A Simpler Method of Proving the Irrationality of Various Infinite Series, + Rationality of $\zeta(s \in \mathbb{N})$, γ and π

Jay Pillai

November 11, 2024

Abstract

Purpose - This paper aims to derive a new method to prove the irrationality of particular cases of infinite series that are less complicated than approaches taken in the past. This could potentially lead to a much more easily teachable method of proving irrationality as well as addressing many open problems.

Design/Methodology/Approach - Using a very simple approach–the limit of the series' partial sums and the behavior of the series as an unsimplified fraction–the proof can be completed using nothing more than the rules of divisibility and modular arithmetic. This can be used to show that the series converges to a value that is impossible to represent with a rational expression. From this theorem, there are also a couple other things that can be derived, like whether or not the partial sums of an infinite series can ever be an integer.

Findings - The results reveal that using this method, a whole class of series can be proven irrational. On top of this, it also results in novel, simpler proofs for older results like the irrationality of π, as well as addressing some relevant open problems.

Originality/Value - This method offers a much easier approach to a topic relevant in many domains of math–particularly number theory and analysis–that is simple enough to be taught to high school math students.

Keywords: irrationality; proof; Euler-Mascheroni, Riemann-Zeta; pi; e

1 Introduction

In this paper, a proof for this method will be unveiled. In more detail, it will start off with a fairly well known infinite series that is known to be irrational and from there a step-by-step proof of that series will be underway. When that is done, a generalized case of this will be explored, completing the proof, as well as some other minor (consequential) theorems that result from this theorem. That being theorems that could be considered less "relevant" but still pertinent to some problems in math. After this, some open problems will be addressed, and discoveries made from this theorem that are not necessarily open questions will be addressed as well.

2 Proof by Example

Consider the zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

A governing axiom of this proof is the fact that $\zeta(2s) \notin \mathbb{Q}, s \in \mathbb{N}$, since they are all of the form $\frac{\pi^n}{m}$. The more difficult question is the rationality of $\zeta(2s+1)$. This proof will utilize a modular arithmetic argument of sorts. Consider $\zeta(2)$:

$$\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \cdots = \frac{\pi^2}{6} \notin \mathbb{Q}.$$

This is already known to be irrational on account of its sum being known, but let's try to prove its irrationality a different way. Consider the ath partial sum of $\zeta(2)$, where a is a prime number.

$$\sum_{n=1}^{a} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \cdots \frac{1}{a^2}$$

Expanding it into a fraction gives

$$1+\frac{(3\cdot 4\cdot 5\cdots a)^2+(2\cdot 4\cdot 5\cdots a)^2+(2\cdot 3\cdot 5\cdots a)^2\cdots}{a!^2}$$

Without "crunching the numbers" you could deduce which factors are not shared between the numerator and denominator based on whichever factors appear everywhere in the numerator but once. In this case, you can say for a fact that a is an unshared factor. That is, until you get to the $2a$th partial sum, then a will become a shared factor. This is relevant because the lower bound for the size of the reduced numerator at any ath partial sum $N(a)$ is given by:

$$N(a)\geq (p_1\cdot p_3\cdot p_4\cdots p_m)^2+(p_1\cdot p_2\cdot p_4\cdots p_m)^2\cdots$$
$$\text{Where } p_1\cdots p_m<2a.$$

Before going on, this lower bound will be proven. You can deduce that for every prime $p_k<2a$,

$$N(a)=n{p_k}^2+(p_1\cdots p_{k-1}\cdot p_{k+1}\cdots)^2$$

Given that p_k is a prime, it can be deduced that

$$(p_1\cdots p_{k-1}\cdot p_{k+1}\cdots)^2\perp p_k.$$

Also keep in mind that p_k can denote absolutely any one of the prime terms $p_1,p_2\cdots p_m$ since they are all examples of terms that appear everywhere but once. Following this same logic, the denominator is divisible by every single one of these terms, given that it equal to $a!^2$. It can be deduced that none of these prime terms will be canceled out in the reduced fraction, leading to the conclusion that for any kth partial sum,

$$N(k)\geq (p_1\cdot p_3\cdot p_4\cdots p_m)^2+(p_1\cdot p_2\cdot p_4\cdots p_m)^2\cdots$$

Where p_b doesn't denote the bth prime, but the bth largest unshared prime factor at any given partial sum. Since there will always be some unshared prime factor, the question is does $p_m\to\infty$? Because then $N(k)=\prod_{n=1}^{m\to\infty}p_n\to\infty$.

This is essentially a function of $\pi(2a)-\pi(a)$

$$\lim_{a\to\infty}\pi(2a)-\pi(a)=\infty$$

Now, if you look at the ath partial sum of any $\zeta(s\in\mathbb{N})$:

$$1+\frac{(3\cdot 4\cdot 5\cdots a)^s+(2\cdot 4\cdot 5\cdots a)^s+(2\cdot 3\cdot 5\cdots a)^s\cdots}{a!^s}$$

We can also deduce that this method of proof is indifferent to the size of s, for two reasons. The first reason being that exponents do not introduce new prime factors, hence the divisibility argument will hold. The second reason being that it is already known that $\zeta(2s)\notin\mathbb{Q}$ for all integer s. This would imply that in addition to this,

$$\zeta(2s+1)\notin\mathbb{Q}.$$

To elaborate on this logic, let's look at a case where a similar logic could not be applied. Take a geometric series:

$$\sum_0^\infty\frac{1}{2^n}$$

Using the same procedure as before, take the ath partial sum in fraction form.

$$\sum_0^a\frac{1}{2^n}=\frac{2+2^2+2^3\cdots 2^a+1}{2^a}$$

Yes, it is true that for every finite partial sum the numerator and the denominator do not share any factors. The only difference between this and the zeta series is that it is only that single $\frac{1}{2^a}$ term that makes it so the numerator and denominator go to infinity. As you go to higher partial sums there are no extra unshared factors introduced. $\frac{1}{2^a}$ obviously goes to 0 as $a\to\infty$, and this series eventually simplifies to 2, a rational number.

Now take the case of the zeta series. We will take 2 partial sums to show a modular change. First, look at the ath partial sum:

$$\sum_{n=1}^{a}\frac{1}{n^s}=1+\frac{(3\cdot 4\cdot 5\cdots a)^s+(2\cdot 4\cdot 5\cdots a)^s+(2\cdot 3\cdot 5\cdots a)^s\cdots}{a!^s}$$

Note that if you were to just look at one term in the denominator, a, the modular relationship between the numerator N and a can be described as

$$N(a)\equiv(3\cdot 4\cdot 5\cdots(a-1))^s \pmod a \equiv k \pmod a, 1\le k<a$$

This is true for any a, and is also true for every prime p in the denominator for which $2p$ has not been reached yet. In the $a+1$th partial sum you have

$$N(a)\equiv(3\cdot 4\cdot 5\cdots(a-1)(a+1))^s \pmod a \equiv k(a+1)^s \pmod a \equiv k \pmod a$$

Now for the $a+2$th.

$$N(a+2)\equiv(3\cdot 4\cdot 5\cdots(a-1)(a+1)(a+2))^s \pmod a \equiv k(a+2)^s \pmod a \equiv 2^s k \pmod a$$

It will cycle through $k, 2^s k, 3^s k, 4^s k, 5^s k\cdots$.

Now let's assume that at the $a+k$th partial sum the all of the unshared factors $p_b\le a-k$ for some k go to 0 like in the geometric series. This still will not matter because in the $a+k$th partial sum there will always be sufficiently large primes such that there will be unshared factors that do not get canceled out by the denominator as you go to higher partial sums. To add to this proof, another one will be explored.

3 Proving Theorem by Contradiction, and Rationality of $\zeta(2s+1)$

In this proof, a contradiction will be made, based on the facts gained from the last section.

Assume that there is some $\zeta(s\in\mathbb{N})\in\mathbb{Q}$. To reconcile this with the divisibility proof from above, this would imply that even though the numerator and denominator go to infinity, there is some part of it (like the geometric series) that "disappears" as you go to higher partial sums that allows it to converge to a rational number. So to mathematically represent this assumption you have

$$\zeta(s)=\lim_{a\to\infty}\frac{c}{d}+\frac{p(a)}{a!^s}$$

$$\text{Where }\lim_{a\to\infty}\frac{p(a)}{a!^s}=0.$$

So then it has to be true, as stated before, of $p(a)$ that for every kth partial sum it approaches some equivalent expression to

$$N(k)\ge(p_1\cdot p_3\cdot p_4\cdots p_m)^s+(p_1\cdot p_2\cdot p_4\cdots p_m)^s\cdots$$

$p(a)$ being all the terms in the numerator of the kth partial sum that are missing primes p_d such that

$$k<2p_d.$$

There is just one simple problem though. The amount of prime number terms that there are infinitely increases. There is no one single term that you can put over $a!^s$ that will then get reduced to 0, because there are no constant terms. The terms will increase proportionally to $a!^s$ and $\frac{p(a)}{a!^s}$ for any terms that you take will itself reduce to a giant fraction with a huge numerator and denominator, and $\frac{p(a)}{a!^s}$ will just end up being an irrational number. Therefore assuming that there is some $p(a)$ such that $\frac{p(a)}{a!^s}=0$ for any rational s cannot be true. To further enforce this logic, notice the fact that if you assume $\frac{p(a)}{a!^s}$ goes to 0 for some s, you are also assuming that it goes to 0 for $\zeta(s+1)$. Since $\zeta(s\to\infty)=1$, the numerator slowly gets lower relative to the denominator until they equal 1. For any ath partial sum, let $N(a)$ denote its unreduced numerator and $D(a)$

its unreduced denominator, or $a!^s$. Note that since $D(a) > N(a)$ for all $\zeta(s \in \mathbb{N})$, $p(a) < N(a)$ and $\lim_{s\to\infty} \frac{N_s(a)}{D_s(a)} = 1$, it will always be true for any ath partial sum that

$$\frac{p_s(a)}{a!^s} \geq \frac{p_{s+1}(a)}{a!^{s+1}}$$

Assuming the for the sake of contradiction that $\frac{p_s(a)}{a!^s}$ could be 0 for some unknown s.

Take the case of $\zeta(4)$. This is known to be $\frac{\pi^4}{90}$, which is irrational, and of course cannot be expressed as $\lim_{a\to\infty} \frac{c}{d} + \frac{p(a)}{a!^s}$ where $\lim_{a\to\infty} \frac{p(a)}{a!^s} = 0$. This same thing is also true of $\zeta(6)$. Now one result that is unknown is the rationality of $\zeta(5)$. Let's assume that $\zeta(5)$ is rational. This would imply that for all of the terms in $\sum_{n=1}^{a} \frac{1}{n^5}$, you have

$$\zeta(5) = \lim_{a\to\infty} \frac{c}{d} + \frac{p(a)}{a!^5}$$

$$\text{Where } \lim_{a\to\infty} \frac{p(a)}{a!^5} = 0.$$

Now refer to the previous result that stated the following:

$$\frac{p_s(a)}{a!^s} \geq \frac{p_{s+1}(a)}{a!^{s+1}}$$

So then it must be true that

$$\frac{p(a)}{a!^5} \geq \frac{p(a)}{a!^6}$$

Which would also additionally imply that $\lim_{a\to\infty} \frac{p(a)}{a!^6} = 0$ as well, meaning that you could also state the following

$$\zeta(6) = \lim_{a\to\infty} \frac{c}{d} + \frac{p(a)}{a!^6} = \frac{c}{d} \in \mathbb{Q}$$

This is obviously impossible since $\zeta(6)$ is an irrational number, as has already been confirmed for all functions $\zeta(2s)$. This is a contradiction and implies that

$$\zeta(5) \notin \mathbb{Q}.$$

Additionally, since you can apply this proof for any $\zeta(2s+1)$ you can also say that

$$\zeta(2s+1) \notin \mathbb{Q}$$

And so finally,

$$\therefore \zeta(s \in \mathbb{N}) \notin \mathbb{Q}.$$

4 Main Theorems

Using the fact that all of the zeta functions of the natural numbers are irrational, the following conditions can be stated:

1. There are infinitely many primes of the form $g(n)$
2. For whatever a such that $f(a)$ is prime and $g(a)$ is the nearest multiple of $f(a)$, $\lim_{a\to\infty} \pi(g(a)) - \pi(f(a)) = \infty$
3. For any kth partial sum where p is a prime, the equation $g(n) = p$ has a finite number of solutions for p.

If these conditions are all satisfied, then the following statements are true:

$$\sum_{n=m}^{\infty} \left(\frac{f(n)}{g(n)}\right)^k \notin \mathbb{Q}.$$

$$\sum_{n=m}^{b} \left(\frac{f(n)}{g(n)}\right)^k \notin \mathbb{Z}.$$

(Provided that $k \in \mathbb{N}$.)

5 Irrationality of π

$$\pi = 4\sum_{n=0}^{\infty}\frac{-1^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}\cdots$$

Besides 2, all primes are of the form 2n + 1. This means that $\pi(n)$ can be used without any sort of algebraic manipulation of the denominator expression. To do this, let's set up the same limit as before and analyze it to see if it is possible. Make this (incorrect) assumption:

$$\pi = \frac{c}{d} + \lim_{a\to\infty}\frac{p(a)}{3\cdot 5\cdot 7\cdots(2a+1)} = \frac{c}{d}$$

$$p(a) = (3\cdot 5\cdot 7\cdots) - (5\cdot 7\cdots) + (3\cdot 7\cdots)\cdots(\textbf{Only terms with missing primes})$$

As we can see, the same thing occurs. There is again no separable term that renders the remaining fraction rational. In this series, since all the denominators are odd, for any prime number $2a+1$ its next multiple doesn't occur until $6a+3$ which would be the $3a+1$th partial sum. $\lim_{a\to\infty}\pi(3a) - \pi(a) = \infty$ so you can say with certainty that $p(a)$ goes to infinity.

$$\therefore \pi \notin \mathbb{Q}.$$

6 Irrationality of γ

An open irrationality problem is the rationality of the Euler-Mascheroni Constant γ.

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln(n)\right)$$

This takes a bit more effort to convert into the $\sum_{n=a}^{\infty}\left(\frac{g(n)}{f(n)}\right)^k$ format.

First, consider the infinite series representation for $\ln(1+n)$:

$$\ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}$$

$$\lim_{n\to\infty}\ln(n+1) = \ln(n)$$

So in the context of a limit to infinity,

$$\ln(n) \sim \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}(n+1)^k$$

Therefore you can rewrite γ like so:

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \sum_{k=1}^{\infty}\frac{(-1)^{k-1}n^k}{k}\right)$$

Which then becomes

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \frac{(-1)^{k-1}n^k}{k}\right)$$

Fortunately the denominator is straightforward. Just k. Of course, there are infinitely many primes of the form $f(k) = k$.

Now to try to apply this theorem to it, let's look at a slightly altered series:

$$\left(\sum_{k=1}^{\infty}\frac{1-(-1)^{k-1}n^k}{k}\right)$$

The main concern with this series is the fact that it almost looks right apart from the $(-1)^{k-1}n^k$. This is simply a matter of proving that it is irrational for every integer that you can plug in for n.

Since you can approach infinity with integers, if that outcome is irrational there's no reason that approaching infinity with different increments would yield a different result.

Consider the behavior of the ath partial sum of this series.

$$\sum_{k=1}^{a} \frac{1-(-1)^{k-1}n^k}{k} = \frac{1-n}{1} + \frac{1+n^2}{2} + \frac{1-n^3}{3} \cdots$$
$$= \frac{(2\cdot 3\cdot 4\cdot (1-n)\cdots a) + (3\cdot 4\cdot (1+n^2)\cdots a) + 2\cdot 4\cdot (1-n^3)\cdots a)}{a!}$$

This luckily does not interfere with the theorem whatsoever. It only does if for some prime number p, it just so happens that the one term $(3\cdot 4\cdot (b+n^a)\cdots (p-1)\cdot (p+1))\ p$ is missing in it is replaced with some n^a+b such that $n^a+b=p$. Luckily, for any prime p, this situation only happens if you have some n such that $1-(-1)^{p-1}n^p = p, -p$. After some simple algebra you get $p, -p = (-n)^p + 1$. Since primes are odd apart from 2, you would need to only solve for $-p$.

$$-p = -n^p + 1$$
$$p = n^p - 1$$
$$n = \pm\sqrt[p]{p+1}$$

So considering that there are only two possible ns that would eliminate 1 single missing factor, the slightly more complicated form of the numerator clearly does not matter. We can then proceed with the general conditions, now that this problem is out of the way.

Criteria 1: there are infinitely many primes of the form k, and there are infinitely many primes between k and $2k$. This condition is satisfied.

Criteria 2: $1-(-1)^{k-1}n^k \in \mathbb{Z}$ for all n. This is also satisfied.

$$\therefore \gamma \notin \mathbb{Q}$$

7 Irrationality of ϕ

While it is not a particularly pressing problem as ϕ is commonly known as $\frac{1+\sqrt{5}}{2}$, you can use a similar logic to prove the irrationality of ϕ in a different way.

It is known that

$$\phi = \lim_{n\to\infty} \frac{F_{n+1}}{F_n}$$

The Fibonacci sequence is as follows:

$$1, 1, 2, 3, 5, 8\cdots$$

Keep in mind if $b \perp a$, then $a+b \perp a$. So since $1 \perp 2$, then $2+1 \perp 2$, and so then $3+2 \perp 2$ and so on, it can be easily proven that $F_{n+1} \perp F_n$. If it is already known that they are coprime then the limit $\lim_{n\to\infty} \frac{F_{n+1}}{F_n}$ needs no simplification, and since both the numerator and denominator go to infinity as a direct result of this limit,

$$\therefore \phi \notin \mathbb{Q}.$$

8 Other Observations

This theorem has a lot of other implications that are not necessarily solutions to actively pursued problems. Below is a list of some of the more interesting ones:

$$\sum_{n=1}^{a} \frac{1}{n} \notin \mathbb{Z}$$

$$\sum_{n=1}^{a} \frac{1}{T_n^k} \notin \mathbb{Z}, \quad k \in \mathbb{Z} \tag{1}$$

$$\sum_{n=1}^{\infty} \frac{1}{T_n^k} \notin \mathbb{Q}, \quad k \in \mathbb{Z} \tag{2}$$

(The irrationality of the twin prime sum only applies if the Twin Prime Conjecture [5] is true.)

$$\sum_{p \in \mathbb{P}} \frac{1}{p^s} \notin \mathbb{Q}, s \in \mathbb{N} \tag{3}$$

The above prime series is the prime zeta function, the zeta function excluding the composites. It is sometimes notated as P(s).

9 Conclusion

This theorem is a much simpler method to accomplish a task that has been historically daunting in the world of mathematics. Not only is it generalized but it is also simple enough to teach in schools and places where people may not have a college level math education, as well as the fact that it can be used on open problems that cannot be practically solved with previously available methods.

References

[1] Bertrand, J. (1852). *Théorème relatif aux nombres premiers*. Comptes rendus des séances de l'Académie des Sciences de Paris, 35, 198-200.

[2] Madhava-Leibniz Pi Series: The Madhava-Leibniz series for calculating the value of π was independently discovered by the Indian mathematician Madhava of Sangamagrama and the German mathematician Gottfried Wilhelm Leibniz. This infinite series converges to π and is given by:

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots\right)$$

[3] Riemann, B. (1859). On the Number of Primes Less Than a Given Magnitude. *Monthly Notices of the Royal Astronomical Society*, 20(2), 27-31.

[4] Goldston, D. A., Pintz, J., and Yıldırım, C. Y. (2005). Primes in tuples I. *Annals of Mathematics*, 170(2), 819-862.

[5] Niven, I. (1947). A simple proof that π is irrational. *Bulletin of the American Mathematical Society*, 53(7), 507.