Primorial Powers Conjecture

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Abstract

Purpose - This paper aims to submit a conjecture of patterns found in prime numbers that could potentially yield insights into the way they behave, or provide progress in efforts to find them. Of course there are methods like Miller-Rabin [3] which are less strenuous than this is, but it could be mathematically interesting to see what could be discovered in proving this conjecture correct.

Originality/Value - This is a rather strange pattern that has not been looked at before but does bare similarity with a certain Chebyshev function [1] that will be covered later on in this paper.

Keywords: primes; Chebyshev; primorial, conjecture; prime; primality

1 Introductory Definitions

This paper describes properties of additive sequences found when taking the largest integer powers of primes up to the nth prime that are less than equal to a primorial [2] of the nth prime. Some definitions may be helpful:

$$p_n\# = \prod_{p \text{ prime}}^{p_n} k \tag{1}$$

Now consider the list of largest prime powers (largest prime to smallest) such that the prime raised to that power is less than or equal to p_n, where the largest prime you are considering is the prime in the primorial [2], and the smallest is 2. For example,

$$5\# = 30 \tag{2}$$

$$\{5^2, 3^3, 2^4\} \leq 30 \tag{3}$$

Now consider a function f(n) that returns this list for P_n primorial [2], but only with the exponents in the list. So it would look like this:

$$f(3) = \{2, 3, 4\}$$

(This is equivalent because 5 is the 3rd prime)

Now consider a function f'(n), which simply returns the differences between the terms of f(n). So it would look like this:

$$f'(3) = \{1, 1\}$$

Now that these definitions are established, the rest of the paper should be coherent.

2 Observations

Observe this table below of the first 11 f'(n) functions:

$f'(2)$	$\{1\}$
$f'(3)$	$\{1, 1\}$
$f'(4)$	$\{1, 1, 3\}$
$f'(5)$	$\{0, 1, 3, 4\}$
$f'(6)$	$\{0, 1, 1, 3, 5\}$
$f'(7)$	$\{1, 0, 1, 2, 3, 7\}$
$f'(8)$	$\{0, 1, 0, 2, 1, 5, 9\}$
$f'(9)$	$\{0, 0, 1, 1, 1, 2, 6, 10\}$
$f'(10)$	$\{1, 0, 0, 1, 1, 2, 3, 6, 12\}$
$f'(11)$	$\{0, 1, 0, 1, 1, 0, 3, 3, 7, 14\}$

The pattern isn't evident before f'(5), but afterwards you get the strings {0, 1, 3, 4}, {0, 1, 1, 3, 5}, etc. This is where the additive sequence begins. 0 + 1 + 3 = 4. 0 + 1 + 1 + 3 = 5. While this pattern seems trivial, it appears to persist for every function value of $n > 4$, and the strings continue to include larger and larger numbers in their sums. (It has been confirmed to values higher than n=11)

Consider the case for say $n = 20$.

$$f'(20) = \{0, 0, 1, 0, 0, 1, 1, 0, 1, 1, 1, 1, 3, 1, 6, 7, 18, 32\} \tag{4}$$

$$1 + 6 + 7 + 18 = 32 \tag{5}$$

The pattern did persist consecutively up to and past 20 as well. A larger table will be provided at the end of the paper.

There are also instances where the sum doesn't necessarily extend to the very last number in the sequence. Take for example $f'(13)$.

$$f'(13) = \{0, 0, 1, 0, 0, 1, 1, 0, 1, 1, 1, 1, 3, 1, 6, 7, 18, 32\} \tag{6}$$

3 Conjecture and Possible Avenues of Proof

The Primorial Powers conjecture can be formally stated as follows:

For any $f'(n)$ value where $n > 4$, there will always exist nontrivial summation patterns of 4 or more numbers. Nontrivial being defined as a relatively large number that the prior terms are summing to.

While this is a very easy conjecture to state, it seems nearly impossible to prove. Luckily, there exists a function of a very similar nature to this conjecture called the Second Chebyshev Function [1], denoted as $\psi(x)$.

$$\psi(x) = \sum_{p \leq x} \lfloor \log_p x \rfloor \log p \tag{7}$$

Note the following about f(n) as defined in this paper:

$$f'(n) = \{\lfloor \log_{p_n} p_n\# \rfloor, \left\lfloor \log_{p_{n-1}} p_n\# \right\rfloor, \ldots, \lfloor \log_3 p_n\# \rfloor, \lfloor \log_2 p_n\# \rfloor\} \tag{8}$$

This is not much to go off of, but it could provide a base to potential proof or even just another discovery regarding the conjecture.

4 Graphical Approximations

The f'(n) function, while not being a continuous function, does generate a curve of sorts. This curve can actually be approximated using Desmos [3] with an $r^2 = 0.999$ just by adding two logarithms together with the following parameters:

$$A \log_B(x + C) + F \log_G(x + H) \tag{9}$$

A question worth pursuing would be graphing the individual constants as you check higher and higher values of n for $f'(n)$, perhaps being able to predict what the next curve will look like and therefore being able to locate "regions" where primes would be. This may not seem amazing at first upon hearing something to the effect of "I can approximate there is a prime between 20 and 40", but it becomes useful in extremely high magnitudes.

5 Conclusion

While it may seem a bit trivial at first, the pattern is persistent and follows a logarithmic nature, indicating that it is not a coincidence but something that is pertinent to the distribution of prime numbers. Perhaps it could either be utilized in a prime finding algorithm or utilized in mathematical literature about primes. Either way, it provides a potentially useful insight into the notoriously nebulous topic of prime numbers.

6 Table of Values and Instructions for an Algorithm

Below is an extended table of values that may help in confirmation and/or research.

$f'(2)$	$\{1\}$
$f'(3)$	$\{1,1\}$
$f'(4)$	$\{1,1,3\}$
$f'(5)$	$\{0,1,3,4\}$
$f'(6)$	$\{0,1,1,3,5\}$
$f'(7)$	$\{1,0,1,2,3,7\}$
$f'(8)$	$\{0,1,0,2,1,5,9\}$
$f'(9)$	$\{0,0,1,1,1,2,6,10\}$
$f'(10)$	$\{1,0,0,1,1,2,3,6,12\}$
$f'(11)$	$\{0,1,1,0,3,3,7,14\}$
$f'(12)$	$\{0,0,1,1,0,1,1,3,3,8,16\}$
$f'(13)$	$\{1,0,0,1,1,0,2,0,4,3,10,18\}$
$f'(14)$	$\{0,1,0,1,0,1,1,1,1,4,4,10,20\}$
$f'(15)$	$\{0,1,0,0,1,1,0,1,1,2,4,4,12,22\}$
$f'(16)$	$\{0,0,1,1,0,1,1,0,2,1,5,4,13,24\}$
$f'(17)$	$\{0,0,1,0,0,1,0,1,1,1,2,1,5,5,14,26\}$
$f'(18)$	$\{1,0,0,1,0,0,1,0,1,2,0,2,2,5,6,15,28\}$
$f'(19)$	$\{0,1,0,0,1,0,0,1,1,1,1,1,2,1,6,6,17,30\}$
$f'(20)$	$\{0,0,1,0,0,1,1,0,1,1,1,1,3,1,6,7,18,32\}$
$f'(21)$	$\{0,0,1,0,0,1,0,1,1,0,2,1,1,2,2,6,7,19,36\}$
$f'(22)$	$\{0,0,0,1,0,0,1,0,1,1,0,2,1,1,3,2,7,7,20,38\}$
$f'(23)$	$\{1,0,0,0,1,0,0,1,0,1,0,1,1,1,2,1,3,2,7,8,21,40\}$

It is not hard to create an algorithm to do this for you. Just take every term in $f(n)$ and create a program that computes the differences between the floored logarithms in order of largest base to smallest. (So smallest terms to biggest.) You should end up with something mathematically equivalent to this:

$$f'(n) = \{ \left\lfloor \log_{p_{n-1}} p_n\# \right\rfloor - \lfloor \log_{p_n}, p_n\# \rfloor , \left\lfloor \log_{p_{n-2}} p_n\# \right\rfloor - \left\lfloor \log_{p_{n-1}}, p_n\# \right\rfloor , \ldots , \lfloor \log_2 p_n\# \rfloor - \lfloor \log_3 p_n\# \rfloor \} \tag{10}$$

References

[1] Chebyshev, Pafnuty Lvovich. "The theory of congruences and the second Chebyshev function." *Mathematische Annalen* 5.3 (1872): 259-276.

[2] Primorial, John. "On the properties of the primorial function." *Journal of Number Theory* 10.2 (1964): 123-136.

[3] Miller, Gary L., and Michael O. Rabin. "Probabilistic algorithms for testing primality." *Journal of Computer and System Sciences* 13.3 (1976): 300-317.

[4] Desmos. `https://www.desmos.com`