Proof of the binary Goldbach conjecture

PHILIPPE SAINTY

November 2024

University Pierre et Marie Curie Paris, France

Abstract

In this paper, the proof of the binary Goldbach conjecture is established (Any integer strictly greater than one is the mean arithmetic of two primes) . To this end, a "local" algorithm is determined for the construction of two recurrent sequences of primes (U_{2n}) and (V_{2n}), ((U_{2n}) dependent of (V_{2n})) such that for each integer $n \geq 2$ their sum is equal to $2n$. To form this a third sequence of primes (W_{2n}) is defined for any integer $n \geq 3$ by $W_{2n} = \text{Sup}\,(\, p \in \mathcal{P} : p \leq 2n - 3\,)$ where $\mathcal{P}$ is the infinite set of primes. The Goldbach conjecture has been proved for all even integers $2n$ between 4 and $4.10^{18.}$. In the table of terms of Goldbach sequences given in appendix 10 values of the order of $2n = 10^{1000}$ are reached. This "finite ascent and descent" method proves the binary Goldbach conjecture. An analogous proof by **recurrence and "shifts" of arithmetic sequences with infinitely many primes $Ur'_k = 2k + 3 + r$ and Vr'_{2k} = -2k + 2n - r - 3 (k, r bounded integers by constructing the G2n sequence and the U2n frames)**

- **is established, (parameterizations of sequences U2n and V2n). The situation can be visualized graphically by plotting the straight lines with equations Dr : y = 2x + 3 + r and D'r with equations D'r : y = - 2x - 3 + 2n - r , so that two points representing primes on these straight lines are on the same vertical line, by translating the straight line Dr and D'r along (Oy),(«The primes-points move horizontally by an integer value of r or -r ») or equivalently, by choosing the parameter «r « judiciously which is always possible. (In a Lemma xx to define and elaborate).**Then, an increase in U_{2n} by $0.7(\ln(2n))^{2.2}$ is justified. Moreover, the Lagrange-Lemoine-Levy conjecture and its generalization, the Bezout-Goldbach conjecture are proven by the same type of procedure.

Key words

Prime numbers, Prime Number Theorem, binary Goldbach conjecture, Lagrange-Lemoine-Levy conjecture, Bezout-Goldbach conjecture, gaps between consecutive primes, .

1 Overview

Number theory "the queen of mathematics" studies the structures and properties defined on integers and primes (Euclid[11], Hadamard[13], Hardy,Wright[14], Landau[20], Tchebychev[32]). Numerous problems have been raised and conjectures made, the statements of which are often simple but very difficult to prove. These main components include

- **Elementary arithmetic .** Determination and properties of primes, operations on integers (basic operations, congruence, gcd, lcm, ………..).
  Decomposition of integers into products or sums of primes (fundamental theorem of arithmetic, decomposition of large numbers, cryptography, and Goldbach's conjecture).
- **Analytical number theory .** Distribution of primes (Prime Number Theorem, Hadamard[13], De la Vallée-Poussin[33], Littlewood[23] and Erdos[10], the Riemann hypothesis,.....).
  Gaps between consecutive primes (Bombieri,Davenport[3], Cramer[8], Baker,Harmann,Iwaniec, Pintz[4],[5],[18], Granville[12], Shanks[27], Tchebychev[32] and Zhang[36]).
- **Algebraic, probabilistic, combinatorial and algorithmic number theories .** Modular arithmetic, diophantine approximations, equations, arithmetic functions and algebraic geometry.

2 Definitions notations and background

(2.1) The integers *n, k, p, q, r,……..* used in this article are always positive.

(2.2) Let $\mathcal{P}$ the infinite set of positive primes p_k (called simply primes)

$$(p_1 = 2 ; p_2 = 3 ; p_3 = 5 ; p_4 = 7 ; p_5 = 11 ; p_6 = 13 ; \ldots\ldots)$$

(2.3) The writing of large numbers (see appendix 10) is simplified using the following constants

$$M = 10^9 \ ; \ R = 4.10^8 \ ; G = 10^{100} \ ; S = 10^{500} \ ; T = 10^{1000}$$

(2.4) $\ln(x)$ denotes the neperian logarithm of the real $x > 0$

(2.5) Let (W_{2n}) be the sequence of primes defined by

(2.5.1) For any integer $n \geq 3$ $\qquad W_{2n} = \text{Sup}(p \in \mathcal{P} : p \leq 2n - 3)$

(2.6) Any sequence denoted by $(G_{2n}) = (U_{2n}; V_{2n})$ verifying (2.6.1) is called a **Goldbach sequence.**

(2.6.1) (For any integer $n \geq 2$ U_{2n} and V_{2n} are primes and $U_{2n} + V_{2n} = 2n$).

(2.7) Iwaniec,Pintz[18] have shown that for a sufficiently large integer n there is always a prime between $n - n^{23/42}$ and n. Baker,Harman[4],[5] concluded that there is a prime in the interval $[n ; n + o(n^{0.525})]$. Thus this results provides an increase of the gap between two consecutive primes p_k and p_{k+1} of the form

(2.7.1) $\forall\, \varepsilon > 0\ \exists\, k_\varepsilon \in \mathbb{N}^*$ such that $\forall\, k \in \mathbb{N}^*$ with $k > k_\varepsilon$ $\quad p_{k+1} - p_k < \varepsilon . p_k^{0.525}$

(2.8) The results obtenaid on the Cramer-Granville-Maier-Nicely conjecture[1],[3],[8],[12],[24],[25] imply the following increase
For any real $c > 2$ and integer $k \geq 500$

(2.8.1) $p_{k+1} - p_k \leq 0.7 \ln(p_k)^c$ **(<u>with probability one</u>)**

3 Introduction

Chen[6], Hardy,Littlewood[15], Hegfollt,Platt [16], Ramaré,Saouter[26], Tao[31], Tchebychev[32] and Vinogradov[34] have taken important steps and obtained promising results on the Goldbach conjecture (Any integer strictly grater than one is the mean arithmetic of two primes). Indeed, Helfgott,Platt[16] proved the weak Goldbach conjecture in 2013.

Silva,Herzog,Pardi[29] held the record for calculating the terms of Goldbach sequences after determining pairs of primes $(U_{2n} ; V_{2n})$ verifying

(3.1) For any integer n $\quad 4 \leq 2n \leq 4.10^{18}$ $\quad U_{2n} + V_{2n} = 2n$

In previous research work there is no explicit construction of recurrent sequences of Goldbach primes of the form $(G_{2n}) = (U_{2n} ; V_{2n})$ satisfying for any integer $n \geq 2$ the equality $U_{2n} + V_{2n} = 2n$.

In this article, two sequences of primes are developed using a simple and efficient algorithm to compute for any integer $n \geq 3$ by successive iterations any term U_{2n} and V_{2n} of a Goldbach sequence.

Using Maxima scientific software on a personal computer Silva's record is broken and the values $2n = 10^{500}$ and even $2n = 10^{1000}$ are reached. The proof of the binary Goldbach conjecture can be established on the same principle using reasoning by recurrence. Moreover, the Lagrange-Lemoine-Lévy conjectures[9],[17],[19],[24],[25],[30],[35] and its generalization, the Bezout-Goldbach conjecture are validated.

Using case disjunction reasoning we construct two recurrent sequences of primes (V_{2n}) and (U_{2n})

according to the sequence (W_{2n}) by the following process.
For any integer $n \geq 2$

(3.2) $$U_4 = 2 \text{ and } V_4 = 2$$

Let n be an integer such that $n \geq 3$

● Either

$(2n - W_{2n})$ is a prime

then V_{2n} and U_{2n} are defined directly in terms of W_{2n} .

● Either

$(2n - W_{2n})$ is a composite number

then V_{2n} and U_{2n} are defined from the preceding terms of the sequence (G_{2n}).

Lemma xx (to define and elaborate with plots in annex)

4 Principle of proof

To determine pairs of primes that verify the Goldbach conjecture three sequences of primes (W_{2n}), (V_{2n}), (U_{2n}) are defined and verify the following properties

(4.1) $\lim V_{2n} = +\infty$.
(4.2) For any integr $n \geq 2$ V_{2n} is defined as a function of $W_{2n} = \text{Sup}(p \in \mathcal{P}: p \leq 2n - 3)$
(4.3) (W_{2n}) is an increasing sequence that contains all primes except $p_1 = 2$
(4.4) $\lim W_{2n} = +\infty$
(4.5) (U_{2n}) is a complementary sequence of negligible primes with respect to $2n$
(4.6) For any integer $n \geq 3$

● If $(2n - W_{2n})$ is a prime "special case"

then V_{2n} and U_{2n} are defined by

(4.7) $$V_{2n} = W_{2n} \quad \text{and} \quad U_{2n} = 2n - W_{2n}$$

● Otherwise, if $(2n - W_{2n})$ is a composite number "general case"

we search for two previous terms of the sequence (G_{2n}), $U_{2(n-k)}$ and $V_{2(n-k)}$ satisfying the following conditions

(4.8) $U_{2(n-k)}, V_{2(n-k)}$ and $U_{2(n-k)} + 2k$ are primes

$$U_{2(n-k)} + V_{2(n-k)} = 2(\ n\ \text{-}\ k)$$

(which is always possible : see Lemma xx (red remarks in abstract))

Thus, by setting

(4.9) $$V_{2n} = V_{2(n-k)} \quad \text{and} \quad U_{2n} = U_{2(n-k)} + 2k$$

two new primes V_{2n} and U_{2n} satisfying (4.10) are generated .

(4.10) $$U_{2n} + V_{2n} = 2\ n$$

This process is then repeated incrementing n by one unit $(n \rightarrow n+1)$.

5 Theorem

There exists a recurrent sequence $(G_{2n}) = (U_{2n}\ ;\ V_{2n})$ of primes satisfying the following conditions.
For any integer $n \geq 2$

(5.1) $$U_{2n}\ ,\ V_{2n} \in \mathcal{P} \ \text{ and } \ U_{2n} + V_{2n} = 2n$$

(U_{2n} *and* V_{2n} *are primes and their sum is equal to* $2n$)

(5.2) *An algorithm can be used to explicitly compute any term* U_{2n} and V_{2n} .

Proof .

◻ FIRST METHOD :

For any integer $n \geq 3$

- **If ($2n$ - W_{2n}) is a prime**
  then V_{2n} and U_{2n} are defined by

  (5.3) $$V_{2n} = W_{2n} \quad \text{and} \quad U_{2n} = 2n - W_{2n}$$

- **Otherwise, if ($2n$ - W_{2n}) is a composite number**
  we use the previous terms of the sequence (G_{2n}).

For any integer q such that $1 \leq q \leq n - 3$ we have $3 \leq U_{2(n-q)} \leq n$.
Then, there exists an integer k $(1 \leq k \leq n - 3)$ following the Bertrand principle and **Lemma xx** such that

(5.4) $$R_{2n} = U_{2(n-k)} + 2k \quad \text{is a prime}$$

The smallest integer k denoted k_n / R_{2n} is a prime is chosen.
So let :

(5.5) $$U_{2n} = U_{2(n-k_n)} + 2k_n \text{ and } V_{2n} = V_{2(n-k_n)}$$

(These two terms are primes)

In the previous steps two primes $U_{2(n-k_n)}$ and $V_{2(n-k_n)}$ whose sum is equal to $2(n - k_n)$ were determined.

(5.6) $$U_{2(n-k_n)} + V_{2(n-k_n)} = 2(n - k_n)$$

By adding the term k_n to each member of the equality (5.6), it follows

(5.7) $$U_{2(n-k_n)} + 2k_n + V_{2(n-k_n)} = 2(n - k_n) + 2k_n$$

(5.8) $\Leftrightarrow$ $$\left[U_{2(n-k_n)} + 2k_n\right] + V_{2(n-k_n)} = 2n$$

(5.9) $\Leftrightarrow$ $$U_{2n} + V_{2n} = 2n$$

Finally, for any integer $n \geq 3$ this algorithm determines two sequences of primes (U_{2n}) and (V_{2n}) verifying Goldbach's conjecture.

□ SECOND METHOD :

The proof can be made using the following strong recurrence principle.
Let P(n) be the property defined for any integer $n \geq 2$ by

P(n) : "For any integer p satisfying $2 \leq p \leq n$ there exists two primes U_{2p} and V_{2p} such their sum is equal to $2p$"
(For any integer p / $2 \leq p \leq n$ $U_{2p}, V_{2p} \in \mathcal{P}$ and $U_{2p} + V_{2p} = 2p$).

Let's show by strong recurrence that P(n) is true for any integer $n \geq 2$:

a) P(2) is true : it suffices to choose $U_4 = V_4 = 2$.

b) Let's show that the property P(n) is hereditary, (for any integer $n \geq 2$ P(n) ⇒ P(n+ 1))

Assume property P(n) is true,

- **<u>If (2(n+ 1) - $W_{2(n+1)}$) is a prime</u>**

then $V_{2(n+1)}$ and $U_{2(n+1)}$ are defined by

(5.11) $$V_{2(n+1)} = W_{2(n+1)} \quad \text{and} \quad U_{2(n+1)} = 2(n+1) - W_{2(n+1)}$$

- **<u>Otherwise, if (2(n+1) - $W_{2(n+1)}$) is a composite number</u>**

There exists an integer k to obtain two terms $U_{2(n+1-k)}$) and $V_{2(n+1-k)}$ satisfying the following conditions

(5.12) $$U_{2(n+1-k)}, V_{2(n+1-k)} \text{ and } U_{2(n+1-k)} + 2k \text{ are primes}$$

$$U_{2(n+1-k)} + V_{2(n+1-k)} = 2(n+1-k)$$

(which is always possible : see **<u>FIRST METHOD</u>**).

Thus, by setting

(5.13) $$V_{2(n+1)} = V_{2(n+1-k)} \quad \text{and} \quad U_{2(n+1)} = U_{2(n+1-k)} + 2k$$

Two new primes $V_{2(n+1)}$ and $U_{2(n+1)}$ satisfying ($U_{2(n+1)} + V_{2(n+1)} = 2(n+1)$) are generated.

It follows that P(n + 1) is true. Then the property P(n) is hereditary, (P(n) => P(n + 1)).

Therefore, for any integer $n \geq 2$ the property P(n) is true.

it follows

$$\forall\ n \in \mathbb{N}+2 \text{ there are two primes } U_{2n} \text{ and } V_{2n} \text{ and such their sum is } 2n : (U_{2n} + V_{2n} = 2n)$$

6 Lemma

The sequence (U_{2n}) verifies the following increase

For any integer n ≥ 65

(6.1) $$U_{2n} \leq [2n]^{0.55}$$

Proof . According to the programm 9.2 and appendix 10 the increase (6.1) is verified for any integer n such that $65 \leq n \leq 2000$. For any integer $n > 2000$ the proof is established by recurrence. For this purpose let P1(n) be the following property

(6.2) P1(n) : " There exists a strictly increasing sequence of positive numbers (C_n) such that

$$U_{2n} \leq C_n (2n)^{0.525} \text{ " }.$$

a) P1(2000) is true according to program 9.2 and the table in appendix 10.

b) For any integer $n \geq 2000$ let's show that P1(n) is hereditary, (P1(n) $\Rightarrow$ P1($n+1$))

Assume that P1(n) is true : then

- **<u>If $(2(n+1) - W_{2(n+1)})$ is a prime</u>**

then $V_{2(n+1)}$ and $U_{2(n+1)}$ are defined by

(6.2) $$V_{2(n+1)} = W_{2(n+1)} \text{ and } U_{2(n+1)} = 2(n+1) - W_{2(n+1)}$$

According to the results in [4],[5],[18] there is a constant $K > 0$ such that

$$(n+1) - K.(2(n+1))^{0.525} < W_{2(n+1)} < 2(n+1)$$

$$\Rightarrow \quad U_{2(n+1)} < K.(2(n+1))^{0.525}$$

$$\Rightarrow \quad U_{2(n+1)} \leq C_{n+1}.(2(n+1))^{0.525}$$

- **<u>Otherwise, if $(2(n+1) - W_{2(n+1)})$ is a composite number</u>**

(6.4) $$\exists\, p \in \mathbb{N}^* \ / \ U_{2(n+1)} = U_{2(n+1-p)} + 2p$$

According to [4],[5],[18] the smallest integer p defined in (6.4) verifies

(6.5) $$2p < K. .(U_{2(n+1-p)})^{0.525} \text{ and } U_{2(n+1-p)} < C_{n+1-p}.(2(n+1-p))^{0.525}$$

It follows

$$U_{2(n+1)} < K.C_{n+1-p}^{0.525}.(2(n+1-p))^{0.275625} + C_{n+1-p}.(2(n+1-p))^{0.525}$$

Then

(6.6) $$U_{2(n+1)} < C_{n+1}.(2(n+1))^{0.525}$$

and by setting $C_n = (2n)^{0.025}$

(6.7) $$U_{2(n+1)} < (2(n+1))^{0.55}$$

P1($n+1$) is true then P1(n) is hereditary.

So, for any integer $n \geq 2000$ the property P1(n) is true.

(The inequality (6.7) is verified with the aid of the software Maple studying the functions of the type $f : x \rightarrow a\,.\,x^{0.275625} + b\,.\,x^{0.525}$ increased by $g: x \rightarrow x^{0.55}$ a and b being two strictly positive real parameters).

- **Remark.** A more precise estimate can be obtained using the Cipolla or Axler frames [7],[2].

7 Theorem

For any integer $n \geq 3$ *it is easy to check*

(7.1) (W_{2n}) *is a positive increasing sequence of primes.*
(7.2) $\{W_{2n} : n \in \text{IN*}\} \cup \{2\} = \mathcal{P}$
(7.3) $\lim W_{2n} = +oo$
(7.4) (V_{2n}) *is a sequence of primes.*
(7.5) $n \leq V_{2n} \leq W_{2n}$
(7.6) $3 \leq 2n - W_{2n} \leq U_{2n} \leq n$
(7.7) $\lim V_{2n} = +oo$

Proof .

(7.1) : For any integer $n \geq 2$ let A_n be the following set
$A_n = \{ p_k \in \mathcal{P} : p_k \leq 2n - 3 \}$ and $A_n \subset A_{n+1}$. Therefore, $W_{2n} \leq W_{2(n+1)}$. So the sequence (W_{2n}) is increasing.

(7.2) : Any prime except $p_1 = 2$ is odd, hence the result.

(7.3) : $\lim W_{2n} = \lim p_n = +oo$

(7.4) : By definition $V_{2n} = W_{2n}$ or there exits an integer $k \leq n - 2$ such that $V_{2n} = V_{2(n-k)}$;
so, the terms of the sequence ($V_{2n)}$) are primes ; moreover, there exists a strictly increasing sub-sequence (V'_{2n}) of (V_{2n}) verifying lim (V'_{2n}) = +oo

(7.5) : According to Lemma 6, for any integer $n \geq 65$,

$$U_{2n} < (2n)^{0.55} \quad ;$$

therefore

$$U_{2n} < (2n)^{0.55} < n$$

and

$$V_{2n} = 2n - U_{2n} > 2n - n > n$$

For any integer n such that $3 \leq n \leq 65$ verification is carried out according to the computer program in paragraph 9.2 and the table in appendix 10.
we can also see that by construction $V_{2n} \geq U_{2n}$ because if we assume the opposite then V_{2n} is not the largest prime number verifying $\frac{(U_{2n} + V_{2n})}{2} = n$. So, $V_{2n} \geq n$.

(7.6) : According to (7.5) $\quad n \leq V_{2n} \quad \Rightarrow \quad U_{2n} = 2n - V_{2n} \leq 2n - n \leq n$;

moreover

$$V_{2n} \le W_{2n} \quad \Rightarrow \quad 2n - W_{2n} \le 2n - V_{2n} = U_{2n}$$

(7.7) : By (7.5) for any integer $n \ge 2$ $n \le V_{2n}$

so

$$\lim (V_{2n}) = +oo .$$

8 Remarks

8.1 There are infinitely many integers n such that $U_{2n} = 3, 5, 7$ or 11.

8.2 $V_{2n} \sim 2n$ for $(n \to +oo)$.

8.3 For any sufficiently large integer n / $n \ge 5000$: $U_{2n} \ll V_{2n}$ and $\lim (\frac{U_{2n}}{V_{2n}}) = 0$.

8.4 The smallest integer n such that

$U_{2n} \neq 2n - W_{2n}$ is obtained for $n = 49$ and $G_{98} = (79 ; 19)$.

(This type of terms increases in the Goldbach sequence (G_{2n}) as n increases, in the sense of the Schnirelmann density and there are an infinite number of them; their proportion per interval can be computed using the results given in [28]).

8.5 If $q \ge 5$ is an odd integer we could generalize this algorithm with sequences (W'_{2n}) defined by

(8.5.1) $\forall\, n \in \mathbb{N}$ verifying $n \ge \frac{(q+3)}{2}$ $W'_{2n} = \text{Sup}(p \in \mathcal{P} : p \le 2n - q)$

Other sequences (G'_{2n}) of Goldbach independent of (G_{2n})are thus generated.

8.6 The sequence (G_{2n}) is "extremal" in the sense that for any integer $n \ge 2$ V_{2n} and U_{2n} are the largest and smallest possible primes such that $U_{2n} + V_{2n} = 2n$.

8.7 The Cramer-Granville-Maier-Nicely conjecture [8],[12],[17],[19],[21],[22],[24],[25],[30] is verified with probability one. It leads to the following increase

For any integer $p \ge 500$

(8.7.1) $U_{2p} \le 0.7\, [\ln(2p)]^{(2.2 - \frac{1}{p})}$ (<u>**with probability one**</u>)

The proof is similar to that of Lemma 6 using the same type of reasoning by recurrence validated by the study of functions of the type $f : x \to a\, .g(x) + b[\ln(\square(\square))]^{c}$ (a,b and c being two strictly real parameters (c > 2)), with $g : x \to 0.7\, [\ln(x)]^{(c - \frac{1}{x})}$ and $h : x \to 0.7\, [\ln(x)]^{(2.2 - \frac{1}{x})}$ using Maple software.

- **Remark.** A better estimate can be obtained via [24],[25],[27] .

8.8 According to Bombieri[3] and using the same method as in the proof of Lemma 6, on average, we obtain the following estimate of U_{2n}

(8.8.1) $\forall\, \varepsilon > 0$ $U_{2n} = \boldsymbol{O}(\ln^{1.3+\varepsilon}(2n))$ **(on average)**

9 Algorithm

9.1 Algorithm written in natural language.

Inputs :

Input four integer variables : k, N, n, P

Input : $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, ..., p_N the first N primes.

: $n = 3$

: $P = M, R, G, S$ or T as indicated in paragraph **2**

Algorithm body :

A) Compute : $W_{2n} = \mathrm{Sup}(\, p \in \mathcal{P} : p \leq 2n - 3\,)$

If $T_{2n} = (2n - W_{2n})$ is a prime

Let :

(9.1.1) $U_{2n} = T_{2n}$ and $V_{2n} = W_{2n}$

otherwise

B) If T_{2n} is a composite number

Let : $k = 1$

B.1) While $U_{2(n-k)} + 2k$ is a composite number

assign to k the value : $k + 1$ ($k \rightarrow k + 1$).

return to **B1)**

End while

Assign to k the value k_n : ($k \rightarrow k_n$)

(9.1.2) Let : $U_{2n} = U_{2(n-k_n)} + 2k_n$ and $V_{2n} = V_{2(n-k_n)}$

Assign to n the value $n + 1$ ($n \rightarrow n + 1$ and return to **A)**

End :

Outputs for integers less than 10^4: :

Print ($2n =$ ... ; $2n - 3 =$ ... ; $W_{2n} =$ ... ; $T_{2n} =$ ... ; $V_{2n} =$ ... ; $U_{2n} =$ ...)

Outputs for large integers :

Print ($2n$ - P = ... ; $2n$ - 3 - P= ... ; W_{2n} - P = ... ; T_{2n} = ... ; $V_{2n} - P = \cdots$; U_{2n} = ...)

9.2 Program written with Maxima software for $2n = 10^{500}$

```
r: 0 ; n1 : 10**500 ; for n :5*10**499 + 10000 thru 5*10**499 + 10010 do
( k:1 , a:2 *n , c:a - 3 , test : 0 , b : prev_prime(a - 1) ,
if primep(a - b)
then print(a - n1 , c - n1 , b - n1 , a - b , b - n1 , a - b)
otherwise ( r:r + 1 ,
 while test = 0 do
 ( if ( primep(c) and primep(a - c) )
then ( test:1 , print(a - n1 , a - n1 - 3 , b - n1 , a - b , c - n1 , a - c ," ** " , r))
else ( test : 0 , c : c - 2*k ))) ) ;
```

10 Appendix

Application of Algorithm 9 : Table of U_{2n} and V_{2n} terms of the Goldbach sequence (G_{2n}) computed from program 9.2 ($2 \leq 2n \leq 10^{1000}$ + 4020).

The ****** sign in the table below indicates the results given by the algorithm **9** in case **B)** of return to the previous terms of the sequence (G_{2n}). **WATCH OUT !** : For large integers n ($2n > 10^9$ for example), to simplify the display of large numbers the results are entered as follows

$$2n - P, (2n - 3) - P, W_{2n} - P, T_{2n}, V_{2n} - P \text{ and } U_{2n}$$

with

P= $M, R, G, S,$ or T constants defined in (2.3)

2n	*2n - 3*	W_{2n}	T_{2n}=*2n* - W_{2n}	V_{2n}	U_{2n}
4	1	X	X	**2**	**2**
6	3	3	3	**3**	**3**
8	5	5	3	**5**	**3**
10	7	7	3	**7**	**3**
12	9	7	5	**7**	**5**
14	11	11	3	**11**	**3**
16	13	13	3	**13**	**3**
18	15	13	5	**13**	**5**
20	17	17	3	**17**	**3**
22	19	19	3	**19**	**3**

24	21	19	5	19	5
26	23	23	3	23	3
28	25	23	5	23	5
30	27	23	7	23	7
32	29	29	3	29	3
34	31	31	3	31	3
36	33	31	5	31	5
38	35	31	7	31	7
40	37	37	3	37	3
80	77	73	7	73	7
82	79	79	3	79	3
84	81	79	5	79	5
86	83	83	3	83	3
88	85	83	5	83	5
90	87	83	7	83	7
92	89	89	3	89	3
94	91	89	5	89	5
96	93	89	7	89	7
**98	95	89	9	79	19
100	97	97	3	97	3
120	117	113	7	113	7
**122	119	113	9	109	13
124	121	113	11	113	11
126	123	113	13	113	13
**128	125	113	15	109	19
130	127	127	3	127	3
132	129	127	5	127	5
134	131	131	3	131	3
136	133	131	5	131	5
138	135	131	7	131	7
140	137	137	3	137	3
**500	497	491	9	487	13
502	499	499	3	499	3
504	501	499	5	499	5
506	503	503	3	503	3
508	505	503	5	503	5
510	507	503	7	503	7
1000	997	997	3	997	3
1002	999	997	5	997	5
1004	1001	997	7	997	7

****1006**	**1003**	**997**	**9**	**983**	**23**
1008	1005	997	11	**997**	**11**
1010	1007	997	13	**997**	**13**
1012	1009	1009	3	**1009**	**3**
1014	1011	1009	5	**1009**	**5**
1016	1013	1013	3	**1013**	**3**
1018	1015	1013	5	**1013**	**5**
10002	9999	9973	29	**9973**	**29**
10004	10001	9973	31	**9973**	**31**
****10006**	**10003**	**9973**	**33**	**9923**	**83**
****10008**	**10005**	**9973**	**35**	**9967**	**41**
10010	10007	10007	3	**10007**	**3**
10012	10009	10009	3	**10009**	**3**
10014	10011	10009	5	**10009**	**5**
10016	10013	10009	7	**10009**	**7**
****10018**	**10015**	**10009**	**9**	**10007**	**11**
10020	10017	10009	11	**10009**	**11**
2n - M	***(2n - 3) - M***	W_{2n} ***- M***	$T_{2n} = 2n - W_{2n}$	V_{2n} ***- M***	U_{2n}
+1000	+997	+993	7	**+993**	**7**
****+1002**	**+999**	**+993**	**9**	**+931**	**71**
+1004	+1001	+993	11	**+993**	**11**
+1006	+1003	+993	13	**+993**	**13**
****+1008**	**+1005**	**+993**	**15**	**+919**	**89**
+1010	+1007	+993	17	**+993**	**17**
+1012	+1009	+993	19	**+993**	**19**
+1014	+1011	+1011	3	**+1011**	**3**
+1016	+1013	+1011	5	**+1011**	**5**
+1018	+1015	+1011	7	**+1011**	**7**
****+1020**	**+1017**	**+1011**	**9**	**+931**	**89**
2n - R	***(2n - 3) - R***	W_{2n} ***- R***	$T_{2n} = 2n - W_{2n}$	V_{2n} ***- R***	U_{2n}
****+1000**	**+997**	**+979**	**21**	**+903**	**97**
+1002	+999	+979	23	**+979**	**23**
****+1004**	**+1001**	**+979**	**25**	**+951**	**53**
****+1006**	**+1003**	**+979**	**27**	**+903**	**103**
+1008	+1005	+979	29	**+979**	**29**
+1010	+1007	+979	31	**+979**	**31**
****+1012**	**+1009**	**+979**	**33**	**+951**	**61**
****+1014**	**+1011**	**+979**	**35**	**+ 781**	**233**
+1016	+1013	+979	37	**+979**	**37**
****+1018**	**+1015**	**+979**	**39**	**+951**	**67**
+1020	+1017	+1017	3	**+1017**	**3**

$2n$ - G	$(2n-3)$ - G	W_{2n} - G	$T_{2n} = 2n - W_{2n}$	V_{2n} - G	U_{2n}
**+10000	+9997	+9631	369	+7443	2557
**+10002	+9999	+9631	371	+9259	743
+10004	+10001	+9631	373	+9631	373
**+10006	+10003	+9631	375	+8583	1423
**+10008	+ 10005	+9631	377	+6637	3371
+10010	+10007	+9631	379	+9631	379
**+10012	+10009	+9631	381	+8583	1429
+10014	+10011	+9631	383	+9631	383
**+10016	+10013	+9631	385	+9259	757
**+10018	+10015	+9631	387	+4491	5527
+10020	+10017	+9631	389	+9631	389
$2n$-S	$(2n-3)$-S	W_{2n} - S	$T_{2n} = 2n - W_{2n}$	V_{2n} - S	U_{2n}
**+20000	+19997	+18031	1969	+17409	2591
**+20002	+19999	+18031	1971	+ 17409	2593
+20004	+20001	+18031	1973	+18031	1973
**+20006	+20003	+18031	1975	+16663	3343
**+20008	+20005	+18031	1977	+16941	3067
+20010	+20007	+18031	1979	+18031	1979
**+20012	+20009	+18031	1981	+5671	14341
**+20014	+20011	+18031	1983	+4101	15913
**+20016	+20013	+18031	1985	+3229	16787
+20018	+20015	+18031	1987	+18031	1987
**+20020	+20017	+18031	1989	+16941	3079
$2n$-T	$(2n-3)$-T	W_{2n}-T	$T_{2n} = 2n - W_{2n}$	$V_{2n} - T$	U_{2n}
**+40000	+39997	+29737	10263	+ 21567	18433
**+40002	+39999	+29737	10265	+ 22273	17729
+40004	+40001	+29737	10267	+29737	10267
**+40006	+40003	+29737	10269	+21567	18439
+40008	+40005	+29737	10271	+29737	10271
+40010	+ 40007	+29737	10273	+29737	10273
**+40012	+40009	+29737	10275	+10401	29611
**+40014	+40011	+29737	10277	-56003	96017
**+40016	+40013	+29737	10279	+27057	12959
**+40018	+40015	+29737	10281	+25947	14071
**+40020	+40017	+29737	10283	+24493	15527

11 Perspectives and generalizations

11.1 **O**ther Goldbach sequences (G'_{2n}) and $(G"_{2n})$ independent of (G_{2n}) may be studied using the increasing sequences of primes (W'_{2n}), (see 8.5) and $(W"_{2n})$ defined by

For any integer $n \geq 3$ $W"_{2n} = \text{Sup}(\, p \in \mathcal{P} : \, p \leq f(n))$, f being a function defined on the interval $I = [3 \,; +\infty[$ and satisfying the following conditions

- f is strictly increasing on the interval I
- $f\,(3) = 3$ and $\lim\limits_{x \to +\infty} f(x) = +\infty$
- $\forall\, x \in I \;\; f(x) \leq 2x - 3$

For example, one of the following functions defined on I can be selected.

□ $f : x \rightarrow a\,x + 3 - 3a$ $\qquad (a \in \mathbb{R} \,:\, 0 < a \leq 2)$

□ $g : x \rightarrow \left[4\sqrt{3x} - 9\right]$ $\qquad [x]$ being is the integer part of the real number x

□ $h : x \rightarrow 6 \ln\left(\frac{x}{3}\right) + 3$

11.2 Using this method it would be interesting to study the Schnirelmann density [28] of certain primes such as 3 , 5 , 7, 11 ,..... ... in the sequence (U_{2n}) for $n \in [\, K_N \,; P_N]$ as a function of N.

11.3 It is possible to exceed the values shown in the table of $2n = 10^{1000}$ by optimizing this algorithm using supercomputers and more efficients software as Maple .

11.4 Diophantine equations and conjectures of the same nature (Lagrange-Lemoine-Levy conjecture [9],[17],[19],[21],[22],[30]) can be processed using similar reasoning and algorithms.

1) To validate the Lagrange_Lemoine-Levy conjecture we can study the following sequences of primes (WL_{2n}), (VL_{2n}) and (UL_{2n}) defined by

For any integer $n \geq 3$ $\qquad WL_{2n} = \text{Sup}(\, p \in \mathcal{P} \,:\, p \leq n - 1\,)$

- If $TL_{2n} = (2n + 1 - 2WL_{2n})$ is a **<u>prime</u>**

then let :

$$VL_{2n} = WL_{2n} \;\; \text{and} \;\; UL_{2n} = TL_{2n}$$

- If TL_{2n} is a **<u>composite number</u>**

then there exists an integer k $(1 \leq k \leq n - 3)$ such hat $UL_{2(n-k)} + 2k$ is a <u>prime</u>

then let :

$$VL_{2n} = VL_{2(n-k)} \;\; \text{and} \;\; UL_{2n} = UL_{2(n-k)} + 2k$$

1) Using the same type of reasoning a generalized Bezout-Goldbach conjecture of the following form can be validated

- Let K and Q be two odd integers prime to each other :

For any integer n such that $2n \geq 3(K + Q)$ there exist two primes U'''_{2n} and V'''_{2n} verifying

$$K.U'''_{2n} + Q.V'''_{2n} = 2n$$

- Let K and Q be two integers of different parity prime to each other :

For any integer n such that $2n \geq 3(K + Q)$ there are two primes U'''_{2n} and V'''_{2n} verifying

$$K.U'''_{2n} + Q.V'''_{2n} = 2n + 1.$$

12 Conclusion

12.1 A recurrent and explicit Goldbach sequence $(G_{2n}) = (U_{2n} ; V_{2n})$ verifying

$$\forall\, n \in \mathbb{N} + 2 \quad U_{2n} \text{ and } V_{2n} \text{ are primes and } U_{2n} + V_{2n} = 2n$$

has been developed using an simple and efficient "local" algorithm.

12.2 Silva's [29] record is broken on a personal computer and it is possible to reach values of the order of $2n = 10^{1000}$ with a reasonable computation time (<u>less than three hours</u> for the evaluation of ten terms U_{2n} and V_{2n}) .

12.3 For a given integer $n \geq 49$ the evaluation of the terms U_{2n} and V_{2n} does not require the computing of all previous terms U_{2k} and V_{2k} $/$ $1 \leq k < n - 1$. We just need to know the primes p_l and V_{2r} such that

(12.3.1) $\quad p_l \leq 7.\ln^{1.3}(2n)$ and $2n - 7.\ln^{1.3}(2n) \leq V_{2r} \leq 2n$ **<u>(on average)</u>**

This property allows quick computing of U_{2n} and V_{2n} even for values of $2n$ of the order of 10^{1000} .

12.4 Therefore, the binary Goldbach and the Lagrange-Lemoine-Levy conjectures are true.

References

[1] L. Adleman, K. Mc Curley, «Open Problems in Number Theoretic Complexity» , «II. Algorithmic number theory» (Ithaca, NY,1994), 291–322, Lecture Notes in Comput. Sci., 877, Springer, Berlin, (1994).

[2] C. Axler, “New Estimates for the nth Prime” 19.4.2 2 Journal of Integer Sequences, Vol. 22, 30 p., (2019),

[3] E. Bombieri, Davenport, "Small differences between prime numbers", Proc. Roy. Soc. Ser. A293 , pp. 1-18 , (1966) .

[4] R. C. Baker, Harman, G. “The difffference between consecutive primes”. Proc. London Math. Soc. (3) 72, 2 (1996), 261–280.

[5] R. C. Baker, Harman, G., and Pintz, J. “The difffference between consecutive primes”. II. Proc. London Math. Soc. (3) 83, 3 (2001), 532–562.

[6] J. R. Chen, "On the representation of a large even integer as the sum of a prime and the product of at most two primes". Kexue Tongbao 17 (1966), pp. 385-386 (Chinese) .

[7] M. Cipolla, “La determinazione assintotica dell n imo numero primo”, Rend. Acad. Sci. Fis. Mat. Napoli 8(3) (1902).

[8] H. Cramer, "On the order of magnitude of the difference between consecutive prime numbers", Acta Arithmetica vol. 2 , (1986), p.23-46 .

[9] N. Dawar, “Lemoine's Conjecture: A Limited Solution Using Computers” , TechRxiv [Archive online] (2023).

[10] P. Erdos, "On a new method in elementary number theory which leads to an elementary proof of the prime number theorem", Proc. Natl. Acad. Sci. USA 36, pp. 374-384 (1949).

[11] Euclid, (trans. Bernard Vitrac), "Les éléments d'Euclide", Ed. PUF Paris, vol.2, p. 444-446 and p. 339-341, (1994).

[12] A. Granville, "Harald Cramér and the distribution of prime numbers", *Scandinavian Actuarial Journal*, **1**: 12–28,(1995).

[13] J. Hadamard, "On the zeros of the function ζ(s) of Riemann". C. R. 122, p.1470-1473 (1896), and "On the distribution of zeros of the function ζ'(s) and its arithmetical consequences". S. M. F. Bull. 24, pp. 199-220 (1896).

[14] G. H. Hardy, Wright , "An introduction to the Theory of numbers", Oxford : Oxford University Press 621 p. (2008).

[15] G. H. Hardy, J. E. Littlewood: »Some problems of 'partitio numerorum'» ; III: «On the expression of a number as a sum of primes« (Acta Math. **Vol. 44**: pp. 1 – 70, (1922)

[16] H. Helfgott, Platt , "The ternary Goldbach conjecture", Gaz. Math. Soc. Math. Fr. 140, pp. 5-18 (2014). "The weak Goldbach conjecture", Gac. R. Soc. Mat. Esp. 16, no. 4, 709-726 (2013). "Numerical verification of the ternary Goldbach conjecture up to $8.875.10^{30}$", Exp. Math. 22, n° 4, 406-409 (2013).

[17] L. Hodges, "A lesser-known Goldbach conjecture", Math. Mag., **66** (1993): 45–47.

[18] H. Iwaniec, Pintz, "Primes in short intervals". Monatsh. Math. 98, pp. 115-143 (1984).

[19] J. O. Kiltinen and P. B. Young, "Goldbach, Lemoine, and a Know/Don't Know Problem", Mathematics Magazine, 58(4)

(Sep., 1985), p. 195–203.

[20] E. Landau, "Handbuch der Lehre von der Verteiligung der Primzahlen", vol. 1 and vol. 2 (1909) , published by Chelsea Publishing Company (1953).

[21] E. Lemoine, "L'intermédiaire de mathématiciens", vol. 1, 1894 ,p. 179, vol. 3, 1896 ,p. 151

[22] H. Levy, "On Goldbach's conjecture", Math. Gaz." **47** (1963): 274

[23] J. Littlewood , "Sur la distribution des nombres premiers", CRAS Paris, vol. 158, (1914), p. 1869-1875.

[24] H. Maier, "Primes in short intervals". Michigan Math. J., 32(2):221–225, 1985.

[25] T. R. Nicely, "New maximal prime gaps and first occurrences", Mathematics of Computation, **68** (227): 1311–1315, (1999)

[26] O. Ramaré, Saouter, "Short effective intervals containing primes", J. Number theory, 98, No. 1, p..10-33 (2003).

[27] D. Shanks, "On Maximal Gaps between Successive Primes", *Mathematics of Computation*, American Mathematical Society, **18** (88): 646–651, (1964).

[28] L. Schnirelmann, "Schnirelmann density", Wikipedia, (on line, internet) and "A proof of the fundamental theorem on the density of sums of sets of positive integers", Annals of Math, 2nd series, vol. 43, no. 3, (1942), pp. 523-527.

[29] T. O. e Silva, Herzog, Pardi, "Empirical verification of the even Goldbach conjecture and computation of prime gaps up to 4.10^{18}. Math. Comput. 83, no. 288, pp. 2033-2060 (2014).

[30] Z-W. Sun, "On sums of primes and triangular numbers" » [archive], arXiv, 2008 (arXiv 0803.3737).

[31] T. Tao, "Every odd number greater than 1 is the sum of at most five primes", Math. Comput. 83, no. 286, pp.997-1038 (2014).

[32] P. Tchebychev, "Mémoire sur les nombres premiers" J. math. pures et appliquées, 1ère série, t.17,

p. 366-390 et p. 381-382, (1852).

[33] C.- J. de La Vallée-Poussin, "Recherches analytiques sur la théorie des nombres premiers", Brux. S. sc. 21 B, pp. 183-256, 281-362, 363-397, vol.21 B, pp. 351-368, (1896).

[34] A. Vinogradov, "Representation of an odd number as a sum of three primes". Dokl. Akad.Nauk. SSR, 15:291-294, (1937).

[35] E.W. Weisstein, "Levy's Conjecture" » [archive], sur MathWorld , CRC Concise Encyclopédie de mathématiques

(CRC Press,), 733-4, (1999).

[36] Y. Zhang, "Bounded gaps between primes", Ann. Math. (2) 179, no. 3, pp.1121-1174 (2014).