The Set Of Definition And The Diophantine Equation Of The Twin Prime Numbers

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"No Hardy , 1729 is a very interesting number , it is the smallest number expressible as a sum of two cubes in two different ways."

Srinivasa Ramanujan

Abstract

Since the ancient Greeks over 2000 years ago , mathematicians have asked the question about the distribution of the twin prime numbers . In this paper we present the set of definition and the Diophantine equation of the twin prime numbers , it is a magic subset of the natural numbers which arranges the twin prime numbers. This theory will provide a deep understanding of the twin prime conjecture and the Riemann hypothesis.

Notation and reminder

$\mathbb{N}^* := \{1,2,3,4,\dots\}$ *the natural numbers*.

$\mathbb{Z} := \{\dots,-4,-3,-2,-1,0,1,2,3,4,\dots\}$ *the integers and* $\mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$.

$\mathbb{P} := \{2,3,5,7,\dots\}$ *the prime numbers*.

A *is a subset of* $\mathbb{N}^*$ *will be determined in the main theorem and* $\overline{\mathrm{A}} := \mathbb{N}^* \setminus \mathrm{A}$.

$\mathrm{B} := \{6a \pm 1 : a \in \mathbb{Z}^*\}$ *and* $\mathrm{C} := \{\alpha.\beta : \alpha, \beta \in \mathrm{B}\}$.

$d|m$ *means that* d *divides* m.

$|x| := \max\{-x, x : x \in \mathbb{Z}^*\}$ *the absolute value of* x.

$p \wedge q := \max\{d \in \mathbb{N}^* : d|p \text{ and } d|q\}$ *the greatest common divisor of* p *and* q.

Introduction

One of the most famous open problems in number theory is the twin prime conjecture also known as Polignac's conjecture, assertion that there are infinitely many pairs of primes which differ by 2 , or there are infinitely many primes p such that $p+2$ is also prime . The first twin primes are (3 , 5),(5 , 7),(11 , 13),(17 , 19), ... There is some debate as to how old the twin prime conjecture is it was certainly considered by Alphonse de Polignac [**1**] , but there has been speculation that it could go back much further, potentially as far back as Euclid and the ancient Greeks over 2000 years ago . For various recent advances on weak forms of the twin prime conjecture , we refer the reader to [**2**] , [**3**] , [**4**] , [**5**] , [**6**] , [**7**] and [**8**] . In this paper we present the set of definition and the Diophantine equation of the twin prime numbers , it is a magic subset of the natural numbers which arranges the twin prime numbers . This theory will provide a deep understanding of the twin prime conjecture and the Riemann hypothesis.

Main Theorem. *Let* $\mathrm{A} := \{6|ab| + a + b : a , b \in \mathbb{Z}^*\}$ *we have*

$\min \mathrm{A} = 4$ *and* $\mathrm{A} \subset \mathbb{N}^*$.

$n \in \overline{\mathrm{A}} \Leftrightarrow 6n-1 \in \mathbb{P}$ *and* $6n+1 \in \mathbb{P}$. $\overline{\mathrm{A}}$ *is called the set of definition of the twin prime numbers. In other words, the equation* $n = 6|ab| + a + b$ *has no solutions when* $a , b \in \mathbb{Z}^*$ *, if and only if ,* $6n-1$ *and* $6n+1$ *are twin primes . This equation is called the Diophantine equation of the twin prime numbers.*

For instance, we present the elements of A and $\overline{\mathrm{A}}$ up to 100 :

A : = { 4 , 6 , 8 , 9 , 11 , 13 , 14 , 15 , 16 , 19 , 20 , 21 , 22 , 24 , 26 , 27 , 28 , 29 , 31 , 34 , 35 , 36 , 37 , 39 , 41 , 42 , 43 , 44 , 46 , 48 , 49 , 50 , 51 , 53 , 54 , 55 , 56 , 57 , 59 , 60 , 61 , 62 , 63 , 64 , 65 , 66 , 67 , 68 , 69 , 71 , 73 , 74 , 75 , 76 , 78 , 79 , 80 , 81 , 82 , 83 , 84 , 85 , 86 , 87 , 88 , 89 , 90 , 91 , 92 , 93 , 94 , 96 , 97 , 98 , 99 , ... }.
$\overline{\mathrm{A}}$: = { 1 , 2 , 3 , 5 , 7 , 10 , 12 , 17 , 18 , 23 , 25 , 30 , 32 , 33 , 38 , 40 , 45 , 47 , 52 , 58 , 70 , 72 , 77 , 95 , 100 , ... }.

In the next section we introduce a number of prerequisite results. Results given here may not be in the strongest forms, but they are adequate for the proof of [**Main Theorem**].

Lemma 1. *The prime numbers except* 2 *and* 3 *are on the form* $6n-1$ *and* $6n+1$ *for some* $n \in \mathbb{N}^*$.

Proof. We have that $6n-1 = 6(n-1)+5$ and $6 \wedge 1 = 1$ and $6 \wedge 5 = 1$, then according to Dirichlet's theorem [**9**] there are infinitely many primes of the form $6(n-1)+5 = 6n-1$ and $6n+1$ for some $n \in \mathbb{N}^*$.
We have $5 = 6.1-1 \in \mathbb{P}$ and $7 = 6.1+1 \in \mathbb{P}$ for $n = 1$, and the set $\{6m+r : m \in \mathbb{N}^* \text{ and } 1 \leq r \leq 6\}$ represents all natural numbers greater than or equal to 7 , then for $r = 2$ or $r = 3$ or $r = 4$ or $r = 6$ we see that $6m+r$ is a composite number , then $6m+r$ can be prime when $r = 1$ or $r = 5$. From these two results the proof is complete.

Lemma 2. *The twin prime numbers except* $(3\ ,5)$ *are on the form* $(6n-1\ ,6n+1)$ *for some* $n \in \mathbb{N}^*$.

Proof. From [**Lemma 1**] the prime numbers except 2 and 3 are on the form $6n-1$ and $6n+1$ for some $n \in \mathbb{N}^*$. Then , we can consider three pairs $(6n-3\ ,6n-1)$, $(6n-1\ ,6n+1)$ and $(6n+1\ ,6n+3)$, and $(5\ ,7)$ is a pair of twin primes on the form $(6n-1\ ,6n+1)$ for $n = 1$, and for $n \geq 2$ we see that $3|6n-3$ and $3|6n+3$, then $(6n-3\ ,6n-1)$ and $(6n+1\ ,6n+3)$ cannot be pairs of twin primes.

Lemma 3. *The set* B *is stable for multiplication*.

Proof. Indeed , let $\alpha\ ,\beta \in$ B , and $\alpha.\beta$ has four possibilities :

$\alpha.\beta = (6a-1)(6b-1) = 6\big(6ab+(-a)+(-b)\big)+1$ and

$\alpha.\beta = (6a-1)(6b+1) = 6\big(6ab+a+(-b)\big)-1$ and

$\alpha.\beta = (6a+1)(6b-1) = 6(6ab+(-a)+b)-1$ and

$\alpha.\beta = (6a+1)(6b+1) = 6(6ab+a+b)+1$ where $a\ ,b \in \mathbb{Z}^*$.

Since $6ab+(-a)+(-b)$, $6ab+a+(-b)$, $6ab+(-a)+b$ and $6ab+a+b \in \mathbb{Z}^* \Rightarrow \alpha.\beta \in$ B.

Lemma 4. *Let* $m \in$ B , *we have* $d|m \Rightarrow d \in$ B.

Proof. Let $m \in$ B , implies that $m \wedge 6 = 1$, then according to the fundamental theorem of arithmetic [**10**] , [**Lemma 1**] and [**Lemma 3**] respectively , the proof is complete.

Lemma 5. *Let $m \in \mathrm{B}$, m is a composite number $\Leftrightarrow m \in \mathrm{C}$.*

Proof. Let $m \in \mathrm{B}$ and m is a composite number , then m has at least two prime divisors identical or different , if m has exactly two prime divisors, then [**Lemma 4**] implies that $m \in \mathrm{C}$, if m has three prime divisors and more , we apply [**Lemma 4**] and [**Lemma 3**] respectively and we obtain $m \in \mathrm{C}$.

Before starting the proof of [**Main Theorem**] , we can easily see that
$6ab + (-a) + (-b) < 0$, $6ab + a + (-b) < 0$, $6ab + (-a) + b < 0$
and $6ab + a + b < 0$ when $ab < 0$, and $6ab + (-a) + (-b) > 0$,
$6ab + a + (-b) > 0$, $6ab + (-a) + b > 0$ and $6ab + a + b > 0$ when $ab > 0$.
On the other hand , it is easy to see that these four expressions are exactly the set A when $ab > 0$.

***Proof of Main Theorem*.** First, $\forall\ a , b \in \mathbb{Z}^*$ we have

$|ab| \geq -a$ and $|ab| \geq -b$ and $|ab| \geq 1$, then $2|ab| \geq -(a+b)$

, then $2|ab| + a + b \geq 0$, then $6|ab| + a + b \geq 4|ab| \geq 4.1 = 4$,

and $4 = 6|-1.-1| + (-1) + (-1) \Rightarrow 4 \in \mathrm{A}$, hence $\min \mathrm{A} = 4$.

It is clear that A is a subset of $\mathbb{Z}^*$ and $\min \mathrm{A} = 4 \Rightarrow \mathrm{A} \subset \mathbb{N}^*$.

Second, we apply an argument by contraposition.

Let $n \in \mathbb{N}^*$ such that $6n - 1 \notin \mathbb{P}$ or $6n + 1 \notin \mathbb{P}$

$\Leftrightarrow 6n - 1 \in \mathrm{C}$ or $6n + 1 \in \mathrm{C}$ according to [**Lemma 5**] ,
then $\alpha.\beta > 0$ and $\alpha.\beta$ has four possibilities :

$\alpha.\beta = (6a-1)(6b-1) = 6\big(6ab + (-a) + (-b)\big) + 1$ and

$\alpha.\beta = (6a-1)(6b+1) = 6\big(6ab + a + (-b)\big) - 1$ and

$\alpha.\beta = (6a+1)(6b-1) = 6(6ab + (-a) + b) - 1$ and

$\alpha.\beta = (6a+1)(6b+1) = 6(6ab + a + b) + 1$ where $a , b \in \mathbb{Z}^*$ and $ab > 0$

$\Leftrightarrow n = 6ab + (-a) + (-b)$ or $n = 6ab + a + (-b)$ or $n = 6ab + (-a) + b$
or $n = 6ab + a + b$ where $a , b \in \mathbb{Z}^*$ and $ab > 0$

$\Leftrightarrow n = 6|ab| + a + b : a , b \in \mathbb{Z}^*$ or $n \in \mathrm{A}$.

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