Recursive Inaccessibility and the Inductive Hypothesis as to Countable Sets in ZFC

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Abstract

The significance of the Inductive Hypothesis is examined with respect to the Principle of Mathematical Induction. A few relevant theorems that involve functions in set theory are specified with respect to the Inductive Hypothesis.

The countability of rational numbers is reviewed, as to Cantor's "intuition" (i.e., the "zig-zag" method of enumerating rational numbers) and constructive formulas that would map the set of natural numbers to a subset of the rational numbers (e.g., a multiplicative inverse function, a divisive function).

Von Neumann and Zermelo ordinals are introduced to support the definition of a non-dense, well-ordered set of numbers. It is determined that for a specific transfinite set of ordinals with a maximal element that is a limit ordinal, the set must contain at least one successor ordinal that cannot be recursively accessed in a finite number of steps from a specified base ordinal.

1.1 The Inductive Hypothesis and the Axiom of Induction

Definition 1.1.1: An open statement $P(n)$, is a statement that associates with each object in a "domain of discourse" D, such that there is a variable n in $P(n)$ that would be replaceable for each object in D.

Essentially, the variable n would be a sort of placeholder for any object in the domain D that is associated with the open statement $P(n)$. The truth value of the open statement would depend on the specific object in the domain of discourse that is substituted with the variable n in the open statement $P(n)$.

If $P(n)$ is true for all $n \in D$, then it means $P(n)$ is true for each object in the specified domain D.

Example 1.1.1: $P(n)$: "n is an integer", is an open statement, where the variable n in $P(n)$ would be substitutable for any object in the domain of discourse D that is associated with the open statement $P(n)$.

For any "domain of discourse" D that associates with an open statement $P(n)$: "n is an integer":

- $P(n)$ would be true for all $n \in D$, if and only if D is a subset of the set of integers, $\mathbb{Z}$
- $P(n)$ would not be true for all $n \in D$, if the set $\{n \in D : n \notin \mathbb{Z}\}$ is non-empty

Definition 1.1.2: A countable set, is a set N that has a finite quantity of elements (i.e., the cardinality of N is finite), or there is an injective function that maps N to the set of non-negative integers $\mathbb{N}_{\geq 0}$.

Definition 1.1.3: A non-dense set of numbers, is a set N, such that for any pair of elements $n_\alpha, n_\beta \in N$, there is strictly a finite quantity of $r \in N$ (or there is no $r \in N$), for which $n_\alpha < r < n_\beta$ holds.

Definition 1.1.4: A well-ordered set N, is a non-empty set in which there is a specific element $a \in N$, such that $a \in N$ precedes (or equals) all other elements in the set, as to a relation on N (e.g., a $\leq$-relation)

The Principle of Mathematical Induction (or Axiom of Induction):

Assume $P(n)$ designates some open statement, where the domain of $P(n)$ is a non-dense, well-ordered set of numbers N, and the cardinality of N is not conclusively finite.

The Principle of Mathematical Induction states that, if:

- The *Base Case*, $P(a)$ is true for some $a \in N$

AND

- $P(k)$ is true, implies $P(k+1)$ is also true, for any $k \in N$, where $k \geqslant a$ (by Regular Induction)

OR

- $P(r)$ is true, implies $P(k+1)$ is also true, for all $\{r \in N\ ;\ a \preccurlyeq r \preccurlyeq k;\ a, k \in N\}$, where $k \geqslant a$ (by Strong Induction)

then $P(n)$ is true for all $\{n \in N : n \geqslant a;\ a \in N\} \subseteq N$.

Regarding a non-dense, well-ordered set of numbers N, where the cardinality of N is not conclusively finite, the **Inductive Hypothesis** is the conjecture that an open statement $P(n)$ is valid for at least one element $k \in N$, where a previous case $P(a)$ has been validated for some $a \in N$, and $k \geqslant a$.

Hypothesis 1.1.1 (Inductive Hypothesis by Regular Induction):

If $P(n)$ is an open statement, where the "domain" of the open statement is a non-dense, well-ordered set of numbers N, and the base case, $P(a)$ holds for a specified element $a \in N$, then there is some arbitrary element $k \in N$, where $k \geqslant a$, such that $P(k)$ also holds.

Hypothesis 1.1.1 (Inductive Hypothesis by Strong Induction):

If $P(n)$ is an open statement, where the "domain" of the open statement is a non-dense, well-ordered set of numbers N, and the base case, $P(a)$ holds for a specified element $a \in N$, then there is a subset $\{r \in N\ ;\ a \preccurlyeq r \preccurlyeq k;\ a, k \in N\} \subsetneq N$, such that $P(r)$ also holds for some $k \geqslant a$.

Definition 1.1.5: **Proof by (Mathematical) Induction,** is a method to prove that a mathematical statement, $P(n)$ holds for countably many elements of a non-dense, well-ordered set of numbers N, where the cardinality of N is not conclusively finite.

To prove a proposition $P(n)$ by mathematical induction, with regard to a non-dense, well-ordered set N:

- Firstly, it must be proven that $P(n)$ is true for at least one element $a \in N$. This would normally be the *Base Case* $P(a)$, and would serve as the initial point of the inductive process.

  Ideally, it would be proven $P(n)$ is true for the minimal element in N. But it does not have to be proven for the smallest element in N; it may be proven $P(n)$ is true for another element in N.

- Secondly, the **Inductive Hypothesis** would have to be invoked (with respect to either Regular Induction or Strong Induction)

- Finally, by the *Inductive Step*, it would be proven $P(k+1)$ is true, if:
  - $P(k)$ is true (with respect to the Inductive Hypothesis by Regular Induction)

    OR
  - $P(r)$ is true (with respect to the Inductive Hypothesis by Strong Induction)

If these steps are completed and the proposition $P(n)$ holds, then it means by mathematical induction, it can be decided that the proposition $P(n)$ is true for all $\{n \in N : n \geqslant a;\ a \in N\} \subseteq N$.

A statement $P(n)$ that can be proven by Strong Induction, can also be proven by Regular Induction, i.e., both methods of induction can obtain the result that $P(n)$ holds for all $\{n \in N : n \geqslant a;\ a \in N\} \subseteq N$.

Remark 1.1.1: Regarding the **Axiom of Induction**, the significance of the **Inductive Hypothesis**, is that:

- The conjecture would be applicable to some unspecified number $k \in N$, such that the number k *could be* preceded by another number that is specified in N (albeit not necessarily)
- The conjecture implicitly alludes to the *potential* existence of a non-dense, well-ordered set of numbers N, where the cardinality of N is transfinite.

  Otherwise, the Inductive Hypothesis would be redundant, because the "truth" of the statement $P(n)$ would be knowable for every specific number $n \in N$ to which it may apply, which would invalidate a requirement for the Inductive Hypothesis to be involved in the proof of $P(n)$.

There are some elementary lemmas specified below, with a proof by Regular or Strong Induction:

Lemma 1.1.1:

$P(n)$: For all $n \in \mathbb{N}$, the product $2 \times n$ is an even number (i.e., an integer that would produce an integer if subtracted by half its value)

Proof of Lemma 1.1.1 (by Regular Induction):

Base Case, $P(1) : 2(1) = 2;\ {}^{2}/_{2} = 1$

Inductive Hypothesis: Assume $P(n)$ is true for any $k \in \mathbb{N};\ k \geq 1$

Inductive Step: $P(k)$ implies $P(k+1)$

$$P(k) : 2k;\ {}^{2k}/_{2} = k, \qquad P(k+1) : 2(k+1) = 2k+2;\ {}^{2k+2}/_{2} = {}^{2(k+1)}/_{2} = k+1$$

Therefore, $P(n)$ is true for all $n \in \mathbb{N}$. ∎

Lemma 1.1.2:

$P(n)$: For all $n \in \mathbb{N}$, the sum of $2n+1$ is equivalent to an odd number (i.e., an integer that would not produce an integer if subtracted by half its value)

Proof of Lemma 1.1.2 (by Strong Induction):

Base Case, $P(1) : 2(1) + 1 = 3;\ {}^{3}/_{2} = 1 + {}^{1}/_{2} \notin \mathbb{Z}$

Inductive Hypothesis: Assume $P(r)$ is true for all $\{r \in \mathbb{N} : 1 \leq r \leq k;\ k \in \mathbb{N}\}$

Inductive Step: $P(r)$ implies $P(k+1)$

$$P(r) : 2r + 1;\ {}^{2r+1}/_{2} = r + {}^{1}/_{2}, \qquad P(k+1) : 2(k+1) + 1 = 2k + 3;\ {}^{2k+3}/_{2} = k + {}^{3}/_{2}$$

If $2r + 1$ is divided in half, there would be an additional value of ${}^{1}/_{2}$ after the integer r

If $2k + 3$ is divided in half, there would be an additional value of ${}^{1}/_{2}$ after the integer sum $k + 1$

The value of ${}^{1}/_{2}$ is not an integer. Therefore, $P(n)$ is true for all $n \in \mathbb{N}$. ■

Lemma 1.1.3:

$P(q)$: For all $\{q \in \mathbb{Q} : q = {}^{n}/_{2}\,;\ n \in \mathbb{N}\}$, the product of $10 \times q$ is equal to a natural number

Proof of Lemma 1.1.3 (by Regular Induction):

Base Case, $P({}^{1}/_{2}) : {}^{1}/_{2}\,(10) = 5$

Inductive Hypothesis: Assume $P(r)$ is true for any $r \in \mathbb{Q} : r = {}^{k}/_{2}\,;\ k \in \mathbb{N};\ r \geq {}^{1}/_{2}$

Inductive Step: $P(r)$ implies $P(r + 1)$

$$P(r) :\ {}^{k}/_{2}\,(10) = {}^{10k}/_{2} = 5k, \qquad P(r+1) : \left({}^{k}/_{2} + 1\right)(10) = {}^{10k}/_{2} + 10 = 5k + 10$$

Therefore, $P(q)$ is true for all $\{q \in \mathbb{Q} : q = {}^{n}/_{2}\,;\ n \in \mathbb{N}\}$. ■

Remark 1.1.2: Regarding the Principle of Mathematical Induction, the significance of the Base Case is that it would involve an explicit construction that would demonstrate the "truth" of the open statement $P(n)$ in at least one case. Additionally, the **Inductive Hypothesis** would follow from the fact that $P(n)$ is provably true in at least one case, and the pertinence of the Inductive Hypothesis in mathematical induction would depend on the condition that $P(n)$ is directly validated for at least one case $P(a)$. The case (or cases) for which $P(n)$ is directly validated would normally include the Base Case.

By the Inductive Hypothesis, a specific statement would be postulated as true for some element of a non-dense, well-ordered set of numbers, even if the cardinality of the set is not conclusively finite. The Inductive Hypothesis is also required to complete the proof of an open statement by mathematical induction, as it would enable the Base Case to connect with the Inductive Step, and by the Inductive Step, it would be decided that the open statement $P(n)$ holds for all $\{n \in N : n \geqslant a;\ a \in N\}$.

1.2 An Overview of Functions (in Set Theory)

1.2.1 Injection, Surjection, Bijection

Definition 1.2.1.1: A cardinal number is a number that corresponds to the total amount of elements in a set. If there exists a set A, its *cardinality* $|A|$ would be the cardinal number of A.

Definition 1.2.1.2: If there exists a pair of sets, A and B, then in set-theoretic terms, a *function* (or a *map*) f would be a sort of systematic process that associates elements of A to elements of B.

Regarding the function $f : A \rightarrow B$, the set A corresponds to the domain of f, whereas the set B corresponds to the range of f. All possible input values of f are contained within the domain of f, and all possible output values of f are contained within the range of f.

By a function f, each element of A must be associated with some element of B. Any pair of distinct elements of A can be associated with the same element in B via a function f, but there cannot exist a single element of A that is associated with two or more distinct elements of B simultaneously.

The fundamental properties of a function $f : A \rightarrow B$ are as follows:

- For each $a \in A$, there is some $b \in B$, such that $f(a) = b$.
- For any pair of distinctive elements $b_i, b_j \in B;\ b_i \neq b_j$, there is no $a \in A$ for which $f(a) = b_i$ and $f(a) = b_j$.

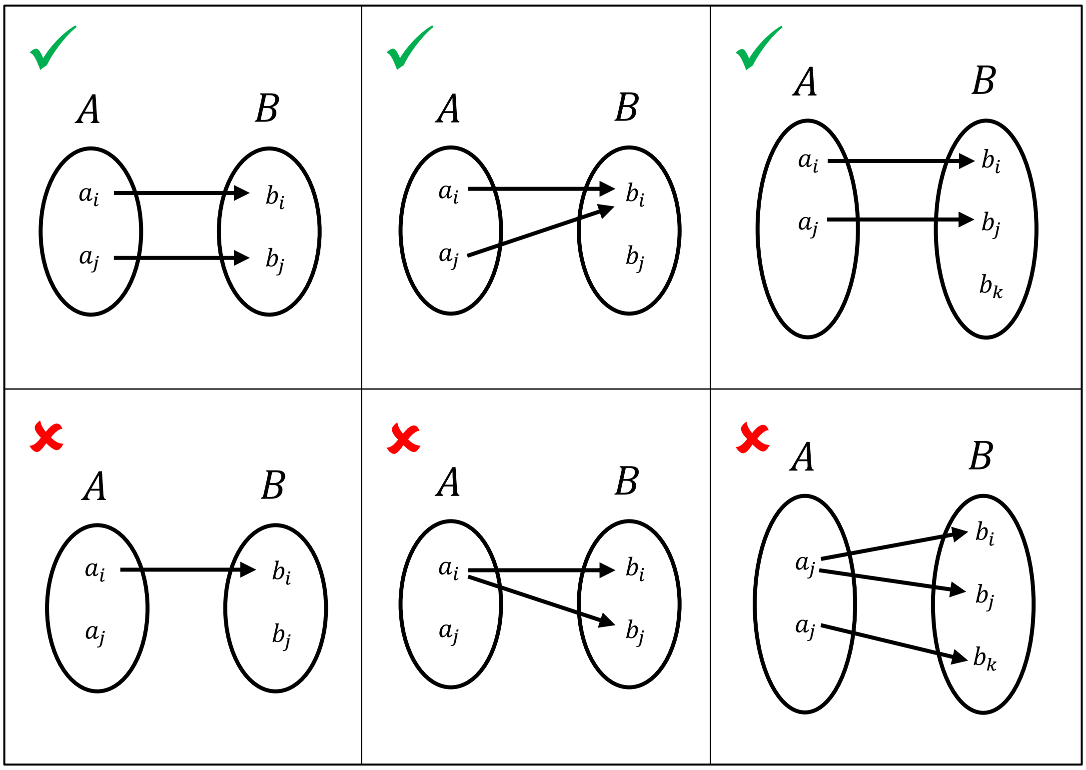

Figure 1.2.1.1: An illustration of functions and non-functions. A check symbol is included in boxes that contain an illustration of a function. A cross symbol is included in boxes that do not contain an illustration of a function.

In theory, there are three fundamental types of functions: the injective function, the surjective function, and the bijective function.

Definition 1.2.1.3 (Injective Function): A function $f : A \to B$ is injective, if and only if each element of the domain A maps *exclusively* to some element of the range B. That is, each $a \in A$ has a one-to-one correspondence with some $b \in B$. (See upper left and upper right boxes in **Fig. 1**).

For all $b \in X$, where $X \subseteq B$ and $|X| = |A|$, there is only one $a \in A$, such that $f(a) = b$.

For all $a_i, a_j \in A$, $f(a_i) = f(a_j)$ implies $a_i = a_j$.

Definition 1.2.1.4 (Surjective Function): A function $f : A \to B$ is surjective, if and only if each element of the range B is mapped from at least one element of the domain A.

For all $b \in B$, there is at least one $a \in A$, such that $f(a) = b$.

Remark 1.2.1.1: A function f would not be injective if and only if there is a pair of distinctive elements $a_i, a_j \in A$; $a_i \neq a_j$, such that $f(a_i) = f(a_j)$. (See upper middle box in **Fig. 1**).

Furthermore, an injective function does not necessarily have to allow each element of the range to be mapped from one or more elements from the domain. But a surjective function must allow each element of the range to be mapped from one or more elements from the domain.

A function can be injective (but would also not be surjective) if there is some element $b_\omega \in B$, such that there is no element $a \in A$ for which $f(a) = b_\omega$. (See upper right box in **Fig. 1**).

Definition 1.2.1.5 (Bijective Function): A function $f : A \to B$ is bijective, if and only if:

- Each element of the domain A maps *exclusively* to some element of the range B
- Each element of the range B is mapped by only one element of the domain A

(See upper left box in **Fig. 1**)

1.2.2 Axioms of Peano Arithmetic & Schroeder-Bernstein Theorem

Definition 1.2.2.1: Regarding a non-dense, well-ordered set of numbers N (See **Definition 1.4.2**), a *successor function* S would map each element $n \in N$ to its immediate successor $S(n)$, such that there would not exist an element $x \in N$, for which $n \prec x \prec S(n)$.

$$\exists(S : N \to N) \to \exists\left[N : n, S(n) \in N \wedge \neg\big(x \in N;\ n \prec x \prec S(n)\big)\right]$$

The successor function S is *recursive*, such that it repeats the same operation on each successor $S(n) \in N$, as it would for the element $n \in N$ that immediately precedes to $S(n) \in N$.

Peano Arithmetic (PA) is a constructive axiomatic framework for the natural numbers (or non-negative integers). The axioms of Peano Arithmetic may be applied in a mathematical inductive process.

Definition 1.2.2.2 (Peano Axioms): The axioms of Peano Arithmetic are defined with respect to the set of non-negative integers $\mathbb{N}_{\geq 0} = \{p \in \mathbb{Z} : p \geq 0\}$, or the set of natural numbers $\mathbb{N}$:

- There exists a pair of characterized elements "0" and "1", where $0,1 \in \mathbb{N}_{\geq 0}$ and $1 \in \mathbb{N}$
- The equality of non-negative integers is reflexive, symmetric and transitive.
  - Reflexivity: $n = n$, for all $n \in \mathbb{N}_{\geq 0}$
  - Symmetry: If $n_i = n_j$, then $n_j = n_i$, for all $n_i, n_j \in \mathbb{N}_{\geq 0}$
  - Transitivity: If $n_i = n_j$ and $n_j = n_k$, then $n_i = n_k$, for all $n_i, n_j, n_k \in \mathbb{N}_{\geq 0}$
- There exists a successor function $S : \mathbb{N}_{\geq 0} \to \mathbb{N}_{\geq 0}$, such that:
  - For each non-negative integer n, there is also a non-negative integer $S(n)$
  - S is injective, i.e., $S(n_i) = S(n_j)$ implies $n_i = n_j$ for any $n_i, n_j \in \mathbb{N}_{\geq 0}$
  - There does not exist an element $n \in \mathbb{N}_{\geq 0}$ for which $S(n) = 0$ (i.e., the number 0 is not a successor of any non-negative integer)

    Likewise, there does not exist any $n \in \mathbb{N}$ for which $S(n) = 1$ (i.e., the number 1 is not a successor of any natural number)

    Essentially, $\{n \in \mathbb{N}_{\geq 0} : S(n) = 0\} = \emptyset$, and $\{n \in \mathbb{N} : S(n) = 1\} = \emptyset$
- **Principle of Mathematical Induction**: If Z is a set, such that:
  - $0 \in Z$
  - $n \in Z$ also implies $S(n) \in Z$, for every non-negative integer $n \in \mathbb{N}_{\geq 0}$

  then the set of non-negative integers $\mathbb{N}_{\geq 0}$ is equal to Z, i.e., $\mathbb{N}_{\geq 0} = Z$.

  Equivalently, if $P(n)$ is a statement, such that:
  - $P(a)$ is true for some $a \in \mathbb{N}_{\geq 0}$
  - If $P(n)$ is true, it implies $P(n + 1)$ is also true, for all $n \in \mathbb{N}_{\geq 0};\ n \geq a$

  then $P(n)$ is true for all elements of the set $\{n \in \mathbb{N}_{\geq 0} : n \geq a;\ a \in \mathbb{N}_{\geq 0}\}$.

Lemma 1.2.2.1: For any pair of countable sets A and B, the following conditions would hold:

- If there exists an injective map from A to B, it means the cardinality of A is either less than or equal to the cardinality of B, i.e., $|A| \leq |B|$
- If there exists a surjective map from A to B, it means the cardinality of A is either larger than or equal to the cardinality of B, i.e., $|A| \geq |B|$
- If there exists a bijective map from A to B, it means the cardinality of A is equivalent to the cardinality of B, i.e., $|A| = |B|$. A bijection implies the equinumerosity of sets A and B

Remark 1.2.2.1: The proof of **Lemma 1.2.2.1** for countable sets would depend on mathematical induction, such that the **Inductive Hypothesis** (Regular Induction) must be invoked as follows:

- For the injective map $f : A \to B$, it must be assumed $P(n)$ is true for some element of a non-dense, well-ordered set of numbers N

  In this case, $P(n)$ states, "There is a one-to-one map from $a_n \in A$ to some $b \in B$ via $f : A \to B$"

- For the surjective map $f : A \to B$, it must be assumed $P(n)$ is true for some element of a non-dense, well-ordered set of numbers N

  In this case $P(n)$ states, "Each $b_n \in B$ is mapped from some $a \in A$ via the function $f : A \to B$"

- For the bijective map $f : A \to B$, it must be assumed $P(n)$ is true for some element of a non-dense, well-ordered set of numbers N

  In this case, $P(n)$ states, "There is a map from $a_n \in A$ to $b_n \in B$ via the function $f : A \to B$"

In theory, the axioms of **Peano Arithmetic** can be used to construct a *witness* (i.e., an example that proves $P(n)$ is valid in a specified case) for each case of an open statement $P(n)$ up to some arbitrary point.

By a successor function $S : N \to N$, the *witnesses* can be $S(n) = n + 1$ for each consecutive case of the open statement $P\big(S(n)\big)$ as to injective, surjective and bijective maps in **Lemma 1.2.2.1.**

Ideally, it would be possible to *deductively* prove a special case of **Lemma 1.2.2.1**, where:

- For the injective function, A is a finite set with a specified cardinality (e.g., $|A| = k \in \mathbb{N}_{\geq 0}$, such that $|B| \geq k$; $|B| \in \mathbb{N}_{\geq 0}$)
- For the surjective function, B is a finite set with a specified cardinality, (e.g., $|B| = k \in \mathbb{N}_{\geq 0}$, such that $|A| \geq k$; $|A| \in \mathbb{N}_{\geq 0}$)
- For the bijective function, A and B are finite sets with specified cardinalities (e.g., $|A| = k \in \mathbb{N}_{\geq 0}$ and $|B| = k \in \mathbb{N}_{\geq 0}$)

Definition 1.2.2.3 (Function Composition): If $f : A \to B$ and $g : B \to C$ are functions, then the composition $g \circ f : A \to C$ is also a function, where $g \circ f(a) = g\big(f(a)\big)$ for all $a \in A$.

Lemma 1.2.2.2 (Schroeder-Bernstein Theorem): If there are injective maps from A to B and B to A respectively, then there is a bijection between A and B.

Remark 1.2.2.2: Ideally, this would be a simple proof of **Lemma 1.2.2.2.** According to **Lemma 1.2.1.1**:

- $|A| \leq |B|$ implies there exists an injection from A to B
- $|A| \geq |B|$ implies there exists an injection from B to A

Furthermore, $|A| = |B|$ implies there exists a bijection between A and B.

Assume there would not exist a bijection between A and B, such that $|A| \neq |B|$. As a result, it would have to be concluded that $|A| > |B|$ and $|A| < |B|$ simultaneously, which leads to a contradiction.

For this reason, the rational conclusion is that there is a bijection between A and B, on the condition there are injective functions that map A to B and B to A respectively. ■

This proof is based on the presupposition that some information about the cardinalities of a pair of sets, can be obtained via the foreknowledge of an injection from at least one of the sets into the other set.

This is also a *proof by contradiction*. A prior lemma (i.e., **Lemma 1.2.2.1**) was used to determine a specific consequence of injections from A to B and B to A. Then, it was asserted that a bijection between sets A and B implies the cardinalities of both sets are equal to each other. Finally, a contradiction was discovered via the assumption that there does not exist a bijection between A and B.

However, this is not a direct proof of **Lemma 1.2.2.2**, as it does not provide a method for constructing a witness that can be used to test the validity of **Lemma 1.2.2.2** (as no such construction is provided). The

Principle of Mathematical Induction would be required to prove the **Schroeder-Bernstein Theorem** (i.e., **Lemma 1.2.2.2**) for transfinite sets, which would involve the **Inductive Hypothesis**.

As to **Lemma 1.2.2.2**, it must be assumed $P(n)$ is true for some $n \in \mathbb{Z}$; the statement $P(n)$ is as follows.

$P(n) =$ "Regarding injections $f : A \to B$ and $g : B \to A$, there exists a corresponding $f(x_n) \in \text{domain}(g)$ for some $g\big(f(x_n)\big) \in \text{domain}(f)$, where $f(x_n) \in \text{range}(f)$"

The **Base Case**, if any, would be for some natural number $1 \in \mathbb{N}$, such that:

- $f(x_1)$ would express the first map from domain(f) to range(f) via the function f
- $g\big(f(x_1)\big)$ would express the first map from range(f) to domain(f) via the composition $g \circ f$, where range$(f) =$ domain(g), and domain$(f) =$ range(g)

The **Inductive Hypothesis,** would obviously be the assumption that $P(n)$ holds for some natural number $n \in \mathbb{N}$, where $n \geq 1$, and $P(1)$ is the base case.

The **Inductive Step**, would prove that for all $n \in \mathbb{N}$, if $P(n)$ holds, then $P(n+1)$ also holds.

The base case is applied to avoid the *viciousness* of infinite regress. It would be impossible to invoke the **Inductive Hypothesis** without a specified base case, because there cannot be a base case in a sequence that is infinitely regressive, and the **Inductive Hypothesis** is derived from a specified base case. For this reason, a proof by mathematical induction would necessitate that a base case $P(a)$ holds.

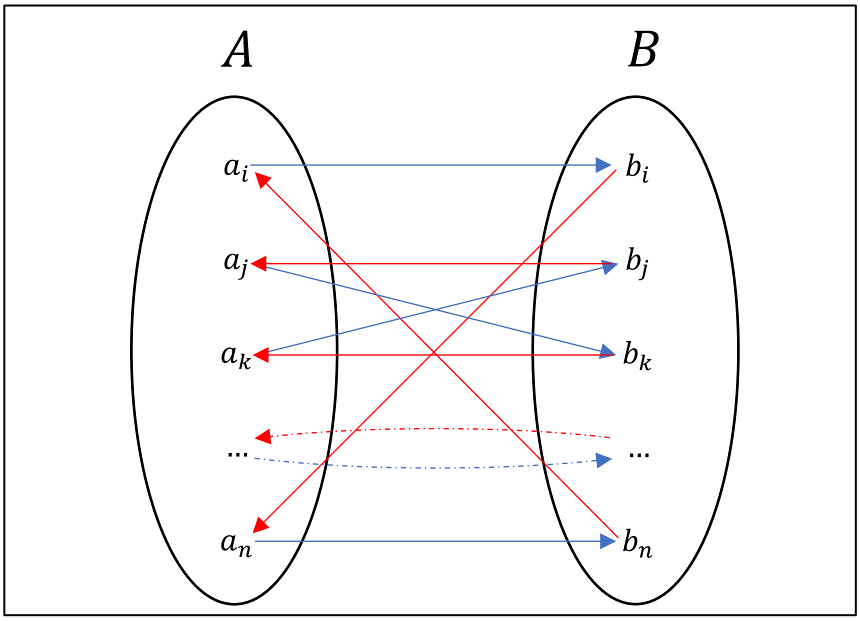

Figure 1.2.2.1: The illustration may appear to be a decorative array of lines. But it is meant to be a non-specific illustration of injections into B from A and injections into A from B. The aim was to provide an impression of how a bijection between sets can be inferred by injections from each set into the other set.

A proof of **Lemma 1.2.2.2 (Schroeder-Bernstein Theorem)** by a special (finite) case example:

Assume $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$. As to the injection $f : A \to B$, assume there is at least one injection that maps $a_1 \in A$ to $b_1 \in B$ via f, such that $f(a_1) = b_1$.

Now the remaining injections can be specified with respect to elements of the index set $I = \{1,2,3\}$ (this does not require the Inductive Hypothesis).

$$P(n) = \text{“Regarding injections } f : A \to B \text{ and } g : B \to A\text{, there exists a corresponding } f(x_n) \in \text{domain}(g) \text{ for some } g\big(f(x_n)\big) \in \text{domain}(f)\text{, where } f(x_n) \in \text{range}(f)\text{”}$$

Base Case: $P(1) : f(a_1) \in \text{range}(f) \Rightarrow f(a_1) \in \text{domain}(g);\ g\big(f(a_1)\big) \in \text{domain}(f)$

Second and Third Case: $f(a_{2,3}) \in \text{range}(f) \Rightarrow f(a_{2,3}) \in \text{domain}(g);\ g\left(f(a_{2,3})\right) \in \text{domain}(f)$

If $f(a_1) = b_1$, then $f(a_{2,3}) = b_{2,3}$. Therefore, $B = \{f(a_1), f(a_2), f(a_3)\}$

As g is an injection from B to A, then $g\big(f(a_1)\big) \in A$.

Furthermore, $g\left(f(a_{2,3})\right) \in A - \left\{a_n \in A : n \in I;\ \big((n \neq 1) \veebar (n \neq 2) \veebar (n \neq 3)\big)\right\}$:

- If $g\big(f(a_1)\big) = a_1$, then $g\left(f(a_{2,3})\right) = a_{2,3}$
- If $g\big(f(a_1)\big) = a_2$, then $g\left(f(a_{2,3})\right) = a_{1,3}$
- If $g\big(f(a_1)\big) = a_3$, then $g\left(f(a_{2,3})\right) = a_{1,2}$

Therefore, $A = \{g\big(f(a_1)\big), g\big(f(a_2)\big), g\big(f(a_3)\big)\}$

Each element of A and B is indexed by a natural number, $n \in I$.

It is also noticeable that the same numbers are shared between the indexes of elements of A and B.

For this reason, a bijection $h : A \to B$ can be constructed, such that for all $n \in I$:

$$h\Big(g\big(f(a_n)\big)\Big) = f(a_n)$$

By the function h, the following maps are all one-to-one:

1. $g\big(f(a_1)\big) \to f(a_1)$
2. $g\big(f(a_2)\big) \to f(a_2)$
3. $g\big(f(a_3)\big) \to f(a_3)$

By the inverse of h, the following maps are all one-to-one:

1. $f(a_1) \to g\big(f(a_1)\big)$
2. $f(a_2) \to g\big(f(a_2)\big)$
3. $f(a_3) \to g\big(f(a_3)\big)$

Therefore, h is a bijection from A to B, where the cardinalities of A and B are equal to three. ■

Remark 1.2.2.3: Ideally, this proof can be extended to other pairs of sets with a cardinality that is countably finite, such that the cardinality of the index set I would also be countably finite.

An extrapolation of the proof to **Lemma 1.2.2.2** for countably finite sets is specified below.

Regarding the index set $I \subsetneq \mathbb{N}$, where I is a countably finite set:

- The statement $P(n)$ would hold for $|I|$ number of cases
- If $f(a_1) \in \text{domain}(g)$, then $g\big(f(a_1)\big) \in \text{domain}(f)$
- For all $n \in I$, it would be demonstrable that $f(a_n) \in \text{domain}(g)$
- $\big\{g\big(f(a_n)\big) \in \text{domain}(f) : f(a_n) \in \text{range}(f)\big\}_{n \neq 1} \subseteq A - \{a_n \in A : n \in K \subseteq I;\ |K| = 1\}$
- The cardinality of $A - \{a_n \in A : n \in K \subseteq I;\ |K| = 1\}$ would be countably finite
- $h\Big(g\big(f(a_n)\big)\Big) = f(a_n)$, for all $n \in I \subsetneq \mathbb{N}$

However, if the proof of **Lemma 1.2.2.2** were to be extended to pairs of sets that have a transfinite cardinality, such that the cardinality of the index set I would be transfinite, then:

- The **Inductive Hypothesis** would have to be invoked, (i.e., assume $P(n)$ holds for some number $n \geq a$, where $P(a)$ is already proven). This means it would have to be shown that $P(n+1)$ holds if $P(n)$ also holds for some $n \geq a$, where n is an element of a non-dense, well-ordered set.
- If $g\big(f(a_1)\big) \in A$, then it would have to be concluded that for all $n \in I$:
  - $\big\{g\big(f(a_n)\big) \in \text{domain}(f) : f(a_n) \in \text{range}(f)\big\}_{n \neq 1} \subseteq A - \{a_n \in A : n \in K \subseteq I;\ |K| = 1\}$
  - The cardinality of $A - \{a_n \in A : n \in K \subseteq I;\ |K| = 1\}$ is larger than or equal to $\aleph_0$
- At least two axioms of **Peano Arithmetic** would have to be applied. Specifically, the axioms, (1) For each $n \in \mathbb{N}$, its immediate successor $S(n)$ is also a natural number, and (2) S is injective
- $h\Big(g\big(f(a_n)\big)\Big) = f(a_n)$, for all $n \in I$, where $|I| \geq \aleph_0$

Apparently, it is not possible to exhaust all possible mappings from A to B in a finite series of cases, where A and B are transfinite sets. For this reason, a more comprehensive proof would be required to prove the Schroeder-Bernstein Theorem for pairs of sets that have a transfinite cardinality.

The proof of **Lemma 1.2.2.2** by finite case example is more deductive, such that less assumptions are required to complete the proof (i.e., the **Inductive Hypothesis** and **Peano Axioms** can be avoided). This might appeal to *finitism*, i.e., a mathematical perspective that accepts only the existence of finite sets.

However, there are inductive proofs of **Lemma 1.2.2.2** that are more comprehensive (i.e., they can be extended to sets with a transfinite cardinality). A more comprehensive proof of Schroeder-Bernstein Theorem can be found in Jech & Hrbacek (1999) [1], Jech (2003) [2] and Gunning (2018) [3].

The **Schroeder-Bernstein Theorem** is a fundamental result in conventional set theory, as it demonstrates a certain kind of relation between pairs of sets with respect to their cardinalities.

To be exact, it demonstrates a sort of anti-symmetric relation between the cardinalities of sets, such that if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$. Otherwise, $|A| \neq |B|$, and either $|A| < |B|$ or $|B| < |A|$.

1.2.3 Cantor's Theorem

Axiom of Power Set: For any set X, there is a power set $P(X)$; all subsets of X are elements of $P(X)$.

$$\forall X : \exists P(X);\ \forall A\left[(A \subseteq X) \Rightarrow \big(A \in P(X)\big)\right]$$

Axiom Schema of Replacement: If there is a class function F, and a non-empty set U, then there exists a set V, such that $V = F(U)$. That is, the image $F(U)$ of a class function F is a set.

Axiom Schema of Restricted Comprehension: As to a predicate ψ (i.e., a property or relation), there is a non-empty set A that is a subset of another set X, such that x would be an element of A if and only if x is also an element of X and the predicate $\psi(x)$ holds.

$$\forall A\ \exists X\ \forall x : \left[(x \in X) \Leftrightarrow \big((x \in A) \wedge \psi(x)\big)\right]$$

The non-empty set A and the predicate $\psi(x)$ are specified below:

$$A = \{x \in X : \psi(x)\};\ A \subseteq X;\ A \neq \emptyset, \qquad \psi(x) \neq (x \notin x), (x \notin \mathrm{X})$$

Cantor's Theorem is another fundamental result in conventional set theory. Essentially, it states the cardinality of any set is strictly smaller than the cardinality of its powerset. For any set X and its powerset $P(X)$, there would exist an injection from X to $P(X)$, but there would not exist a surjection from X to $P(X)$.

A corollary of Cantor's Theorem is that there does not exist a "largest cardinal number", i.e., any cardinal number can always be used to produce a larger cardinal number via the powerset operation.

It is clear that Cantor's Theorem relies on the **Powerset Axiom.**

It is also clear that Cantor's Theorem relies on the **Axiom Schema of Replacement**, as the specified axiom allows the image of a function (i.e., the output value of a function) to be a set.

By the Axiom Schema of Replacement, the image of a set under a function is also a set. This is essential for a function that would map a set to its own subsets.

Cantor's Theorem asserts the existence of a function f, such that each element of the range of f is a subset of the domain of f (and the range of f is equivalent to the powerset of the domain of f).

Lemma 1.2.3.1 (Cantor's Theorem): For any set X and its powerset $P(X)$, the function $f : X \to P(X)$ is not surjective. Therefore, the cardinality of $P(X)$ is strictly larger than the cardinality of X.

Proof of Lemma 1.2.3.1: The formal proof of Cantor's Theorem declares the existence of a set that is a subset of the domain of f and not a subset of the range of f. The set is notated below:

$$A = \{x \in X : \psi(x)\}, \qquad \psi(x) : x \notin f(x)$$

By the Powerset Axiom, a set that is a subset of X would simultaneously be an element of $P(X)$.

As A is a subset of X, this means A is an element of $P(X)$.

It also means A is a well-defined set by **the Axiom Schema of Separation**, as:

- It is clear there are elements of A that are also elements of another set X (i.e., A is a subset of X)
- There is a property ψ that applies to elements of A. That is, the property that elements of A are not elements of $f(x)$, where $f(x) \in \text{range}(f)$, and $\text{range}(f) = P(X)$

If it were to be assumed f is surjective, it would mean there exists some element $\alpha \in X$ that maps to a specific element of $P(X)$ via f, such that $f(\alpha) = A$.

But if $f(a) = A$, then it would infer a contradiction, where $a \in A$ or $a \notin A$.

Case I: If $\alpha \in A$, then $\alpha \in f(\alpha)$, since $f(a) = A$, which means $f(\alpha)$ would be an element of the range of f. But the property $\psi(x)$ determines that all elements of A cannot be elements of $f(x)$ in the range of f. Therefore, α would simultaneously not be an element of $f(\alpha)$, via the property $\psi(\alpha) : \alpha \notin f(\alpha)$.

Case II: If $\alpha \notin A$, then $\alpha \in f(\alpha)$, as by definition, A is the set of all elements $x \in X$ that are not elements of $f(x)$. However, $f(\alpha) = A$, which means α would simultaneously be an element of A.

For this reason, it would be contradictory for f to be surjective.

However, there is an injective map from X to $P(X)$ via f, which may be defined as $f(x) = x \cup \{x\}$ or $f(x) = \{x\}$. Therefore, the cardinality of X is strictly smaller than the cardinality of its powerset $P(X)$. ■

1.3 The Axiom of Infinity and Ordinal Numbers

Definition 1.3.1: A finite set, is a set that has a cardinality of finite magnitude; the total quantity of elements of a finite set would be limited.

Definition 1.3.2: By the Axiom of Infinity, an "infinite" (or transfinite) set, is a set of which its cardinality is not limited. That is, a set A, of which its cardinality is larger than the cardinality of any finite set.

Axiom of Infinity: There is an "infinite" set.

An "infinite" set may be defined recursively (or inductively) with respect to the von Neumann representation of non-negative integers

$$\exists N : \neg(N = \emptyset);\ \forall n \left[(n \in N) \Rightarrow \big((n \cup \{n\}) \in N\big)\right]$$

An “infinite” set may also be defined recursively (or inductively) with respect to the Zermelo representation of non-negative integers

$$\exists N : \neg(N = \emptyset);\ \forall n\, [(n \in N) \Rightarrow (\{n\} \in N)]$$

Definition 1.3.3: A partially ordered set (or poset) P, is a set of which there is significance about the order by which elements of P are arranged.

Specifically, a partially ordered set is a structure, $P = (X, \leq)$, where the set X designates the domain of P, and the relational operator $\leq$ designates the signature that is applied on the domain of P.

Definition 1.3.4: A partial order is a homogeneous relation $\leq$ on a set P, such that reflexivity, symmetry and transitivity are essential properties of the relation on P [2,3]. For all $r_\alpha, r_\beta, r_\gamma \in P$:

- Reflexivity: $r \preccurlyeq r$ (i.e., each element of P relates to itself)
- Symmetry: If $r_\alpha \preccurlyeq r_\beta$ and $r_\beta \preccurlyeq r_\alpha$, then $r_\alpha = r_\beta$
- Transitivity: If $r_\alpha \preccurlyeq r_\beta$ and $r_\beta \preccurlyeq r_\gamma$, then $r_\alpha \preccurlyeq r_\gamma$

A total order (or chain) $K \subseteq P$, is a partial order that also has the extra property of *strong connectedness*, i.e., any pair of elements of K are “comparable” [1,2].

If there exists a total order on a set $K \subseteq P$, then the following properties would hold as to elements of K:

- Reflexivity, Symmetry and Transitivity (the properties of a partial order)
- Strong Connectedness: For all $r_\alpha, r_\beta \in K$, $r_\alpha \preccurlyeq r_\beta$ or $r_\beta \preccurlyeq r_\alpha$

Definition 1.3.5: A well order is a homogeneous relation $\preccurlyeq$ on a set P, such that there is a least element p in each non-empty of subset of P [1,2].

For all $r \in A$ where $A \subseteq P$, there is some $p \in A$ where $p \preccurlyeq r$ (i.e., either p precedes r, or p equals r)

Axiom of Union: There exists a set X, such that the union of all elements in X is also a set.

That is, for any set X, there is a set U of which its elements, are the elements of the elements of X.

$$\forall \mathrm{X}\, \exists U\, \forall a : (a \in U) \Rightarrow \exists A : (a \in A) \wedge (A \in \mathrm{X})$$

Definition 1.3.6 (Powerset Chain): $P^{|X|}(\emptyset)$ designates a chain of powersets of the empty set.

The total number of powersets P in the chain $P\big(P(\ldots\ P(\emptyset)\ \ldots)\big)$, is equivalent to the cardinality of a non-empty set X, such that:

$$\left|P^{|X|}(\emptyset)\right| = 2^{|X|-1}, \qquad |X| \geq 1$$

Example 1.3.1 (for Powerset Chain):

The cardinality of a chain of two powersets of $\emptyset$: $|P^2(\emptyset)| = \left|P\big(P(\emptyset)\big)\right| = 2^{2-1}$

The cardinality of a chain of three powersets of $\emptyset$: $|P^3(\emptyset)| = \left|P\left(P\big(P(\emptyset)\big)\right)\right| = 2^{3-1}$

Definition 1.3.7: An *ordinal number* α is a number that corresponds to the numerical (a^{th}) position of an element in a well-ordered set.

A *limit ordinal* is an ordinal number that is neither zero nor a successor ordinal. Essentially, a non-zero ordinal number that is also not the immediate successor of another ordinal number [1,2].

As for a pair of ordinal numbers α and β:

- If an ordinal number β precedes another ordinal number α, then the successor ordinal $S(\beta)$ would precede or equal α. That is, if $\beta \prec \alpha$, then $S(\beta) \preccurlyeq \alpha$
- If α is not a limit ordinal, its immediate predecessor $S(\beta)$ would also not be a limit ordinal
- If α is a limit ordinal, then it does not have an immediate predecessor ordinal

By the **von Neumann representation**, the ordinal numbers are defined as below:

- $0 = \emptyset$
- $\alpha + 1 = \alpha \cup \{\alpha\}$
- Each ordinal α is well-ordered by the $\in$-membership relation, where $\beta \in \alpha$ implies $\beta < \alpha$
- For all $\beta < \alpha + 1$, either $\beta \in \alpha$ or $\beta = \alpha$
- If α is a limit ordinal, then

$$\alpha = \bigcup_{\beta<\alpha} \beta$$

(i.e., α is equal to the union of all ordinal numbers β that are smaller than α)

The first several von Neumann ordinal numbers would appear as below:

$$\emptyset = 0, \quad \{\emptyset\} = 1, \quad \big\{\emptyset, \{\emptyset\}\big\} = 2, \quad \Big\{\emptyset, \{\emptyset\}, \big\{\emptyset, \{\emptyset\}\big\}\Big\} = 3, \quad \Big\{\emptyset, \{\emptyset\}, \big\{\emptyset, \{\emptyset\}\big\}, \big\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\big\}\Big\} = 4, \ldots$$

By the **Zermelo representation**, the ordinal numbers are defined as below:

- $0 = \emptyset$
- $\alpha + 1 = \{\alpha\}$

  Each ordinal α is well-ordered by the $\in$-membership relation of some chain of powersets $P^{|X|}(\emptyset)$, where $Z = \big\{z \in P^{|X|}(\emptyset) : |z| \leq 1;\ |X| \geq 1\big\}$

  For all $\alpha, \beta \in Z$, if $\alpha, \beta \in P^{|X|}(\emptyset)$, then $\alpha \in P^{|X|+1}(\emptyset);\ \beta \notin P^{|X|+1}(\emptyset)$, implies $\beta < \alpha$
- For all $\beta < \alpha + 1$, either $\beta < \alpha$ (as shown in previous point) or $\beta = \alpha$
- If α is a limit ordinal, then

$$\alpha \notin \big\{z \in P^{|X|}(\emptyset) : |z| \leq 1;\ 1 \leq |X| < \aleph_0\big\}$$

(i.e., α would at least not be an element of a powerset chain $P^{|X|}(\emptyset)$, where $1 \leq |X| < \aleph_0$)

The first several Zermelo ordinal numbers would appear as below:

$$\emptyset = 0, \quad \{\emptyset\} = 1, \quad \big\{\{\emptyset\}\big\} = 2, \quad \Big\{\big\{\{\emptyset\}\big\}\Big\} = 3, \quad \Big\{\Big\{\big\{\{\emptyset\}\big\}\Big\}\Big\} = 4, \quad \Big\{\Big\{\Big\{\big\{\{\emptyset\}\big\}\Big\}\Big\}\Big\} = 5, \ldots$$

Definition 1.3.8: If β is an ordinal number, then a *successor ordinal* $S(\beta)$ is the smallest ordinal number that succeeds β, such that there is no ordinal α that satisfies the inequality $\beta \prec \alpha \prec S(\beta)$.

Axiom of Foundation (or Regularity): Every non-empty set has an $\in$-minimal element.

That is, for any non-empty set X, there is a set n that is an element of X and disjoint with X simultaneously, such that there would not exist an element $x \in n$, if and only if $n = \emptyset$:

$$\forall \mathrm{X} : \neg(\mathrm{X} = \emptyset) \Rightarrow \exists n \left[(n \in \mathrm{X}) \wedge \big((n \cap \mathrm{X}) = \emptyset\big)\right];\ \forall x : \neg(x \in n) \Rightarrow (n = \emptyset)$$

Remark 1.3.1: The ordinal numbers are essentially well-ordered sets; they are well-ordered via the $\in$-membership relation. The Axiom of Regularity supports the structure of ordinal numbers as it ensures the ordinal numbers are well-founded. By the Axiom of Regularity, the set-theoretic construction of non-negative integers would correspond to the Zermelo or von Neumann representation of ordinal numbers, as Zermelo and von Neumann ordinals correspond to a well order on the set of non-negative integers.

As the set of ordinal numbers contains a least element (i.e., the empty set), the ordinal numbers are also well-founded (i.e., there is a least ordinal number). If not for the Axiom of Regularity, the definition of ordinal numbers would be *viciously* circular (i.e., it would lead to an indefinite regression of $\in$-memberships of a non-empty set). For this reason, the Axiom of Regularity is required for the ordinal numbers to be well-founded in order to avoid a vicious cycle in ordinal number representation.

Definition 1.3.9: Assume there is a total order $\leq$ on a set X, where A is a proper subset of X. Then A is an initial segment of X under $\leq$, if and only if [1]:

$$\forall(a \in A)\ \forall(x \in X) : (x \leq a) \Rightarrow (x \in A)$$

As for **von Neumann ordinal numbers** V, defined by elements of the union of $P^{|X|}(\emptyset)$, an initial segment may be notated as below (for von Neumann ordinals that precede or equal to 4):

$$A = \Big\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \big\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\big\}, \big\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \big\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\big\}\big\}\Big\} \subsetneq V$$

As for **Zermelo ordinal numbers** Z, defined by elements $a \in P^{|X|}(\emptyset)$, where $|a| = 1$, an initial segment may be notated as below (for Zermelo ordinals that precede or equal to 5):

$$A = \Big\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \big\{\{\{\emptyset\}\}\big\}, \big\{\big\{\{\{\emptyset\}\}\big\}\big\}, \Big\{\big\{\big\{\{\{\emptyset\}\}\big\}\big\}\Big\}\Big\} \subsetneq Z$$

By definition, the cardinality of any non-zero Zermelo ordinal is 1. For this reason, any non-zero Zermelo ordinal $\alpha \in \left[P^{|X|}(\emptyset)\right]^1$ can be injected into a non-empty set, and this would include non-empty sets of which their cardinalities are smaller than the ordinal number that corresponds to it. However, this is not possible with von Neumann ordinals, because unlike Zermelo ordinals, the cardinality of a von Neumann ordinal would correspond to the ordinal number itself, such that there are von Neumann ordinals with cardinalities that are larger than or equal to 2, i.e., there are some $\alpha \in V$, where $|\alpha| \geq 2$.

1.4 The Countability of Rational Numbers

Definition 1.4.1: A rational number can be represented as either zero, or the quotient of two non-zero integers p and q. In set-builder notation, the set of rational numbers $\mathbb{Q}$ is defined as below:

$$\mathbb{Q} = \{\pm {}^{p}/_{q}\, , 0 : p, q \in \mathbb{N}\}$$

Evidently, the set of natural numbers $\mathbb{N}$, would be a proper subset of $\mathbb{Q}$, as all natural numbers can be defined as the quotient of p and q, where $p \in \mathbb{N}$ and $q = 1$. In set-builder notation:

$$\mathbb{N} = \{{}^{p}/_{q} : p \in \mathbb{N};\ q = 1\} \subsetneq \mathbb{Q}$$

Definition 1.4.2 (Dense Set): A dense set of numbers, is a set Q, such that for any pair of distinctive elements $p, q \in Q$, there exists some $r \in Q$, where $p < r < q$.

Lemma 1.4.1: The set $\{x \in \mathbb{N}_{\geq 0} : a \leq x \leq b;\ a, b \in \mathbb{N}_{\geq 0}\}$ is not dense (See **Definition 1.1.4**).

Lemma 1.4.2: The set $\{x \in \mathbb{Q} : a \leq x \leq b;\ a, b \in \mathbb{Q};\ a \neq b\}$ is dense. For this reason, $\mathbb{Q}$ is a dense set.

Proof of Lemma 1.4.2:

For any natural number $n \in \mathbb{N};\ n > 1$, there exists a least common multiple.

This is a corollary of the **Fundamental Theorem of Arithmetic**, which states that any natural number $n \in \mathbb{N};\ n > 1$, is either a prime number or a product of a unique combination of prime numbers.

A pair of rational numbers $a = {}^{p_\alpha}/_{q_\alpha}$ and $b = {}^{p_\beta}/_{q_\beta}$ can be multiplied by a pair of natural numbers, $n_\alpha, n_\beta \in \mathbb{N}$, such that the denominators of ${}^{(p_\alpha \times n_\alpha)}/_{(q_\alpha \times n_\alpha)}$, ${}^{(p_\beta \times n_\beta)}/_{(q_\beta \times n_\beta)}$, equal $lcm(n_\alpha, n_\beta)$.

$${}^{p(\times n)}/_{q(\times n)} \Rightarrow {}^{(p_\alpha \times n_\alpha)}/_{lcm(n_\alpha, n_\beta)} \text{ and } {}^{(p_\beta \times n_\beta)}/_{lcm(n_\alpha, n_\beta)}$$

As a result, there would exist another integer $z \in \mathbb{Z}$, such that:

$$(p_\alpha \times n_\alpha) < z < (p_\beta \times n_\beta)$$

It is unclear whether there is a way to show that such an integer $z \in \mathbb{Z}$ exists.

But by **mathematical induction**, it can be shown that a non-negative integer $n_\gamma \in \mathbb{N}_{\geq 0}$ exists, where:

$$(p_\alpha \times n_\alpha) < n_\gamma < (p_\beta \times n_\beta)$$

$P(n)$: There is a non-negative integer $n \in \mathbb{N}_{\geq 0}$ that satisfies the inequality $z_\alpha < n < z_\beta$, where $z_\alpha, z_\beta \in \mathbb{Z}$; $z_\alpha \neq z_\beta$

Base Case: $P(0) : -1 < 0 < 1$, where $\pm 1 \in \mathbb{Z}$, and $z_\alpha = -1;\ z_\beta = +1$

$P(1) : 0 < 1 < 2$, where $0, +2 \in \mathbb{Z}$, and $z_\alpha = 0;\ z_\beta = +2$

Inductive Hypothesis: Assume $P(n)$ is true for any $k \in \mathbb{N}_{\geq 0};\ k \geq 0$

Inductive Step: $P(k)$ implies $P(k+1)$

$$P(k) : k - 1 < k < k + 1;\ k \pm 1 \in \mathbb{Z}, \qquad P(k+1) : k < k + 1 < k + 2;\ k, k + 2 \in \mathbb{Z}$$

If $j < n < l$, it implies $j + 1 < n + 1 < l + 1$, where $z_\alpha = j$ or $j + 1$, and $z_\beta = l$ or $l + 1$.

It also implies $j + 1 \leq n$ and $n + 1 \leq l$, where $z_\alpha = j + 1$, and $z_\beta = l$.

Therefore, $z_\alpha < n$ if and only if $z_\alpha + 1 \ngtr n$, and $n < z_\beta$ if and only if $n + 1 \ngtr z_\beta$

Therefore, $P(n)$ is true for all $n \in \mathbb{N}_{\geq 0}$, as long as $z_\alpha + 1 \ngtr n$ and $n + 1 \ngtr z_\beta$.

Presumably, if some $n_\gamma \in \mathbb{N}_{\geq 0}$ exists, where $(p_\alpha \times n_\alpha) < n_\gamma < (p_\beta \times n_\beta)$, then it would imply some $z \in \mathbb{Z}$ exists also, where $(p_\alpha \times n_\alpha) < z < (p_\beta \times n_\beta)$, and $z = n_\gamma$ or $z = (-1)n_\gamma = -n_\gamma$.

Furthermore, an integer $\pm n_\gamma \in \mathbb{Z}$, divided by another natural number, $lcm(n_\alpha, n_\beta) \in \mathbb{N}$, would be the equivalent of a rational number $\pm {}^{n_\gamma}/_{lcm(n_\alpha, n_\beta)} \in \mathbb{Q}$.

Therefore, for any ${}^{p_\alpha}/_{q_\alpha}, {}^{p_\beta}/_{q_\beta} \in \mathbb{Q};\ lcm(n_\alpha, n_\beta) \in \mathbb{N}$, where $(q_\alpha \times n_\alpha), (q_\beta \times n_\beta) = lcm(n_\alpha, n_\beta)$, and ${}^{p_\alpha}/_{q_\alpha} \neq {}^{p_\beta}/_{q_\beta}$, the following inequality would hold:

$$(p_\alpha \times n_\alpha)/_{lcm(n_\alpha, n_\beta)} < \pm {}^{n_\gamma}/_{lcm(n_\alpha, n_\beta)} < (p_\beta \times n_\beta)/_{lcm(n_\alpha, n_\beta)}$$

Also, there would exist some $r, s \in \mathbb{Q}$, such that:

$$(p_\alpha \times n_\alpha)/_{lcm(n_\alpha, n_\beta)} < r < \pm {}^{n_\gamma}/_{lcm(n_\alpha, n_\beta)} < s < (p_\beta \times n_\beta)/_{lcm(n_\alpha, n_\beta)}$$

where $\pm {}^{n_\gamma}/_{lcm(n_\alpha, n_\beta)}$ is a rational number that is approximated by $r \in \mathbb{Q}$,

and $(p_\beta \times n_\beta)/_{lcm(n_\alpha, n_\beta)}$ is a rational number that is approximated by $s \in \mathbb{Q}$.

Furthermore, if the contrary of Lemma 1.4.2 were to be assumed, such that $\mathbb{Q}$ is not a dense set, and there are only finitely many rational numbers larger than $a \in \mathbb{Q}$ and smaller than $b \in \mathbb{Q}$, where:

$$a = (p_\alpha \times n_\alpha)/_{lcm(n_\alpha, n_\beta)}, \qquad b = (p_\beta \times n_\beta)/_{lcm(n_\alpha, n_\beta)}$$

it would lead to a contradiction.

Assume $(p_\beta \times n_\beta) - (p_\alpha \times n_\alpha) = 2$. If $\mathbb{Q}$ is not dense, it then can be decided that:

$$(p_\alpha \times n_\alpha)/_{lcm(n_\alpha, n_\beta)} < (p_\alpha \times n_\alpha) + 1/_{lcm(n_\alpha, n_\beta)} < (p_\beta \times n_\beta)/_{lcm(n_\alpha, n_\beta)}$$

However, if all numerators and denominators were to be multiplied by 2, then there would be some $2(p_\alpha \times n_\alpha) + 1 \in \mathbb{Z};\ 2\big((p_\alpha \times n_\alpha) + 1\big) + 1 \in \mathbb{Z}$, such that:

$$\frac{2(p_\alpha \times n_\alpha)}{2\left(lcm(n_\alpha, n_\beta)\right)} < \frac{2(p_\alpha \times n_\alpha) + 1}{2\left(lcm(n_\alpha, n_\beta)\right)} < \frac{2\big((p_\alpha \times n_\alpha) + 1\big)}{2\left(lcm(n_\alpha, n_\beta)\right)} < \frac{2\big((p_\alpha \times n_\alpha) + 1\big) + 1}{2\left(lcm(n_\alpha, n_\beta)\right)} < \frac{2(p_\beta \times n_\beta)}{2\left(lcm(n_\alpha, n_\beta)\right)}$$

This would contradict the assumption that $\mathbb{Q}$ is not dense, as where $(p_\beta \times n_\beta) - (p_\alpha \times n_\alpha) = 2$, there should only be a constant $k \in \mathbb{N}$ many rational numbers within the specified inequality:

$$(p_\alpha \times n_\alpha)/_{lcm(n_\alpha, n_\beta)} < x < (p_\beta \times n_\beta)/_{lcm(n_\alpha, n_\beta)}$$

But as previously shown, more than k many rational numbers that share the same denominator as the upper and lower limits of the inequality, would become accessible within the inequality if the numerator and denominator of all values (and limits) in the inequality were to be multiplied by two.

This is not possible for any non-dense set of numbers. Therefore, a contradiction.

Ultimately, it can be decided that $\{x \in \mathbb{Q} : a \leq x \leq b;\ a, b \in \mathbb{Q};\ a \neq b\}$ is dense, as for any $a, b \in \mathbb{Q}; a \neq b$, there would exist some $x \in \mathbb{Q}$, such that the statement, "There are only finitely many $x \in \mathbb{Q}$ larger than a and smaller than b", would be indeterminate, otherwise invalid. Therefore, $\mathbb{Q}$ is dense. ■

Remark 1.4.1: In theory, rational numbers can be recursively enumerated to some arbitrary point, via a successor function $S : \mathbb{Q} \to \mathbb{Q}$. For any $r \in \mathbb{Q}, x \in \mathbb{N}$, the rational numbers can be enumerated as below:

$$S(r) = r + x : (r + x), (r + 2x), (r + 3x), (r + 4x), \dots$$

$$S(r) = r + {}^{1}\!/_{x} : (r + {}^{1}\!/_{x}), (r + {}^{2}\!/_{x}), (r + {}^{3}\!/_{x}), (r + {}^{4}\!/_{x}), \dots$$

Example 1.4.1: A sequence of rational numbers can be recursively enumerated to some arbitrary extent, via the successor function $S : \mathbb{Q} \to \mathbb{Q}$.

If $r = 0$, and $x = 1$:

$$S(r) = r + 1 : 1, 2, 3, 4, 5, \dots$$

If $r = 0$, and $x = 2$:

$$S(r) = r + 2 : 2, 4, 6, 8, 10, 12, \dots$$

$$S(r) = r + {}^{1}\!/_{2} : {}^{1}\!/_{2}, 1, {}^{3}\!/_{2}, 2, {}^{5}\!/_{2}, 3, {}^{7}\!/_{2}, 4 \dots$$

As stated in **Definition 1.1.2**, a set X is countable if it has a finite quantity of elements (i.e., the cardinality of X is finite), or there is an injective function that maps X to the set of non-negative integers $\mathbb{N}_{\geq 0}$.

In set theory, it is apparent that there is a justification for the countability of rational numbers. For convenience, this justification will be named *Cantorian Countability*.

Essentially, Cantorian Countability alludes to the countability of rational numbers, i.e., there is a method to map the rational numbers to the set of natural numbers via a bijection.

By Cantorian Countability, the smallest transfinite cardinal, is equivalent to the cardinality of the set of natural numbers, which is also equal to the cardinality of the set of non-negative integers.

The symbol $\aleph_0$, may be used to designate this transfinite number.

By a bijective function, it can be shown that at least one kind of proper subset of $\mathbb{Q}$ is equinumerous to $\mathbb{N}$, e.g., the subset $A = \{a \in \mathbb{Q} : a = {}^{1}\!/_{n};\ n \in \mathbb{N}\}$, such that there is a bijection from $\mathbb{N}$ to A via a constructive formula. Specifically, the bijection $f : \mathbb{N} \to A$ would be defined by the formula:

$$\forall n \in \mathbb{N} : f(n) = {}^{1}\!/_{n}$$

Through a process of mathematical induction, it can be shown that the cardinality of A is *at least* $|\mathbb{N}|$ with respect to the function f. That is, if $f(k) \in A;\ k \geq n_0$, then $f(k + 1) \in A$.

It can also be shown that at least one kind of proper subset of $\mathbb{N}$ is equinumerous to $\mathbb{N}$, e.g., the subset $A = \{n \in \mathbb{N} : {}^{n}/_{2} \in \mathbb{N}\}$, such that there is a bijection $S : \mathbb{N} \to \{n \in \mathbb{N} : {}^{n}/_{2} \in \mathbb{N}\}$ by the constructive formula:

$$\forall n \in \mathbb{N} : S(n) = 2n$$

Through a process of mathematical induction, it can be shown that the cardinality of some proper subset $\{n \in \mathbb{N} : {}^{n}/_{2} \in \mathbb{N}\} \subsetneq \mathbb{N}$ is *at least* $|\mathbb{N}|$ with respect to the successor function S.

That is, if $S(k) \in \{n \in \mathbb{N} : {}^{n}/_{2} \in \mathbb{N}\};\ k \geq n_0$, then $S(k+1) \in \{n \in \mathbb{N} : {}^{n}/_{2} \in \mathbb{N}\}$ also.

Furthermore, it can be implied via Cantorian Countability, that there exists a bijection $f : \mathbb{N} \to \mathbb{Q}$, which would infer that the set of rational numbers $\mathbb{Q}$ is equinumerous to the set of natural numbers $\mathbb{N}$.

However, it is not clear whether there is a constructive formula that defines a bijection $f : \mathbb{N} \to \mathbb{Q}$. It is also not clear whether there would be a "systematic difference" between each pair of rational numbers $f(n), f(n+1) \in \mathbb{Q}$, with respect to the range of $f : \mathbb{N} \to \mathbb{Q}$, where f is a bijection from $\mathbb{N}$ to $\mathbb{Q}$.

Hypothesis 1.4.1: There is a map from the natural numbers to the non-negative rational numbers, by a bijective function $f : \mathbb{N} \to \{r \in \mathbb{Q} : r \geq 0\}$

Intuition 1.4.1 (as to Hypothesis 1.4.1): A non-specific illustration of a bijection $f : \mathbb{N} \to \{r \in \mathbb{Q} : r \geq 0\}$, is specified below, with respect to Cantor's diagonal method:

$$f(n)_{n \in \mathbb{N}} = \left\{ \begin{array}{cccccccccccccc}
0_1 \to & 1_2 & & 2_4 & \to & 3_5 & & 4_{10} & \to & 5_{11} & & 6_{18} & \cdots \\
& \downarrow & \nearrow & & \swarrow & & \nearrow & & \swarrow & & \nearrow & & \iddots \\
& {}^{1}/_{2_3} & & ({}^{2}/_{2}) & & {}^{3}/_{2_9} & & ({}^{4}/_{2}) & & {}^{5}/_{2_{17}} & & ({}^{6}/_{2}) & \\
& & \swarrow & & \nearrow & & \swarrow & & \nearrow & & \swarrow & & \iddots \\
& {}^{1}/_{3_6} & & {}^{2}/_{3_8} & & ({}^{3}/_{3}) & & {}^{4}/_{3_{16}} & & {}^{5}/_{3_{20}} & & ({}^{6}/_{3}) & \\
& \downarrow & \nearrow & & \swarrow & & \nearrow & & \swarrow & & \nearrow & & \iddots \\
& {}^{1}/_{4_7} & & ({}^{2}/_{4}) & & {}^{3}/_{4_{15}} & & ({}^{4}/_{4}) & & {}^{5}/_{4_{26}} & & ({}^{6}/_{4}) & \\
& & \swarrow & & \nearrow & & \swarrow & & \nearrow & & \swarrow & & \iddots \\
& {}^{1}/_{5_{12}} & & {}^{2}/_{5_{14}} & & {}^{3}/_{5_{21}} & & {}^{4}/_{5_{25}} & & ({}^{5}/_{5}) & & {}^{6}/_{5} & \\
& \vdots & \iddots & & \iddots & & \iddots & & \iddots & & \iddots & &
\end{array} \right\}$$

$n \in \mathbb{N}$

$f(n) \in (\mathbb{Q} \geq 0)$

$$\begin{array}{lccccccccccccccccc}
\mathbb{N} : & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & \cdots \\
& \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
\{r \in \mathbb{Q} : r \geq 0\} : & 0 & 1 & \frac{1}{2} & 2 & 3 & \frac{1}{3} & \frac{1}{4} & \frac{2}{3} & \frac{3}{2} & 4 & 5 & \frac{1}{5} & \frac{1}{6} & \frac{2}{5} & \frac{3}{4} & \frac{4}{3} & \cdots
\end{array}$$

Hypothesis 1.4.2: There is a map from the natural numbers to the rational numbers, by a bijective function $f : \mathbb{N} \to \mathbb{Q}$

Intuition 1.4.2 (as to Hypothesis 1.4.2): A non-specific illustration of a bijection $f : \mathbb{N} \to \mathbb{Q}$ is specified below, with respect to Cantor's diagonal method:

$$
f(n)_{n\in\mathbb{N}} = \left\{\begin{array}{cccccccc}
0_{1} \rightarrow & 1_{2} & & -1_{4} \rightarrow & 2_{5} & -2_{10} \rightarrow & 3_{11} & -3_{19} \;\cdots \\
& \downarrow \;\nearrow & & \swarrow & \nearrow & \swarrow & \nearrow & \iddots \\
& {}^{1}/_{2_{3}} & & -{}^{1}/_{2_{6}} & \left({}^{2}/_{2}\right) & \left(-{}^{2}/_{2}\right) & {}^{3}/_{2_{18}} & -{}^{3}/_{2_{21}} \\
& \swarrow & & \nearrow & \swarrow & \nearrow & \swarrow & \iddots \\
& {}^{1}/_{3_{7}} & & -{}^{1}/_{3_{9}} & {}^{2}/_{3_{12}} & -{}^{2}/_{3_{17}} & \left({}^{3}/_{3}\right) & \left(-{}^{3}/_{3}\right) \\
& \downarrow \;\nearrow & & \swarrow & \nearrow & \swarrow & \nearrow & \iddots \\
& {}^{1}/_{4_{8}} & & -{}^{1}/_{4_{13}} & \left({}^{2}/_{4}\right) & \left(-{}^{2}/_{4}\right) & {}^{3}/_{4_{28}} & -{}^{3}/_{4} \\
& \swarrow & & \nearrow & \swarrow & \nearrow & \swarrow & \iddots \\
& {}^{1}/_{5_{14}} & & -{}^{1}/_{5_{16}} & {}^{2}/_{5_{22}} & -{}^{2}/_{5_{27}} & {}^{3}/_{5} & -{}^{3}/_{5} \\
& \vdots \;\iddots & & \iddots & \iddots & \iddots & \iddots &
\end{array}\right\}
$$

$n \in \mathbb{N}$

$f(n) \in \mathbb{Q}$

$$
\begin{array}{lccccccccccccccccc}
\mathbb{N}: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & \ldots \\
& \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
\mathbb{Q}: & 0 & 1 & \frac{1}{2} & -1 & 2 & -\frac{1}{2} & \frac{1}{3} & \frac{1}{4} & -\frac{1}{3} & -2 & 3 & \frac{2}{3} & -\frac{1}{4} & \frac{1}{5} & \frac{1}{6} & -\frac{1}{5} & \ldots
\end{array}
$$

Remark 1.4.2: If the map from $\mathbb{N}$ to $\mathbb{Q}$ were to include duplicate elements of $\mathbb{Q}$, then f would be a non-injective, surjective function. But if the map were to exclude duplicate elements of $\mathbb{Q}$ (as shown for elements of $\mathbb{Q}$ that are enclosed in curve brackets), then f would be a bijective function.

From a Cantorian perspective, it could be suggested that Cantor Countability is intuitive, as it implies the existence of an original (albeit not evidently constructive) method to map $\mathbb{N}$ to $\mathbb{Q}$ via a bijection.

From a contrarian perspective, it could be suggested that Cantor Countability is counterintuitive, as it does not seem to identify a constructive formula that would map $\mathbb{N}$ to $\mathbb{Q}$ via a bijection.

But there are constructive formulas that would map $\mathbb{N}$ to some proper subsets of $\mathbb{Q}$ via a bijection (e.g., $f : \mathbb{N} \to \{a \in \mathbb{Q} : a = {}^{1}/_{n};\ n \in \mathbb{N}\}$; $f(n) = {}^{1}/_{n}$). Also, there is a function, $h : \mathbb{Q} \to \mathbb{N}_{\geq 0}$ that is somewhat constructive, such that it would indicate the cardinality of $\mathbb{Q}$ is at least the cardinality of $\mathbb{N}_{\geq 0}$.

The function $h : \mathbb{Q} \to \mathbb{N}_{\geq 0}$, would be a composite of the function $f : \mathbb{Q} \to \mathbb{N}_{\geq 0} \times \mathbb{N}_{\geq 0}$, and the Cantor pairing function $g : \mathbb{N}_{\geq 0} \times \mathbb{N}_{\geq 0} \to \mathbb{N}_{\geq 0}$ [4]. For all ${}^{p}/_{q} \in \mathbb{Q}$:

$$f\left({}^{\pm p}/_{\pm q}\right) = (+p, +q), \qquad (+p, +q) \in \mathbb{N}_{\geq 0} \times \mathbb{N}_{\geq 0}$$

$$g(p, q) = {}^{1}/_{2}\,(p+q)(p+q+1) + q, \qquad g(p, q) \in \mathbb{N}_{\geq 0}, \qquad g\left(f\left({}^{\pm p}/_{\pm q}\right)\right) \in \mathbb{N}_{\geq 0}$$

$$h\left({}^{\pm p}/_{\pm q}\right) = g\left(f\left({}^{\pm p}/_{\pm q}\right)\right)$$

Both f and g would be non-injective, surjective functions, such that $h = g \circ f$ would also be surjective. By the composition h, it would be implied that $|\mathbb{Q}| \geq |\mathbb{N}_{\geq 0} \times \mathbb{N}_{\geq 0}|$ and $|\mathbb{N}_{\geq 0} \times \mathbb{N}_{\geq 0}| \geq |\mathbb{N}_{\geq 0}|$.

Ultimately, the composition h would indicate $|\mathbb{Q}|$ is at least $|\mathbb{N}_{\geq 0}|$, if and only if $|\mathbb{N}_{\geq 0} \times \mathbb{N}_{\geq 0}| \ngtr |\mathbb{N}_{\geq 0}|$.

1.5 Upper Bounds, Lower Bounds, Cofinality and Fixed Points

Definition 1.5.1: For any partially ordered set $(P, \preccurlyeq_P)$, where there exists some element $p \in P$ [1,2,5]:

- p is the **minimal element** of $(P, \preccurlyeq_P)$ if there is no $q \in P$ such that $q \prec p$
- p is the **maximal element** of $(P, \preccurlyeq_P)$ if there is no $q \in P$ such that $p \prec q$

Example 1.5.1: If $P = \{1,2,3\}$, such that there is a $\geq$-relation on P, then "1" is the minimal element of P, and "3" is the maximal element of P, such that there is no $p \in P$ for which $p < 1$ or $p > 3$.

Remark 1.5.1: A "min" function and a "max" function may be applied to sets in order to return the minimum element or maximum element of a specified set. As to **Example 1.5.1**:

- $\min P \in \{p \in P : \neg(q \in P : q < p)\}$, such that $\min P = 1$
- $\max P \in \{p \in P : \neg(q \in P : p < q)\}$, such that $\min P = 3$

Definition 1.5.2: If K is a subset of a partially ordered set $(P, \geq_P)$, then [1,2,5]:

- The **lower bound** of K, is an element $s \in P$ that precedes or is equivalent to the minimal element of K. That is, all elements $s \in P$, where $s \leq k$; $\min K = k$
- A **strict lower bound** of K, is an element $s \in P$ that strictly precedes the minimal element of K, i.e., all elements $s \in P$, where $s < k$; $\min K = k$
- The **infimum** of the subset $K \subseteq P$, is the **largest lower bound** of K, such that the infimum would effectively be the minimal element of $(K, \geq_K)$

  If k is the infimum of $K \subseteq P$, then k succeeds or is equal to all lower bounds of K
- The **upper bound** of K, is an element $s \in P$ that succeeds or is equivalent to the maximal element of K. That is, all elements $s \in P$, where $s \geq k$; $\max K = k$
- A **strict upper bound** of K, is an element $s \in P$ that is strictly succeeds the maximal element of K, i.e., all elements $s \in P$, where $s > k$; $\max K = k$
- The **supremum** of the subset $K \subseteq P$, is the **smallest upper bound** of K, such that the supremum would effectively be the maximal element of $(K, \geq_K)$

  If k is the supremum of $K \subseteq P$, then k precedes or is equal to all upper bounds of K

Definition 1.5.3: A finite set, is a set that has a finite quantity of elements, such that the cardinality of a finite set would be an element of the set of non-negative integers $\mathbb{N}_{\geq 0} = \mathbb{N} \cup \{0\}$.

Lemma 1.5.1: The minimal element of $K = \{n, k \in \mathbb{N}_{\geq 0} : 0 \leq n \leq k\}$ is equal to the infimum of $K \subsetneq \mathbb{N}_{\geq 0}$.

Lemma 1.5.2: The maximal element of $K = \{n, k \in \mathbb{N}_{\geq 0} : 0 \leq n \leq k\}$ is equal to the supremum of $K \subsetneq \mathbb{N}_{\geq 0}$.

Remark 1.5.2: The cardinality of K is strictly smaller than the cardinality of the set of non-negative integers $\mathbb{N}_{\geq 0}$, because (1) there is a minimal element and maximal element of K, and (2) $\mathbb{N}_{\geq 0}$ is not a dense set (as specified in **Lemma 1.4.1**), which also means K is not a dense set, as K is a subset of $\mathbb{N}_{\geq 0}$

Lemma 1.5.3: The set $\mathbb{N}_{\geq 0}$ would not have a maximal element, if and only if $\mathbb{N}_{\geq 0}$ is closed under addition. That is, if the sum of any pair of non-negative integers, is also a non-negative integer.

Definition 1.5.4: A *cofinal* set, is a subset $K \subseteq X$, such that for all $x \in X$, there is some $k \in K$ where $k \geq x$.

Definition 1.5.5: A *coinitial* set, is a subset $K \subseteq X$, such that for all $x \in X$, there is some $k \in K$ where $k \leq x$.

Definition 1.5.6: The *cofinality* of a set X is the minimum cardinality of the cofinal subsets of X.

Example 1.5.2: Regarding a finite set $X = \{x \in \mathrm{N} : 85 \leq x \leq 90\}$:

- Any subset K of X, where $90 \in K$, would be a cofinal subset of X
- Any subset K of X, where $85 \in K$, would be a coinitial subset of X
- If $\{90\} \subsetneq X$, then the cofinality of X is 1, because $\{90\}$ is the cofinal subset of X that has the lowest cardinality, with respect to all cofinal subsets of X, such that $|\{90\}| = 1$

Regarding the set of natural numbers, $\mathbb{N}$:

- Any subset K of $\mathbb{N}$, where $k \in K;\ k \geq n$, for each $n \in \mathbb{N}$, would be a cofinal subset of $\mathbb{N}$ (assuming the Inductive Hypothesis), e.g., $K = \{k \in \mathbb{N} : k \geq n;\ n \in \mathbb{N}\}$, or $K = \left\{n \in \mathbb{N} : {}^{n}\!/_{2} \in \mathbb{N}\right\}$
- Any subset K of $\mathbb{N}$, where $1 \in K$, would be a coinitial subset of $\mathbb{N}$
- Assuming the Axiom of Infinity and the PA-based axiom that $n \in \mathbb{N}$ implies $S(n) \in \mathbb{N}$ for all $n \in \mathbb{N}$, the cofinality of $\mathbb{N}$ would be equal to the cardinality of any subset of $\mathbb{N}$ that is equinumerous to $\mathbb{N}$

  Essentially, the cofinality of $\mathbb{N}$ would be equivalent to the cardinality of $\mathbb{N}$ itself

Definition 1.5.7: A co-finite set, is a subset $K \subseteq X$, such that the relative complement of K in X is finite, i.e., $|(X - K)| \in \mathbb{N}_{\geq 0}$. By definition, a set would necessarily be co-finite in itself, because the cardinality of the relative complement of a set X in itself would be zero, which is the least non-negative integer.

Example 1.5.3: $\{x \in \mathbb{N} : x \neq 5\}$ is co-finite in $\mathbb{N}$, as $|(\mathbb{N} - \{x \in \mathbb{N} : x \neq 5\})| = 1$

$\{x \in \mathbb{N} : x > 90\}$ is a co-finite in $\mathbb{N}$, as $|(\mathbb{N} - \{x \in \mathbb{N} : x > 90\})| = 90$

Definition 1.5.8: A sequence $(a_n \dots a_k)$ is *non-decreasing* if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$

Lemma 1.5.4: For any non-decreasing, upper-bounded sequence of natural numbers, there exists a maximal value a_k, such that it is equivalent to the supremum of $\{a_n \in \mathbb{N} : n \leq k \in \mathbb{N}\} \subseteq \mathbb{N}$.

$$\forall a, n, k \in \mathbb{N} : \max(a_n)_{n \leq k} = \sup\{a_n \in \mathbb{N} : n \leq k \in \mathbb{N}\}$$

For all $a, n, k \in \mathbb{N}$, the sequence $(a_n)_{n \leq k}$ would be finite as long as it has a maximal value.

A proof of **Lemma 1.5.4** may be summarized as below:

Assume there is a non-decreasing sequence $(a_n)_{n \leq k}$. If the sequence $(a_n)_{n \leq k}$ is finite, then the sequence would necessarily have a maximal value. Otherwise, $(a_n)_{n \leq k}$ would not be a finite sequence, as a non-decreasing sequence that is upper-bounded must have a least upper bound as a potential maximal value.

Assume a_k is the maximal value of $(a_n)_{n \leq k}$. Then a_k would be an upper bound of $(a_n)_{n \leq k}$, because all values of the sequence $(a_n)_{n \leq k}$ would be less than or equal to a_k.

If there were to be a smaller upper bound of the sequence $(a_n)_{n \leq k}$, e.g., $(a_k - 1) \in \mathbb{N}_{\geq 0}$, it would contradict the assumption that a_k is not the maximal value of $(a_n)_{n \leq k}$.

For this reason, the only reasonable conclusion would be that:

- a_k is the least upper bound (supremum) of the set $\{a_n \in \mathbb{N} : n \leq k \in \mathbb{N}\}$
- The maximal value of $(a_n)_{n \leq k}$, is equal to the supremum of $\{a_n \in \mathbb{N} : n \leq k \in \mathbb{N}\}$. ■

Definition 1.5.9 (Fixed Point, Prefixed Point, Postfixed Point):

As to a function $f : X \rightarrow X$, a *fixed point* is effectively a specific element $k \in X$, such that $f(k) = k$.

- For any $x \in X$, if $f(x) = x$, then x would be a fixed point of the function f
- For any $x \in X$, if $f(x) \leq x$, then x would be a *pre-fixed point* of the function f
- For any $x \in X$, if $f(x) \geq x$, then x would be a *post-fixed point* of the function f

Lemma 1.5.5 (Bourbaki-Witt Theorem): Assume there exists a partially ordered set $(P, \leq_P)$, such that:

- There exists a minimal element in P
- There exists a supremum (i.e., least upper bound) in each totally ordered subset of P

If there exists a function $f : (P, \leq_P) \rightarrow (P, \leq_P)$, such that:

- $p \leq q$ implies $f(p) \leq f(q)$, for all $p, q \in P$
- $p \leq f(p)$, for all $p \in P$ (i.e., there is a pre-fixed point for all $p \in P$ in the function f)

Then the function f has a fixed point, $p_\omega = f(p_\omega)$.

Remark 1.5.3: To prove **Lemma 1.5.5** for sets that have a *maximal element*, it would have to be shown that $\max K_i \leq f(\max K_i)$, where $\{K_i\}_{i \in I}$ is the set of all chains in P, such that $K_i \subseteq P$ for each $K_i \in \{K_i\}_{i \in I}$.

Example 1.5.4 (Successor Function as to Lemma 1.5.5): If $P = \{p \in \mathbb{N} : p \leq k;\ k \in \mathbb{N}\}$, it can meet the conditions for the poset of the Bourbaki-Witt Theorem, as:

- There exists a least element in P, that is $1 \in P$
- There would exist a least upper bound in each totally ordered subset of P

It is not clear whether there is a constructive formula to prove that each chain in P has a supremum. But it would be reasonable to conclude there exists a supremum for each chain in P, because, (1) P is not a dense set, and (2) there exists a maximal element k in P, which would also be the supremum of any chain $K_n \subseteq P$ where $k \in K_n$. For this reason, (1) and (2) at least indicate that some $p \in P$ would be the least upper bound of some chain $K_n \subseteq P$, where $p \leq \max K_n \leq k$.

Assume there is a function $f : (P, \leq_P) \rightarrow (P, \leq_P)$:

$$f(p) = \begin{Bmatrix} p + 1, & \text{if } p + 1 \in P \\ p, & \text{otherwise} \end{Bmatrix}$$

As for the function f, it can be decided that f meets the conditions in **Lemma 1.5.5**:

- $p \leq q$ implies $f(p) \leq f(q)$, as $p \leq q$ implies $p + 1 \leq q + 1$
- $p \leq p + 1$ for all $p \in P$
- There is a fixed point $f(k) = k$, as $k + 1$ is not an element of P, where $k \in \mathbb{N}$; $\max P = k$

Even though it is already stated that $k \in P$ is the maximal element of P, the function f also implies the existence of a maximal element in P, because k is the only fixed point in f, where each integer $p \in P$ is mapped via f to its immediate successor $p + 1$ if its successor is also an element of P.

Evidently, k would be the only element of P of which its immediate successor $k + 1$ is not an element of P.

Remark 1.5.4: To prove **Lemma 1.5.5** for a transfinite set $V_\omega = \bigcup_{|X|=\aleph_0} P^{|X|}(\emptyset)$, where $\max V_\omega = \omega$, the Inductive Hypothesis would have to be invoked, with respect to the statement:

$P(n)$: "Regarding the set of ordinals V_ω, there exists a pair of chains $K_\omega \subseteq V_\omega$ and $K_{n\in\mathbb{N}} \subseteq V_\omega$, such that $\sup(K_\omega \subseteq V_\omega) \succ \sup(K_{n\in\mathbb{N}} \subsetneq V_\omega)$"

By the **Inductive Hypothesis**, it would be assumed there is a chain $K_k \subseteq V_\omega$, where $k \in \mathbb{N}$, such that:

- $\max K_k \succcurlyeq \max K_0$, where $K_0 = \{v \in V_\omega : \neg(w \in V_\omega : w \prec v)\}$; $\max K_0 = \min V_\omega = \emptyset$

  OR

- $\max K_k \succcurlyeq \max K_1$, where $K_1 = \{v \in V_\omega : |v| = 1\}$; $\max K_1 = \{\emptyset\}$

K_0 is an empty chain in V_ω such that K_0 *vacuously* has the properties of reflexivity, symmetry, transitivity and strong connectedness. Also, K_1 is a non-empty chain in V_ω, as K_1 has the properties of a total order.

From this assumption, it would be shown via **Inductive Step** that:

If $\sup(K_\omega \subseteq V_\omega) > \sup(K_k \subseteq V_\omega)$ would hold for $P(k)$, then $\sup(K_\omega \subseteq V_\omega) > \sup(K_{k+1} \subseteq V_\omega)$ would hold for $P(k+1)$, where $\sup(K_{k+1} \subseteq V_\omega) > \sup(K_k \subseteq V_\omega)$

K_{k+1} would be a chain in V_ω, where its supremum supersedes the supremum of K_k in V_ω.

By the **Inductive Step**, it would be shown that $P(n)$ holds for countably many chains in V_ω.

As for the function $f : (V_\omega, \preccurlyeq_{V_\omega}) \to (V_\omega, \preccurlyeq_{V_\omega})$:

$$f(v) = \begin{Bmatrix} v \cup \{v\}, & \text{if } v \cup \{v\} \in V_\omega \\ v, & \text{otherwise} \end{Bmatrix}$$

it would be decidable that f meets the conditions in **Lemma 1.5.5**, with regard to von Neumann ordinals. That is, $v \preccurlyeq w$ implies $f(v) \preccurlyeq f(w)$, and $v \preccurlyeq f(v)$, for all $v, w \in V_\omega$.

Finally, it would be decidable that there is a fixed point $f(\omega) = \omega$, because $\omega \cup \{\omega\}$ would be the immediate successor of ω, as to the von Neumann definition of ordinal numbers, such that $\omega \neq \omega \cup \{\omega\}$. But the maximal element of V_ω is ω. Therefore, $f(\omega) = \omega$, as $\omega \cup \{\omega\} \notin V_\omega$.

The proof of **Lemma 1.5.5** seems to be connected to the idea that *sets cannot be larger than numbers*, because numbers are necessarily attributed to the cardinalities of sets.

If the function f did not have a fixed point, it would allow any set of ordinal or natural numbers to only inject into its range, and not surject into its range (even though there could be a set of numbers that can surject into a proper subset of the range of f). Ultimately, this would infer the existence of a non-empty "set", such that no cardinal number can be attributed to the total amount of elements in the "set".

Obviously, this kind of "set" would not be well-defined. At least, it would not be well-defined under the set-theoretic axioms of ZFC. For this reason, there would have to exist at least one set of numbers that can surject into any arbitrary set. There cannot be a set, such that any collection of numbers can only be injected into the set, and not surjected into the set. **Lemma 1.5.5** depends on this to some extent.

A standard proof of **Lemma 1.5.5** can be found in Lang (2002) [6].

Lemma 1.5.6 (Tarski-Knaster Theorem):

If P is a partially ordered set, where every subset of P has an infimum and a supremum, then there exists a function $f : (P, \leq_P) \to (P, \leq_P)$, such that the set of fixed points $\{p \in P : p = f(p)\} \subseteq P$ is also a poset where every subset of $\{p \in P : p = f(p)\}$ has an infimum and a supremum.

A standard proof of **Lemma 1.5.6** can be found in Tarski (1955) [7].

Remark 1.5.5: In theory, **Lemma 1.5.5** would imply **Lemma 1.5.6**, as the poset of all fixed points in P, would be a collection of specific elements from the range of f in **Lemma 1.5.5.**

The poset would include all fixed points in $f : (P, \leq_P) \to (P, \leq_P)$, such that there would be a minimal element and a maximal element for each subset in $\{p \in P : p = f(p)\}$, as long as range$(f) \subseteq P$.

1.6 Limits of Cardinal Numbers

Definition 1.6.1: A cardinal number κ is a weak limit cardinal, if κ is a non-zero cardinal that is larger than any successor cardinal.

- $\kappa > S(\alpha)$ for any cardinal number $\alpha < \kappa$
- $\kappa \neq 0$

Definition 1.6.2: A cardinal number κ is a strong limit cardinal, if κ is a non-zero cardinal that is larger than the cardinality of the powerset of any set that has a cardinality less than κ.

- $\kappa > 2^\alpha$ for any cardinal number $\alpha < \kappa$
- $\kappa \neq 0$

Definition 1.6.3: A cardinal κ is *weakly inaccessible* if the following conditions hold:

$$\kappa \neq \sum_{i \in I}^{|I|<\kappa} \alpha_i = \alpha_i + \alpha_{i'} + \alpha_{i''} + \alpha_{i'''} + \cdots$$

- κ is not countably finite
- $\alpha < \kappa \Rightarrow S(\alpha) < \kappa$

Definition 1.6.4: A cardinal κ is *strongly inaccessible* if the following conditions hold:

$$\kappa \neq \sum_{i \in I}^{|I|<\kappa} \alpha_i = \alpha_i + \alpha_{i'} + \alpha_{i''} + \alpha_{i'''} + \cdots$$

- κ is not countably finite
- $\alpha < \kappa \Rightarrow 2^\alpha < \kappa$

Lemma 1.6.1: A strong limit cardinal would also be a weak limit cardinal

Lemma 1.6.2: A cardinal that is strongly inaccessible would also be weakly inaccessible

Lemma 1.6.3: The cardinality of $\bigcup_{|X|<\aleph_0} P^{|X|}(\emptyset)$ (i.e., the set of all von Neumann ordinals that precede the first von Neumann limit ordinal) is strongly inaccessible

Lemma 1.6.4: The cardinality of $\{z \in P^{|X|}(\emptyset) : |z| \leq 1;\ 1 \leq |X| < \aleph_0\}$ (i.e., the set of all Zermelo ordinals that precede the first Zermelo limit ordinal) is strongly inaccessible

Lemma 1.6.5: If X is a transfinite set, where $|X| = \aleph_0$, then there does not exist a cardinal number κ for which $2^\kappa = |X|$. This also infers there is no cardinal number κ for which $S(\kappa) = |X|$.

Essentially, the "smallest transfinite cardinal" $\aleph_0$, is a strong limit cardinal.

1.7 A Successor Ordinal that is not Recursively Accessible

Definition 1.7.1: For any successor function $S : N \to N$, where $N \subsetneq P^{|X|}(\emptyset)$:

The set of all elements of N that can be accessed in a finite number of recursive steps from a specified base n, would be a *countably finite* subset $K \subseteq N$, where $|K| < \aleph_0$.

Definition 1.7.2: Assuming it is impossible for an infinite set of ordinal numbers

$$N = \{n \in S(n) : S(n) \in P^{|X|}(\emptyset) \text{ or } S(n) \subsetneq P^{|X|}(\emptyset);\ |X| \geq \aleph_0\}$$

to be completely enumerated, then a *recursively inaccessible ordinal* $\kappa \in N$, would be an ordinal number that cannot be accessed (or constructed) in a finite series of recursive steps from a specified base $n = \emptyset$.

By the Zermelo successor function $S(n) = \{n\}$, or the von Neumann successor function, $S(n) = n \cup \{n\}$, it is possible to recursively construct a well-ordered set of ordinal numbers via S.

By definition, all limit ordinals are recursively inaccessible. However, if the cardinality of a well-ordered set of ordinal numbers were to be transfinite, it would infer the existence of a successor ordinal $S(\kappa)$ that also cannot be accessed in a finite series of recursive steps via the successor function S.

Lemma 1.7.1: As to a successor function $S : N_{V,Z} \to N_{V,Z}$:

$$\textbf{Case I} : N_V = \left\{ n \in S(n) = n \cup \{n\} : S(n) \subseteq \bigcup_{|X| \leq \aleph_0} P^{|X|}(\emptyset) \right\}$$

$$\textbf{Case II} : N_Z = \{n \in S(n) = \{n\} : S(n) \in P^{|X|}(\emptyset);\ |S(n)| \leq 1;\ |X| \leq \aleph_0\}$$

there would exist at least one successor ordinal in $N_{V,Z}$ that cannot be accessed in a finite series of recursive steps from a specified base $n = \emptyset$, if and only if the cardinality of $N_{V,Z}$ is transfinite, and there exists a limit ordinal ω in $N_{V,Z}$, such that $\max N_{V,Z} = \omega$.

Proof of Lemma 1.7.1:

As $N_{V,Z}$ is a poset, it infers there exists a least upper bound for each *finite* chain K_i in $N_{V,Z}$.

Assume T is the set of all least upper bounds of each finite chain $K_i \subsetneq N_{V,Z};\ |K_i| < \aleph_0$.

$$T = \{\sup(K_i \subsetneq N_{V,Z}) \in K_i : |K_i| < \aleph_0\}_{i \in X} \subsetneq N_{V,Z}$$

It is evident that $N_{V,Z}$ would be a transfinite, well-ordered collection of ordinal numbers if it has a limit ordinal. By the surjective function $f : \{K \in P(N_{V,Z}) : |K| < \aleph_0\} \rightarrow T$, the set of all least upper bounds of each finite chain in $N_{V,Z}$ can be mapped from the set of all finite chains in $N_{V,Z}$.

For all $K_i \in P(N_{V,Z})$, where $|K| < \aleph_0$:

$$f(K_i) = \sup(K_i \subsetneq N_{V,Z})$$

The cardinality of the set of all suprema $\left|\{f(K_i) \in T : K_i \in P(N_{V,Z})\}_{i \in X}\right|$ of each finite chain $K_i \subsetneq N_{V,Z}$, would not be countably finite, because each ordinal in $N_{V,Z} - \{\omega\}$ would be the least upper bound of at least one finite chain in $N_{V,Z}$, and the function S is recursive. So, there is no specified limit in the sequence of ordinal numbers that would be mapped via the von Neumann or Zermelo successor function.

T would not have a maximal element, even though $N_{V,Z}$ has a maximal element where $T \subsetneq N_{V,Z}$, as there is not a finite chain K_i in $N_{V,Z}$ that is co-finite in $N_{V,Z}$. The cardinality of any chain in $N_{V,Z}$ (other than $N_{V,Z}$ itself) where the infimum is a successor ordinal and the supremum is $\omega \in N_{V,Z}$, cannot be a finite chain.

The set of all suprema of each finite chain $K_i \in T$ would be well-ordered with regard to the precondition that each ordinal in $N_{V,Z}$ (besides ω) would be the supremum of some finite chain in $N_{V,Z}$.

For this reason, $\omega \in N_{V,Z}$ would be a strict upper bound of $T \subsetneq N_{V,Z}$, which means the supremum of each finite chain $K_i \subsetneq N_{V,Z}$; $|K_i| < \aleph_0$, cannot be immediately succeeded by $\omega \in N_{V,Z}$.

Furthermore, if T itself does not have a maximal element, it means a supremum of at least one finite chain $K_\kappa \subsetneq N_{V,Z}$ cannot be accessed via the successor function $S_T : (T, \geq_T) \rightarrow (T, \geq_T)$.

For all $\sup(K_{i,j} \subsetneq N_{V,Z}) \in T$, where $|K_{i,j}| < \aleph_0$; $\sup(K_i \subsetneq N_{V,Z}) \neq \sup(K_j \subsetneq N_{V,Z})$:

$$S_T(\sup(K_i \subsetneq N_{V,Z})) = \left\{\begin{array}{ll} \sup(K_j \subsetneq N_{V,Z}), & \text{if } \neg(x \in T) : \sup(K_i \subsetneq N_{V,Z}) < x < \sup(K_j \subsetneq N_{V,Z}) \\ \sup(K_i \subsetneq N_{V,Z}), & \text{otherwise} \end{array}\right\}$$

As for the successor function S_T, there would be a finite chain $K_j \subsetneq N_{V,Z}$, where $\sup(K_j \subsetneq N_{V,Z})$ is the immediate successor to the supremum of some $K_i \in T$. But there would not be a fixed point in S_T, because for the supremum of any finite chain $K_n \in T$, there would be a supremum of another finite chain $K_{n+1} \in T$, such that the supremum of K_{n+1} would be a strict upper bound of K_n.

$$\forall K_n, K_{n+1} \in T : \sup(K_n \subsetneq N_{V,Z}) \prec \sup(K_{n+1} \subsetneq N_{V,Z})$$

There cannot be a finite chain in $N_{V,Z}$ that does not have a strict upper bound. That is, there cannot be a finite chain in $N_{V,Z}$, of which its supremum (as a subset of $N_{V,Z}$) is equal to or preceded by the maximal element of $N_{V,Z}$, because for all $n \in N_{V,Z}$, there is a limit ordinal $\omega \in N_{V,Z}$, such that $\omega \succcurlyeq n$.

This means the cofinality of $N_{V,Z}$ would not be a countably finite cardinal; the least co-finite subset of $N_{V,Z}$ would not be a countably finite set. For this reason, there must be at least one successor $S(\kappa) \in N_{V,Z}$ that is the supremum of a finite chain $K_{S(\kappa)} \in T$, such that $S(\kappa) \in N_{V,Z}$ cannot be accessed in a finite series of recursive steps from the base ordinal $\emptyset \in N_{V,Z}$, as long as $|N_{V,Z}|$ is transfinite, and $\max N_{V,Z} = \omega$. ■

Corollary 1 of Lemma 1.7.1: Regarding a set of ordinals $N_{V,Z}$, where $|N_{V,Z}| \geq \aleph_0$, there is a non-empty, well-ordered subset $K \subsetneq N_{V,Z}$ that is *unenumerable by recursion*, where:

- K is a countable subset of $N_{V,Z}$
- The cardinality of K is finite

- A successor ordinal $S(\kappa) \in N_{V,Z}$ is the supremum of K, and $\emptyset \in N_{V,Z}$ is the infimum of K
- At least one of the strict upper bounds of K would be a limit ordinal ω. That is, at least one strict upper bound of K would not be an immediate successor of the supremum of K

Ultimately, $K \subsetneq N_{V,Z}$ would not be a fully constructable set, due to the recursive inaccessibility of its supremum via its infimum (or minimal element).

Corollary 2 of Lemma 1.7.1: If $P(n)$ is an open statement, and the domain of $P(n)$ is a countable subset T of a set of ordinals $N_{V,Z}$, where $|N_{V,Z}| \geq \aleph_0$, there would be at least one successor $S(\kappa) \in T$, such that it would be impossible to construct a witness to determine the truth value of the individual case $P\big(S(\kappa)\big)$.

Remark 1.7.1:

For any finite chain K_i in $N_{V,Z}$, there would be a successor ordinal $S(\kappa) \in N_{V,Z}$ that is an upper bound of K_i, as long as there is a limit ordinal $\omega \in N_{V,Z}$ that is also a strict upper bound of each K_i in $N_{V,Z}$.

Essentially, there would be a successor ordinal in $N_{V,Z}$ that cannot be recursively accessed via any finite chain K_i in $N_{V,Z}$, as long as there exists a strict upper bound of every finite chain K_i in $N_{V,Z}$.

Even though $K \subsetneq N_{V,Z}$ would be a finite chain, K would not be constructible, due to the recursive inaccessibility of its least upper bound (as an element of a transfinite, well-ordered set of ordinals). But the existence of K would at least be indirectly provable with respect to the axioms of ZFC, and the PA-based axiom, "For each ordinal number β, its successor $S(\beta)$ is also an ordinal number".

Remark 1.7.2:

It appears that "infinite" sets may not be constructible without some kind of non-terminating, recursive process. From the finitist perspective, this may be an inconvenience, as the mathematical inductive method can prove an open statement *only to a degree of virtual precision* (at most).

This means the accuracy of a result that is obtained through a proof by mathematical induction can be doubted to some extent, as the proof would have to account for at least one indefinite variable, without reaching a limit that can only be reached via the exhaustion of an unlimited series of objects. Apparently, it is impossible for an unlimited series of objects to be enumerated in a finite series of recursive steps. Therefore, the constructability of "infinite" sets may be liable to criticism from the finitist perspective.

The point is, the existence of a recursively inaccessible ordinal number cannot be directly proven as long as the ordinal is inaccessible via recursion. If an ordinal is recursively inaccessible, then certain axioms must be invoked in order for the existence of the inaccessible ordinal to be (indirectly) validated (i.e., the Axiom of Infinity, and the PA-based axiom, $\beta \in N_{V,Z}$ implies $S(\beta) \in N_{V,Z}$ for each ordinal $\beta \in N_{V,Z}$).

Lemma 1.7.1 also alludes to the **Axiom of Induction**, as it indirectly demonstrates the indefinite aspect of the inductive process. That is, for any well-ordered, non-dense set of numbers N, there would be a successor $S(n) \in N$ for each element $n \in N$, such that at least one successor $S(\kappa) \in N$ would not be accessible in a finite series of recursive steps if the cardinality of N were to be transfinite.

Furthermore, **Lemma 1.7.1** alludes to the undecidability of a particular statement, with regard to a structure that involves ordinal arithmetic and a successor function

For example, a structure $\big(P^{|X|}(\emptyset), X, S, \in\big)$, where:

- $P^{|X|}(\emptyset)$ designates the domain of the structure

- X designates a set, such that $|X| \leq \aleph_0$
- S designates a successor function, with respect to either von Neumann ordinals, $S(n) = n \cup \{n\}$ for all $n \in \bigcup P^{|X|}(\emptyset)$, or Zermelo ordinals, $S(n) = \{n\}$ for all $n \in \left[P^{|X|}(\emptyset)\right]^1$

Proposition A: As for $N_{V,Z} \subsetneq P^{|X|}(\emptyset)$, the cardinality of N is less than the cardinality of $P^{|X|}(\emptyset)$

Proposition B: As for $N_{V,Z} \subsetneq P^{|X|}(\emptyset)$, the cardinality of N is not less than the cardinality of $P^{|X|}(\emptyset)$

By the set-theoretic axioms of ZFC, Proposition B and the negation of Proposition A are valid.

But as indicated by **Lemma 1.7.1**, it would not be possible to prove Proposition B or the negation of Proposition A without the Axiom of Infinity and the PA-based axiom that $\beta \in N_{V,Z}$ implies $S(\beta) \in N_{V,Z}$ (for ordinal numbers). Essentially, neither Proposition B nor the negation of Proposition A would be provable under the axioms of ZFC – Infinity, if and only if $P^{|X|}(\emptyset)$ is not a countably finite set.

1.8 Final Remarks

The Inductive Hypothesis has a sort of significance as to recursion in mathematics. A proof of an open statement $P(n)$ by mathematical induction would partially depend on the Inductive Hypothesis. For the inductive method, the Inductive Hypothesis allows the base step to connect to the recursive step.

By the assumption that an open statement $P(n)$ holds for any number $k \in N;\ k \geq a$, the recursive process would be simplified, as it would enable the process to bypass a condition to prove that $P(n)$ holds for each individual case, e.g., $P(1)$, $P(2)$, $P(3)$, $P(4)$, $P(5)$, … etc. Otherwise, it may be inconvenient, if not impossible, to prove that some open statement $P(n)$ would hold for each distinct case, if the number of cases by which $P(n)$ holds, is unspecified, transfinite, or enormous to the extent of virtual innumerability.

The Inductive Hypothesis allows the recursive process to be simplified with respect to the proof of an open statement $P(n)$, where the "domain of discourse" is equivalent to a well-ordered, non-dense set of numbers N, such that it would not need to be proven that $P(n)$ holds for each distinct case. In other words, the Inductive Hypothesis would enable a proof by mathematical induction to obtain the conclusion that an open statement $P(n)$ holds for an indefinite (if not infinite) number of cases.

Even though a proof by induction would depend on a conjecture (i.e., Inductive Hypothesis), it would be a specific conjecture that pertains to the details of the open statement $P(n)$. If associated with the base step and the recursive step, the Inductive Hypothesis would assist in the discovery of a somewhat reasonable proof of the open statement $P(n)$ for all $n \in N$ that applies to it. The Inductive Hypothesis also alludes to the existence of a *potentially infinite* set (as stated in **Remark 1.1.1**).

As implied by **Lemma 1.7.1**, it would not be possible to directly prove the distinct case $P\big(S(\kappa)\big)$ holds with respect to an open statement $P(n)$, where the domain of $P(n)$ is a set of ordinal numbers $N_{V,Z}$, such that there is a specific $S(\kappa) \in N_{V,Z}$ that is recursively inaccessible from a specified base ordinal.

As the Inductive Hypothesis is a conjecture about an open statement $P(n)$, a proof by mathematical induction would lead to the conclusion that $P(n)$ holds to some approximate or arbitrary degree of accuracy; it would not determine the "truth" of the open statement $P(n)$ with absolute certainty.

The validity of a proof by mathematical induction would not be completely exempt from doubt, as it would necessarily concern at least one variable that has not been accessed (or cannot be accessed) via recursion (as implied by **Lemma 1.7.1**). The Inductive Hypothesis allows the proof of $P(n)$ to advance to the recursive step, but it does not eliminate the possibility that $P(n)$ might not hold for all cases.

The key limitation of the inductive method, is that it cannot provide a direct proof of $P(n)$ for each distinct case $n \in N$ to which it would apply, *if* the domain N is not a countably finite set. As improbable as it may be, it would be reasonable to assume there is a potential for some exception or counterexample to hold in at least one case for which the open statement $P(n)$ has not been directly tested. For this reason, as long as the cardinality of the domain of $P(n)$ is not conclusively finite, the "truth" of $P(n)$ can be determined to a degree of virtual certainty at most, but not absolute certainty.

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