ and $(n+1)^2 - p_f = p'_f - (n+1)^2$, where, $n = 1, 2, 3, ... - \{21, 23, 30, 33, 36, 48, 49, 117, 141\}$.

Conjecture 3: There is always existing a Goldbach first prime pair (p_f, p'_f), such that, $(n)(n-1) < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - (n)(n-1)$ and $p_f + p'_f = 2(n+1)^2$ and $(n+1)^2 - p_f = p'_f - (n+1)^2$, where, $n = 1, 2, 3, ...$

Conjecture 4: The Gap $(p'_f - p_f)$ between the Goldbach first prime pair for the even integer $2(n+1)^2$ is always less than the corresponding value of n, *where*, $n = 2539, 2540, 2541, ...$

Further, based on the results, the new conjectures were proposed for the even integers of form $2(n+1)^2$, *where*, $n = 1, 2, 3, ...$ and the distribution of its Goldbach first prime pairs. Further, the proposed new conjectures were tested using the computer algorithm till $n = 1, 2, 3, ..., 1,000,000$. Very first the **Conjecture 2** ($(n)^2 < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - (n)^2$) was proposed and tested, based on its results the narrow and broad interval were tried, which became **Conjecture 1** ($(n)(n+1) < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - (n)(n+1)$) and **Conjecture 3** (

$(n)(n-1) < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - (n)(n-1)$) respectively. Moreover, based on the observations, **Conjecture 4** has also been suggested. The proposed conjectures and its schematic illustration have been outlined in **Figure 1** for the quick overview. The detailed step-by-step explanation have been given in the further sections.

2. Goldbach conjecture, Legendre's conjecture and Bertrand's postulate

In this section, step-by-step, some of the useful concepts which thought to be necessary for the explanation and understanding of the proposed conjectures have been explained/described.

2.1 Goldbach conjecture, Goldbach prime pairs and Goldbach first prime pair

In this section, taking Goldbach conjecture as benchmark, various definitions and statements have been outlined which thought to be necessary in explaining the conjectures. Further, notes and remarks have been given at appropriate places wherever it was thought to be necessary for clearer explanation of concepts.

Note 1: In this article the term integer means positive integer only.

Statement 1 (Goldbach conjecture): *"Every even integer greater than 2 can be written as the sum of two prime numbers."*

Further, from the **Statement 1**, we can easily understand that, for every even integer, $2k$, there will always be two prime numbers exists such that their sum is equal to $2k$. Hence, the Goldbach conjecture can be expressed in the sense of existence of prime numbers as shown in **Statement 2**.

Statement 2: There is always existing prime numbers p *and* p' such that, $2k = p + p'$, *where* $k = 2, 3, 4, ...$

Lets understand **Statement 1/2** with the help of number-line. For example, all the possible ways that even integer "16" can be written as the sum of two numbers are: $(1+15), (2+14), (3+13), (4+12), (5+11), (6+10), (7+9)$ and $(8+8)$, which can be illustrated on the number-line as shown in **Figure 2**. Now, the pair of the prime-numbers whose sum gives the sum 16 needs to be from these list of pairs only as these are all the possible ways of writing 16 as sum of two numbers.

The integer, 16, is an even number, hence can also be written as $2k = 16$. Hence, the value of k will be 8, as $k = 16/2 = 8$. Further, since every positive integer is of form $2k$, it can also be written in the form of $2k = k + k$. Hence, integer, 16, can be expressed as $8 + 8$.

$$2k = k + k \tag{2.1}$$

$$Hence,\ for\ k = 8 : 2(8) = 8 + 8 \tag{2.2}$$

$$16 = 8 + 8 \tag{2.3}$$

Now, in the equation 2.1, adding and subtracting i from the right-hand side will not change the value of left-hand side ($2k$), hence, the equation 2.1 can be written as,

$$2k = k + k - i + i \tag{2.4}$$

$$2k = (k - i) + (k + i) \tag{2.5}$$

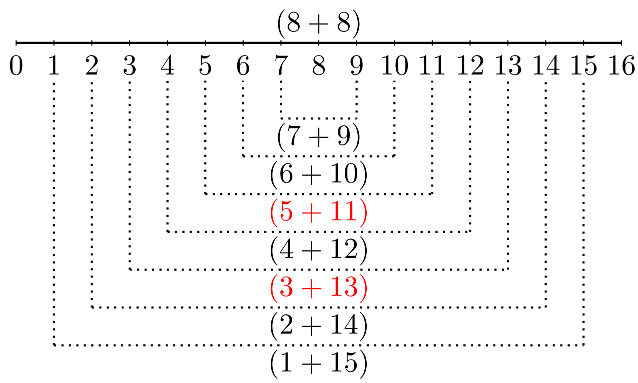

Figure 2. Schematic illustration of all pairs of writing "16" as sum of two integers on the number-line, that is, $(1+15),(2+14),(3+13),(4+12),(5+11),(6+10),(7+9)$ and $(8+8)$.

Hence, for $n=8$, the equation 2.4 and 2.5 will be,

$$2(8) = 8+8-i+i \tag{2.6}$$
$$16 = (8-i)+(8+i) \tag{2.7}$$

Now, in the equation 2.7, the value of i, we cannot take greater than 8, because, doing so, makes the term $(8-i)$ negative. And also, we do not need to take the value of i equal to 8, which makes the term $(8-i)=0$. Hence, we need to take the values of i from 0... to ...$n-1$, that is, I = { 0,1,2,3,4,5,6,7 }, so for each value of i, equation 2.7 can be written as,

$$for\ i=0:\ 16=(8-0)+(8+0)=8+8 \tag{2.8}$$
$$for\ i=1:\ 16=(8-1)+(8+1)=7+9 \tag{2.9}$$
$$for\ i=2:\ 16=(8-2)+(8+2)=6+10 \tag{2.10}$$
$$for\ i=3:\ 16=(8-3)+(8+3)=5+11 \tag{2.11}$$
$$for\ i=4:\ 16=(8-4)+(8+4)=4+12 \tag{2.12}$$
$$for\ i=5:\ 16=(8-5)+(8+5)=3+13 \tag{2.13}$$
$$for\ i=6:\ 16=(8-6)+(8+6)=2+14 \tag{2.14}$$
$$for\ i=7:\ 16=(8-7)+(8+7)=1+15 \tag{2.15}$$

Hence, from the above example, if we need to generate all the pairs in which the given even integer can be written as sum of two numbers can be given in the following expression,

$$2(k)\ =\ f(k)\ = k \pm (i),\ where,\ i=0,1,2,3,....k-1 \tag{2.16}$$

Further, it is noted that, the value of i starts from $i=0$ and ends at $i=k-1$, hence there are total $k-1+1$ (plus 1 for the item 0) items in the set I, and $k-1+1=k$. Hence, it can be stated that, there will be k pairs of numbers in which even integer can be written as sum of two integers. Moreover, the distance (d) between the each pair will be $d=2(i)$, as $d=(k+i)-(k-i)=i+1=2(i)$. Hence, these facts can be put as a statement in the following way.

Statement 3: If $k=1,2,3,...$, there are total k pairs of integers to write $2(k)$ as sum of two integers, and all the pairs are obtained by the function $f(k)$,
$2(k)\ =\ f(k)\ = k \pm (i),\ where,\ i=0,1,2,3,...,k-1.$

And, the distance (d $value$) between each pairs is $d = (k+i) - (k-i) = 2(i)$.

Further, from the **statement 3**, the **statement 3a** can be derived as below.

Statement 3a: Based on **Statement 3**, all the pairs to write the $2(k)$ as sum of two integers are always located at equal distance from the k.

Further, it can also be understand that, **Figure 1**, is nothing but the graphical representation of the **Statement 3** for the even integer $2k = 2(8) = 16$. Further, from the **equation 2.16**, it is also true that, $(k-i) < k < (k+i)$.

Note 2: If $k = 2, 3, 4, ...$ then for any value of k, the expression $(k-i) < k < (k+i)$ is always true for all the values of i, where $i = 0, 1, 2, 3, ..., k-1$.

Hence, from the **Statement 2/3** and **Note 2**, the Goldbach conjecture can also be written as,

Statement 4: If $k = 2, 3, 4, ...$ then for every integer $2k$, there is always existing atleast one i from the values $0, 1, 2, 3, ..., (k-1)$ such that, $(k-i) = p < (k) < (k+i) = p'$, where, $p + p' = 2k$ and p, p' are prime numbers.

Now, the definition of Goldbach prime pairs can also be derived from the **Statement 4** as below,

Definition 1: Goldbach prime pairs (G_{2k}): If $k = 2, 3, 4, ...$ then for integer $2k$, there is always existing atleast one or more i from $0, 1, 2, 3, ..., (k-1)$ such that, $(k-i) = p < (k) < (k+i) = p'$, where, $p + p' = 2k$ and p, p' are prime numbers. And, the prime pairs p, p' are defined as the Goldbach prime pairs for $2k$.

Now, from the **Definition 1**, it also implies that, for each Goldbach prime pair for integer $2k$, there will be only one and one i value associated with that particular Goldbach prime pair. Hence, it also implies that, for value $2k$, number of Goldbach prime pairs will equal to number of i values, $where$, $(k-i) = p$ and $(k+i) = p'$. Further, we can put this discussion in the form of Definition as below,

Definition 2: Total number of Goldbach prime pairs (G_{2k}^T): If $k = 2, 3, 4, ...$ then for integer $2k$, there is always existing atleast one or more i from $0, 1, 2, 3, ..., (k-1)$ such that, $(k-i) = p < (k) < (k+i) = p'$, where, $p + p' = 2k$ and p, p' are prime numbers. And, the prime pairs p, p' are defined as the Goldbach prime pairs. And, Total Number of Goldbach prime pairs (G_{2k}^T) = Total number of i value for which Goldbach prime pairs are existing.

Now, to define the Goldbach first prime pair (G_{2k}^f) for even integer $2k$, lets see the **Figure 1** again. We knew that for integer 16, there are two Goldbach prime pairs, that is, $(5+11)$ and $(3+13)$. The $(5+11)$ is associated with the i value 3 and $(3+13)$ is associated with the i value 5 as per the **Definition 1**. There is no Goldbach prime pair for i values less then 3. Hence, the Goldbach prime pair which is associated with the least i value among all the Goldbach prime pair for that particular even integer is defined as the Goldbach first prime pair (G_{2k}^f). In other words, the Goldbach prime pairs which is most near to the k value on the number-line is defined as the Goldbach first prime pair. In another example, as shown in the **Figure 3**, integer 50 has total 4 Goldbach prime pairs.

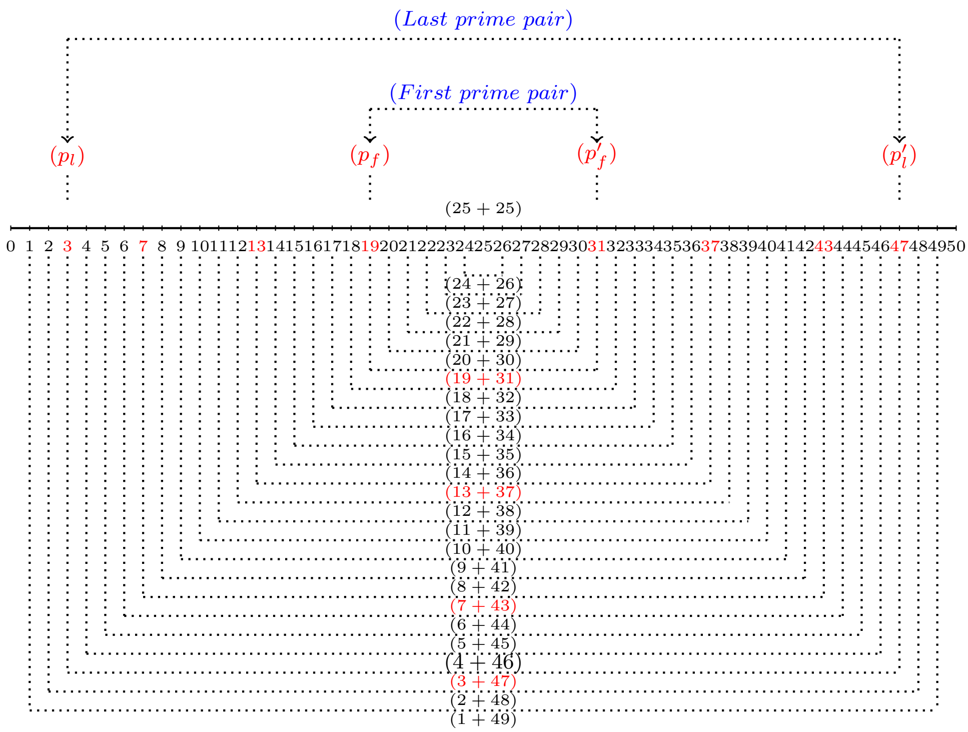

FIGURE 3. Goldbach prime pairs and Goldbach first prime pair for integer "50".

Definition 3: Goldbach first prime pair (G^f_{2k}): If $k = 4, 5, 6, ...$ then for integer $2k$, there is always existing atleast one or more i from $1, 2, 3, ..., (k-1)$, such that, $(k-i) = p < (k) < (k+i) = p'$, where, $p + p' = 2k$ and p, p' are prime numbers. And, the prime pair p, p' which is associated with the lowest i value (note: $i \neq 0$) is defined as Goldbach first prime pair (G^f_{2k}). In other words, the Goldbach first prime pair is the prime pair which has the lowest d *value*, that is, $d = (k+i) - (k-i) = 2(i)$. To distinguished it, this pair is denoted as (p_f, p'_f).

For example, as shown in **Figure 3**, for the integer 50, the total Goldbach prime pairs are 4, that is, $(19+31), (13+37), (7+43), (3+47)$. Among them, the d *value* for $(19+31)$ is $d = 31-19 = 12$, which is lowest among all the Goldbach prime pairs. Hence for integer 50, the Goldbach first prime pair is $(19+31)$. While, $(3+47)$ is the last Goldbach prime pair.

Note 3: All the integers, $2k$, where the k is the prime numbers, will automatically becomes sum of two prime numbers, for example, integer, $26 = 2(13) = 13 + 13$. In this article, we are not counting these pairs as way of writing the even integer as sum of two prime numbers, and has not been counted as Goldbach prime pair or Goldbach first prime pair as these are the obvious prime pairs. Hence, in **Definition 3**, $i \neq 0$ is taken and k value was started from the 4 as 4 and 6 has the only one Goldbach prime pair, that is, $6 = (3+3)$, $4 = (2+2)$.

Now, as per the **Statement 3** and **Definition 1**, all the pair of writing $2(k)$ as sum of two integers will be located in the range 1 to $2k$ only, as putting $i = k - 1$ in the **expression 2.17** will give this range, where the value of $k-1$ gives the very last pairs, that will always be $(1, 2k-1)$. As previously stated, this last pair is not considered as the prime pairs as 1 is not a prime.

$$(k-i) <(k)< (k+i) \tag{2.17}$$
$$(k-(k-1)) <(k)< (k+(k-1)) \tag{2.18}$$
$$1 <(k)< (2k-1) \tag{2.19}$$

Hence, we can write that, all the Goldbach prime pairs is located in the range of 1 to 2k. Hence, the Goldbach first prime pair will also be between these range/interval ($1\ to\ 2k$) only.

Definition 4: Distribution of Goldbach prime pair (G_{2k}): If $k = 4, 5, 6, ...$ then for integer $2k$, all the Goldbach prime pairs are located in the range/interval of $1 < (k) < 2k$.

Definition 5: Distribution of Goldbach first prime pair (G^f_{2k}): If $k = 4, 5, 6, ...$ then for integer $2k$, the Goldbach first prime pair is located the range of $1 < (k) < 2k$.

Now, **Definitions 4/5**, have been derived for all the even integers greater than 7 as we have taken $2(k)$, $where$, $k = 4, 5, 6, ...$ Hence all the above is true for all the even numbers or any set of even numbers or any selected even numbers that is greater than 7 on the number-line. For example, instead of $2k$, if we take $2(n)^2$, $where$, $n = 2, 3, 4, ...$ that is nothing but even integer $8, 18, 32, ...$ Broadly, $2k = 2(Integer)$, and $2(Integer)$ will always gives the even integer. Hence, we can put any mathematical expression at the place of k, whose value is from $k = 4, 5, 6, ...$
For example, all the previous statements and definitions are true to following expression of k as well, for example, $k = n\ or\ (n+1)\ or\ (n+1)^2$ etc.

$$for\ n = 4, 5, 6, ... :\ 2(n) \Rightarrow\ is\ even\ integer \tag{2.20}$$
$$hence,\ for\ n = 3, 4, 5, ... :\ 2(n+1) \Rightarrow\ is\ even\ integer \tag{2.21}$$
$$and\ for\ n = 1, 2, 3, 4, ... :\ 2(n+1)^2 \Rightarrow\ is\ even\ integer \tag{2.22}$$

Now, we can write $2(n+1)^2 = 2[n^2 + 2(n) + 1] = 2k$, where k is from $k = 2, 3, 4, ...$ only, hence, $2(n+1)^2$ is nothing but even integer.
Hence, from the **Definition 5**, the distribution of the Goldbach first prime pair for even integers of form $2(n+1)^2$, $where$, $n = 1, 2, 3, ...$ can be shown as,

$$1 < p < \quad (k) \quad < p'_f < 2k \tag{2.23}$$
$$1 < p_f <(n+1)^2< p'_f < 2(n+1)^2 \tag{2.24}$$

Hence, as per the equation 2.24, the distribution of the Goldbach first prime pair for $2(n+1)^2$ can be given as per the **Definition 6**.

Definition 6: Distribution of Goldbach first prime pair for $2(n+1)^2$ ($G^f_{2(n+1)^2}$): If $n = 1, 2, 3, 4, ...$ then for integer $2(n+1)^2$, the Goldbach first prime pair will be in the range of $1 < p_f < (n+1)^2 < p'_f < 2(n+1)^2$.

2.2 Legendre's conjecture

As mentioned in the introduction section, the Legendre's conjecture states that, *there is a prime number between n^2 and $(n+1)^2$ for every positive integer n*. Hence, we can write as,

$$(n)^2 < p < (n+1)^2, where\ n = 1, 2, 3, ... \tag{2.25}$$

2.3 Bertrand's postulate

The Bertrand's postulate, which has been proven and the less restrictive form states that *for any $n > 1$, there is always at least one prime p such that $n < p < 2n$.* Hence,

$$(n) < p < 2(n), \ where\ n = 1, 2, 3, ... \tag{2.26}$$

Now, if we put the value of n as $2(n + 1)^2$, hence,

$$(n + 1)^2 < p < 2(n + 1)^2, \ where\ n = 1, 2, 3, ... \tag{2.27}$$

3. Establishing inter-connection between three conjecture(/postulate)

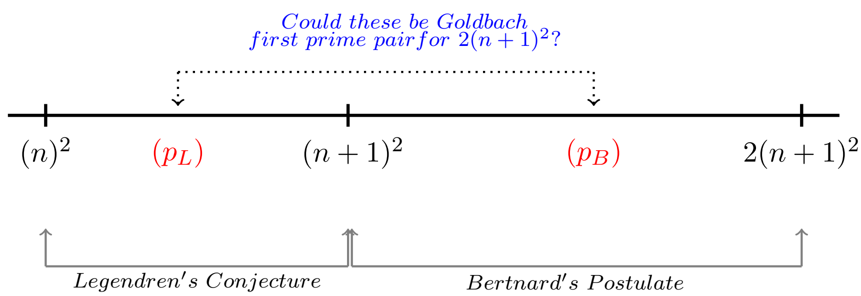

Figure 4. Schematic illustration of the inter-connection of the Legendren's conjectures, Bertnard's postulate, and Goldbach conjecture.

Now, in above mentioned three equations (equation, 2.24, 2.25, 2.27) we can clearly see, in these expressions, the term $(n + 1)^2$ is common, hence we can make the common expression as shown in Figure 4.
Re-writing all equations.

$$\begin{aligned} Goldbach\ conjecture:&\ 1 < p_f < (n + 1)^2 < p'_f < 2(n + 1)^2 \\ Legendre's\ conjecture:&\ (n)^2 < p_L < (n + 1)^2 \\ Bertrand's\ postulate:&\ (n + 1)^2 < p_B < 2(n + 1)^2 \\ where\ n = 1, 2, 3, ... \end{aligned}$$

Hence, lets first write, Legendre's and Bertrand's conjecture as shown below and lets compare it with the Goldbach's expression.

$$\begin{aligned} Legendre's\ +\ Bertrand's:&\ (n)^2 < p_L < (n + 1)^2 < p_B < 2(n + 1)^2 \\ Goldbach\ conjecture:&\ 1 < p_f < (n + 1)^2 < p'_f < 2(n + 1)^2 \end{aligned}$$

Hence, above expressions will be written in one combined expression as below,

$$(n)^2 < p_f < (n + 1)^2 < p'_f < 2(n + 1)^2 \tag{3.1}$$

Now, as shown in the **Definition 3**, $(k - i) = p_f\ and\ (k + i) = p'_f$, Hence, if the p_f will be in the range, $(n)^2 < p_f < (n + 1)^2$, then p'_f will be in the range $(n + 1)^2 + (n + 1)^2 - (n)^2 = 2(n + 1)^2 - (n)^2$, Hence replacing $2(n + 1)^2$ with the $2(n + 1)^2 - (n)^2$ as we do not need to extend the interval till $2(n + 1)^2$.

Hence, equation 3.1 could be written as follows,

$$(n)^2 < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - (n)^2 \quad where,\ n = 1,2,3,4,... \tag{3.2}$$

4. Experimental proof Part 1: Checking the validity of the conjectures

4.1 For expression: $(n)^2 < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - (n)^2$

The expression of **Equation 3.2**, has been checked till $n = 1,000,000$, using the code and results have been outlined in the **Table 1**. Out of 1,000,000 values following 9 values were found to be falling out of the interval.

$$n = 1,2,3,4... - \{21,23,30,33,36,48,49,117,141\} \tag{4.1}$$

Hence, the above results have been checked till $n = 1,000,000$, which has been found true after 141 and till 1,000,000, and may be true beyond $n = 1,000,000$. Hence, we can formulate the conjecture based on these results as **Conjecture 2**.

Conjecture 2: There is always existing a Goldbach first prime pair (p_f, p'_f), such that, $(n)^2 < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - (n)^2$ and $p_f + p'_f = 2(n+1)^2$ and $(n+1)^2 - p_f = p'_f - (n+1)^2$, where, $n = 1,2,3,... - \{21,23,30,33,36,48,49,117,141\}$.

Table 1: Data for proposed conjecture 2

n	n^2	p	$(n+1)^2$	p'	$2(n+1)^2 - n^2$	$2(n+1)^2$	$p+p'$	$G^T_{2(n+1)^2}$	$G^{Interval}_{2(n+1)^2}$
21	441	421	484	547	527	968	968	17	0
23	529	521	576	631	623	1152	1152	36	0
30	900	883	961	1039	1022	1922	1922	30	0
33	1089	1063	1156	1249	1223	2312	2312	35	0
36	1296	1291	1369	1447	1442	2738	2738	37	0
48	2304	2281	2401	2521	2498	4802	4802	64	0
49	2401	2383	2500	2617	2599	5000	5000	76	0
117	13689	13627	13924	14221	14159	27848	27848	217	0
141	19881	19819	20164	20509	20447	40328	40328	283	0

4.2 For expression: $(n)(n-1) < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - (n)(n-1)$

Further, since total 9 values were found to be deviating in conjecture 2, the interval was slightly widened in hope to incorporate these values by changing the left-hand side interval from $(n)^2$ to $(n)(n+1)$. Hence, the expression of the conjecture is to be written as,

$$(n)(n-1) < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - (n)(n-1) \quad where,\ n = 1,2,3,4,... \tag{4.2}$$

The expression of the equation 4.2 was checked for the n=1,000,000, and none of the values were falling out of the interval (**Table 2**). Hence, the equation 4.2 is to be written in the form of conjecture as below,

Conjecture 3: There is always existing a Goldbach first prime pair (p_f, p'_f), such that, $(n)(n-1) < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - (n)(n-1)$ and $p_f + p'_f = 2(n+1)^2$ and $(n+1)^2 - p_f = p'_f - (n+1)^2$, where, $n = 1,2,3,....$

Table 2: Data for proposed conjecture 3

n	*n(n-1)*	*p*	*(n+1)^2*	*p'*	*2(n+1)^2 − n(n-1)*	*2(n+1)^2*	*p + p'*	$G^T_{2(n+1)^2}$	$G^{Interval}_{2(n+1)^2}$
21	420	421	484	547	548	968	968	17	1
23	506	521	576	631	646	1152	1152	36	2
30	870	883	961	1039	1052	1922	1922	30	1
33	1056	1063	1156	1249	1256	2312	2312	35	1
36	1260	1291	1369	1447	1478	2738	2738	37	2
48	2256	2281	2401	2521	2546	4802	4802	64	1
49	2352	2383	2500	2617	2648	5000	5000	76	1
117	13572	13627	13924	14221	14276	27848	27848	217	2
141	19740	19819	20164	20509	20588	40328	40328	283	2

4.3 For expression: $(n)(n+1) < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - (n)(n+1)$

Further, from the observation of the Conjecture 2/3, it was observed that, beyond one particular n value, the conjecture holds true. Hence, the question arises, can we shorten the interval more around the $(n+1)^2$, although with the expense of some initial n values which falls outside these interval, but becomes true for all the number n thereafter. Hence, the interval $(n)^2$, changed to $(n)(n+1)$, and the expression can be written as below,

$$(n)(n+1) < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - (n)(n+1)^2 \quad where,\ n = 1, 2, 3, 4, ... \tag{4.3}$$

The expression of the equation 4.3 was checked for the n=1,000,000, and the 32 values were found to be falling out of the range (**Table 3**),
$n = 1, 2, 3, 4... - \{4, 6, 10, 16, 19, 21, 22, 23, 30, 33, 36, 43, 48, 49, 56, 57, 61, 66, 72, 76, 81, 106, 117, 127, 130, 132, 141, 210, 276, 289, 333, 418.\}$ While, from the 419, the equation was found to be true. Hence, in the form of the conjecture, it can be written as below,

Conjecture 1: There is always existing a Goldbach first prime pair (p_f, p'_f), such that, $(n)(n+1) < p_f < (n+1)^2 < p'_f < 2(n+1)^2 - (n)(n+1)$ and $p_f + p'_f = 2(n+1)^2$ and $(n+1)^2 - p_f = p'_f - (n+1)^2$, where, $n = 1, 2, 3, ... - \{4, 6, 10, 16, 19, 21, 22, 23, 30, 33, 36, 43, 48, 49, 56, 57, 61, 66, 72, 76, 81, 106, 117, 127, 130, 132, 141, 210, 276, 289, 333, 418\}$.

Table 3: Data for proposed conjecture 1

n	*n(n+1)*	*p*	*(n+1)^2*	*p'*	*2(n+1)^2 − n(n+1)*	*2(n+1)^2*	*p + p'*	$G^T_{2(n+1)^2}$	$G^{Interval}_{2(n+1)^2}$
4	20	19	25	31	30	50	50	4	0
6	42	37	49	61	56	98	98	3	0
10	110	103	121	139	132	242	242	8	0
16	272	271	289	307	306	578	578	12	0
19	380	379	400	421	420	800	800	21	0
21	462	421	484	547	506	968	968	17	0
22	506	487	529	571	552	1058	1058	19	0
23	552	521	576	631	600	1152	1152	36	0
30	930	883	961	1039	992	1922	1922	30	0
33	1122	1063	1156	1249	1190	2312	2312	35	0
36	1332	1291	1369	1447	1406	2738	2738	37	0
43	1892	1879	1936	1993	1980	3872	3872	52	0
48	2352	2281	2401	2521	2450	4802	4802	64	0
49	2450	2383	2500	2617	2550	5000	5000	76	0
56	3192	3191	3249	3307	3306	6498	6498	151	0
57	3306	3271	3364	3457	3422	6728	6728	76	0
61	3782	3769	3844	3919	3906	7688	7688	78	0
66	4422	4357	4489	4621	4556	8978	8978	89	0

Continued on the next page...

Table 3: Data for proposed conjecture 1 (contd...)

n	*n(n+1)*	*p*	*(n+1)*^{2}	*p'*	*2(n+1)*^{2} − *n(n+1)*	*2(n+1)*^{2}	*p + p'*	$G^T_{2(n+1)^2}$	$G^{Interval}_{2(n+1)^2}$
72	5256	5227	5329	5431	5402	10658	10658	105	0
76	5852	5851	5929	6007	6006	11858	11858	142	0
81	6642	6619	6724	6829	6806	13448	13448	130	0
106	11342	11311	11449	11587	11556	22898	22898	184	0
117	13806	13627	13924	14221	14042	27848	27848	217	0
127	16256	16249	16384	16519	16512	32768	32768	244	0
130	17030	17029	17161	17293	17292	34322	34322	267	0
132	17556	17551	17689	17827	17822	35378	35378	316	0
141	20022	19819	20164	20509	20306	40328	40328	283	0
210	44310	44269	44521	44773	44732	89042	89042	544	0
276	76452	76441	76729	77017	77006	153458	153458	864	0
289	83810	83737	84100	84463	84390	168200	168200	1259	0
333	111222	111043	111556	112069	111890	223112	223112	1129	0
418	175142	175129	175561	175993	175980	351122	351122	1670	0

5. Experimental proof Part 2: Analysis of Gap $(p'_f - p_f)$

In the previous part systemic study has been carried out to study the distribution of the goldbach first prime pair. The interval of the proposed Conjecture 1 is $(n+1)^2 \pm (n+1)$, while for the conjecture 2, it is $(n+1)^2 \pm (2n+1)$ and for conjecture 3, it is $(n+1)^2 \pm (3n+1)$. Further, from these results, we found that, if we tighten the interval around $(n+1)^2$, few n values falls out of the interval, however, after one particular n value, the conjecture becomes true, as we have seen previously. Now, with the objective of even further narrow-down the interval, the interval $(n+1)^2 \pm (n/2)$ were tried. In other words, it can be stated that, the $Gap(p'_f - p_f)$ is lower than the corresponding value of n or not. Hence, the following expression were tested using computer code,

$$Gap\ (p'_f - p_f) < n, where,\ n = 1, 2, 3, 4, ... \tag{5.1}$$

As, shown in the **Table 4**, after the $n = 2539$, the $Gap(p'_f - p_f)$ is found to be less than the corresponding value of n. Hence, the Conjecture 4 was proposed, which has been tested till $n = 1,000,000$ and found to be true.

Conjecture 4: *The Gap $(p'_f - p_f)$ between the Goldbach first prime pair for the even integer $2(n+1)^2$ is always less than the corresponding value of n, where, $n = 2539, 2540, 2541, ...$*

Table 4: Data for proposed conjecture 1

n	*Gap*	*n*	*Gap*	*n*	*Gap*	*n*	*Gap*	*n*	*Gap*
1	2	46	60	109	114	280	480	576	840
2	4	48	240	117	594	285	294	621	630
3	6	49	234	121	126	289	726	642	756
4	12	52	84	125	230	292	360	732	744
5	10	53	74	127	270	330	516	742	924
6	24	56	116	130	264	331	414	1278	1320
10	36	57	186	132	276	333	1026	1360	1524
12	24	61	150	136	180	336	420	1596	1896
15	30	62	76	141	690	348	384	2538	2616
16	36	66	264	156	264	357	426	2539	822
18	24	69	138	157	246	364	384	2540	116
19	42	72	204	172	336	366	636	2541	990
21	126	75	150	175	210	399	402	2542	240

Continued on the next page...

Table 4: Data for proposed conjecture 1 (contd...)

n	*Gap*	*n*	*Gap*	*n*	*Gap*	*n*	*Gap*	*n*	*Gap*
22	84	76	156	183	210	405	426	2543	34
23	110	81	210	186	240	418	864	2544	336
25	30	85	126	198	204	420	504	2545	186
27	54	87	90	204	228	424	492	2546	364
29	38	93	114	210	504	426	540	2547	66
30	156	95	130	218	236	447	534	2548	276
33	186	97	114	228	240	451	594	2549	94
36	156	99	138	238	276	460	720	2550	180
39	42	101	146	252	456	505	654	.	.
41	46	105	126	262	336	537	546	.	.
43	114	106	276	270	504	549	558	.	.
45	54	107	134	276	576	556	636		

6. Discussion

The present conjectures, provides the significant input in certain the prime numbers for the Goldbach conjecture. Proof of this conjecture not only, help in provind the Goldbach conjecture, but also, how the prime numbers are distributed. Conjecture 1/2/3 provides the systemic approach for ascertaining the interval in which there is sure that, the Goldbach prime pair.

7. Limitation

The present study has been done on very limited number of n values, that is, $1,000,000$ due to the limited computation power. Further, as mentioned in the introduction section, this study is focused on the even integers of form $2(n+1)^2$, not all the form of integers.

8. Conclusion & Outlook

Goldbach conjecture deals with writing the the every even number greater than 2 as sum of two prime numbers. However, there is entirely area oper for ascertaining that, how this pair of prime numbers are distributed. further, these conjectures has been derived with the help of otehr conjectures, that is, Legendre's Conjecture and Bernard's postulate, which provides the scope of combining these conjecture all together and can be new generally conjecture can be made.

This study can be extended for the other power value as well. For example, in the previous sections, we have seen for the $Power = 2$. In the following table, the Conjecture 4 has been checked for $Power = 2, 3, ..., 10$. Here, we can see, as the power is increasing the value of the n from where the expression, $\boldsymbol{Gap}(p'_f - p_f)$, holds true is gradually increasing.

Table 5: Data for proposed Conjecture 4 with various power in equation $2(n+1)^{Power}$

Power	$\boldsymbol{Gap}$ $(p'_f - p_f) < n$
$2(n+1)^2$	$n = 2539, 2540, 2541, ...1,000,000$
$2(n+1)^3$	$n = 8614, 8615, 8616, ...1,000,000$
$2(n+1)^4$	$n = 20869, 20870, 20871, ...1,000,000$
$2(n+1)^5$	$n = 32072, 32073, 32074, ...1,000,000$
$2(n+1)^6$	$n = 72593, 72594, 72595, ...1,000,000$
$2(n+1)^7$	$n = 81440, 81441, 81442, ...1,000,000$

Continued on the next page...

Table 5: Data for proposed conjecture 3 (contd...)

n	$Gap(p'_f - p_f)$
$2(n+1)^8$	$n = 131482, 131483, 131484, ...1,000,000$
$2(n+1)^9$	$n = 160744, 160745, 160746, ...1,000,000$
$2(n+1)^{10}$	$n = 257987, 257988, 257989, ...1,000,000$
.	.
.	.
$2(n+1)^n$	$n = ?, ?, ?, ..., n$

For example, for $Power = 2$, the value of n from where the expression starts true is 2539, while for the $power = 3$, the value of n is 8614. How these values are increasing as the power increasing and what could be for the $power = n$, that is, $2(n+1)^n$. Hence, this could be different set of study, the relationship of power and the first prime pair gap.

References

1 Weisstein, Eric W. "Goldbach Conjecture." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/GoldbachConjecture.html (p. 1)

2 Weisstein, Eric W. "Legendre's Conjecture." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/LegendresConjecture.html (p. 1)

3 Sondow, Jonathan and Weisstein, Eric W. "Bertrand's Postulate." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/BertrandsPostulate.html (p. 1)

Annexure I
(Code for Conjecture 1)

Listing 1. File-1

```
#Code starts
from left import *
from right import *
from decimal import *
import math

#Creating list
list_items = []

def creating_list(start_number, end, r):
    while(end >= r):
        item_of_list = 2*((r + 1)*(r + 1))
        list_items.append(item_of_list)
        r = r+1
creating_list(4, 1000000, 1)
print(f"$ n $ & $ n*(n+1) $ & $ p $ & $ (n+1)2 $ & $ p' $ & $ 2(n+1)2 - n*(n+1)
   $ & $ 2(n+1)2 $ & $ p+p' $ & $ total p-pairs $ & $ pairs in bound $")

# Applying conjecture on each number of list
def accessing_items():
    for number in list_items:
        def items_a (number):
            list_A = []
            list_z = []
            def adding_range(number):
                i=0
                half_num = int(number/2)
                while(i < half_num):
                    left_side = int(half_num - 1*(i))
                    right_side = int(half_num + 1*(i))
                    if (left_side + right_side) == number:
                        if (left_side != 1):
                            if (left_side_check(left_side) != 0) & (
                               right_side_check(right_side) != 0):
                                    pair = (left_side,right_side)
                                    list_A.append(pair)
                                    #n*(n+1)
                                    if ((int(math.sqrt(int(number/2))))-1)*(((
                                        int(math.sqrt(int(number/2))))-1)+1) <=
                                        left_side:
                                         list_z.append(pair)
                    i=i+1
                # n
                A = ((int(math.sqrt(int(number/2))))-1)
                # p, first component of pair
                C = list_A[0][0]
                # p', second component of pair
                D = list_A[0][1]
                # (n+1)
                E = (((int(math.sqrt(int(number/2))))-1)+1)
                # (n+1)^2
                F = (((int(math.sqrt(int(number/2))))-1)+1)*(((int(math.sqrt(int
                   (number/2))))-1)+1)
                # 2(n+1)^2
                G = 2*((((int(math.sqrt(int(number/2))))-1)+1)*(((int(math.sqrt(
                   int(number/2))))-1)+1))
                # 2(n+1)2 - n(n+1)
                L = G - A*E
```

```
                #gap
                gap = D - C
                t_pairs = len(list_A)
                t_l_pairs = len(list_z)
                #if A <= 1000:
                print(f"$ {A} $ & $ {A*E} $ & $ {C} $ & $ {F} $ & $ {D} $ & $ {L
                    } $ & $ {G} $ & $ {C+D} $ & $ {t_pairs} $ & $ {t_l_pairs} $
                    \\\\")
                if (C- A*E) < 0:
                    print("Pre_fail")
                if (L - D) < 0:
                    print("Post_fail")
                if (C == A*E):
                    print("Similar")
                if (L == D):
                    print("Similar")
            adding_range(number)
        items_a(number)
accessing_items()
#Code ends
```

Note: After $n = 418$, the above code can be easily modified to not to check Total prime pairs, instead, just check those prime pairs which is falling in the interval to check the Conjecture. These is no need to check all the prime pairs for all the n values.

LISTING 2. File-1a

```
#Code starts
import sympy
from sympy import isprime

def left_side_check(left_side):
    if isprime(left_side):
        return 1
    else:
        return 0
#Code ends
```

LISTING 3. File-1b

```
#Code starts
import sympy
from sympy import isprime

def right_side_check(right_side):
    if isprime(right_side):
        return 1
    else:
        return 0
#Code ends
```

Annexure II
(Code for Conjecture 2)

Listing 4. File-2

```
#Code starts
from left import *
from right import *
from decimal import *
import math

#Creating list
list_items = []

def creating_list(start_number, end, r):
    while(end >= r):
        item_of_list = 2*((r + 1)*(r + 1))
        list_items.append(item_of_list)
        r = r+1
creating_list(4, 500000, 1)
print(f"$ n $ & $ n^2 $ & $ p $ & $ (n+1)2 $ & $ p' $ & $ 2(n+1)2 - n^2 $ & $ 2(
    n+1)2 $ & $ p+p' $ & $ total p-pairs $ & $ pairs in bound $")

# Applying conjecture on each number of list
def accessing_items():
    for number in list_items:
        def items_a (number):
            list_A = []
            list_z = []
            def adding_range(number):
                i=0
                half_num = int(number/2)
                while(i < half_num):
                    left_side = int(half_num - 1*(i))
                    right_side = int(half_num + 1*(i))
                    if (left_side + right_side) == number:
                        if (left_side != 1):
                            if (left_side_check(left_side) != 0) & (
                                right_side_check(right_side) != 0):
                                    pair = (left_side,right_side)
                                    list_A.append(pair)

                                    #n^2
                                    if ((int(math.sqrt(int(number/2))))-1)*((int
                                        (math.sqrt(int(number/2))))-1) <=
                                        left_side:
                                         list_z.append(pair)

                    i=i+1

                # n
                A = ((int(math.sqrt(int(number/2))))-1)
                # p, first component of pair
                C = list_A[0][0]
                # p', second component of pair
                D = list_A[0][1]
                # (n+1)
                E = (((int(math.sqrt(int(number/2))))-1)+1)
                # (n+1)^2
                F = (((int(math.sqrt(int(number/2))))-1)+1)*(((int(math.sqrt(int
                    (number/2))))-1)+1)
                # 2(n+1)^2
                G = 2*((((int(math.sqrt(int(number/2))))-1)+1)*(((int(math.sqrt(
```

```
                    int(number/2))))-1)+1))
                # 2(n+1)2 - n^2
                L = G - A*A

                #gap
                gap = D - C
                t_pairs = len(list_A)
                t_l_pairs = len(list_z)

                print(f"$ {A} $ & $ {A*A} $ & $ {C} $ & $ {F} $ & $ {D} $ & $ {L
                    } $ & $ {G} $ & $ {C+D} $ & $ {t_pairs} $ & $ {t_l_pairs} $
                    \\\\")
                if (C- A*A) < 0:
                    print("Pre_fail")
                if (L - D) < 0:
                    print("Post_fail")
                if (C == A*A):
                    print("Similar")
                if (L == D):
                    print("Similar")
            adding_range(number)
        items_a(number)
accessing_items()
#Code ends
```

LISTING 5. File-2a

```
#Code starts
import sympy
from sympy import isprime

def left_side_check(left_side):
    if isprime(left_side):
        return 1
    else:
        return 0
#Code ends
```

LISTING 6. File-2b

```
#Code starts
import sympy
from sympy import isprime

def right_side_check(right_side):
    if isprime(right_side):
        return 1
    else:
        return 0
#Code ends
```

Annexure III
(Code for Conjecture 3)

Listing 7. File-3

```
#Code starts
from left import *
from right import *
from decimal import *
import math

#Creating list
list_items = []

def creating_list(start_number, end, r):
    while(end >= r):
        item_of_list = 2*((r + 1)*(r + 1))
        list_items.append(item_of_list)
        r = r+1
creating_list(4, 500000, 1)
print(f"$ n $ & $ n*(n-1) $ & $ p $ & $ (n+1)2 $ & $ p' $ & $ 2(n+1)2 - n*(n-1)
    $ & $ 2(n+1)2 $ & $ p+p' $ & $ total p-pairs $ & $ pairs in bound $")

# Applying conjecture on each number of list
def accessing_items():
    for number in list_items:
        def items_a (number):
            list_A = []
            list_z = []
            def adding_range(number):
                i=0
                half_num = int(number/2)
                while(i < half_num):
                    left_side = int(half_num - 1*(i))
                    right_side = int(half_num + 1*(i))
                    if (left_side + right_side) == number:
                        if (left_side != 1):
                            if (left_side_check(left_side) != 0) & (
                                right_side_check(right_side) != 0):
                                    pair = (left_side,right_side)
                                    list_A.append(pair)

                                    #n*(n-1)
                                    if ((int(math.sqrt(int(number/2))))-1)*(((
                                        int(math.sqrt(int(number/2))))-1)-1) <=
                                        left_side:
                                         list_z.append(pair)
                    i=i+1

                # n
                A = ((int(math.sqrt(int(number/2))))-1)
                # p, first component of pair
                C = list_A[0][0]
                # p', second component of pair
                D = list_A[0][1]
                # (n-1)
                E = (((int(math.sqrt(int(number/2))))-1)-1)
                # (n+1)^2
                F = (((int(math.sqrt(int(number/2))))-1)+1)*(((int(math.sqrt(int
                    (number/2))))-1)+1)
                # 2(n+1)^2
                G = 2*((((int(math.sqrt(int(number/2))))-1)+1)*(((int(math.sqrt(
                    int(number/2))))-1)+1))
```

```
                # 2(n+1)2 - n(n-1)
                L = G - A*E

                #gap
                gap = D - C
                t_pairs = len(list_A)
                t_l_pairs = len(list_z)

                print(f"$ {A} $ & $ {A*E} $ & $ {C} $ & $ {F} $ & $ {D} $ & $ {L
                    } $ & $ {G} $ & $ {C+D} $ & $ {t_pairs} $ & $ {t_l_pairs} $
                    \\\\")
                if (C- A*E) < 0:
                    print("Pre_fail")
                if (L - D) < 0:
                    print("Post_fail")
                if (C == A*E):
                    print("Similar")
                if (L == D):
                    print("Similar")
            adding_range(number)
        items_a(number)
accessing_items()
#Code ends
```

Listing 8. File-3a

```
#Code starts
import sympy
from sympy import isprime

def left_side_check(left_side):
    if isprime(left_side):
        return 1
    else:
        return 0
#Code ends
```

Listing 9. File-3b

```
#Code starts
import sympy
from sympy import isprime

def right_side_check(right_side):
    if isprime(right_side):
        return 1
    else:
        return 0
#Code ends
```

Annexure IV
(Code for Conjecture 4)

Listing 10. File-4

```
#Code starts
from left import *
from right import *
from decimal import *
from math import sqrt

#Creating item list
list_items = []

def creating_list(start_number, end, r):
    while(end >= r):
        item_of_list = 2*((r + 1)*(r + 1))
        list_items.append(item_of_list)
        r = r+1
creating_list(4, 1000000, 1)


def accessing_items_from_list_items():
    for number in list_items:
        def items_a (number):
            list_A = []
            list_z = []
            def adding_range(number):
                i=0
                half_num = int(number/2)
                while(i < half_num):
                    left_side = int(half_num - 1*(i))
                    if (left_side != 1) and ((left_side % 2) != 0):
                        if (left_side_check(left_side) != 0):
                            if (right_side_check(int(2*(half_num) - left_side))
                                != 0):
                                pair = (left_side,(int(2*(half_num) - left_side)
                                    ))
                                list_A.append(pair)
                                if len(list_A) >= 1:
                                    break
                    i=i+1
                # n
                A = ((int(sqrt(int(number/2))))-1)
                # p, 1st component of pair
                C = list_A[0][0]
                # p', second component of pair
                D = list_A[0][1]
                #gap
                gap = D - C
                if (gap >= A):
                    print(f"$ {A} $ & $ {gap} $")
                if A == 1000000:
                    print(f"{A}")
            adding_range(number)
        items_a(number)
accessing_items_from_list_items()
#Code ends
```

Listing 11. File-4a

```
#Code starts
import sympy
```

```
from sympy import isprime

def left_side_check(left_side):
    if isprime(left_side):
        return 1
    else:
        return 0
#Code ends
```

Listing 12. File-4b

```
#Code starts
import sympy
from sympy import isprime

def right_side_check(right_side):
    if isprime(right_side):
        return 1
    else:
        return 0
#Code ends
```

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