Growth of eigenvalues of Floer Hessians

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Abstract

In this article we prove that the space of Floer Hessians has infinitely many connected components.

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1 Introduction

1.1 Main results

We consider a Hilbert space pair $H := (H_0, H_1)$, that is H_0 and H_1 are both infinite dimensional Hilbert spaces such that $H_1 \subset H_0$ and inclusion $\iota\colon H_1 \hookrightarrow H_0$ is compact and dense. Then there exists an unbounded monotone function

$$h = h(H_0, H_1)\colon \mathbb{N} \to (0, \infty),$$

called **pair growth function**, such that the pair (H_0, H_1) is isometric to the pair (ℓ^2, ℓ^2_h), see Appendix A, and from now on we identify the pairs

$$(H_0, H_1) = (\ell^2, \ell^2_h), \qquad h = h(H_0, H_1).$$

Here ℓ^2_h is defined as follows. In general, for $f\colon \mathbb{N} \to (0, \infty)$ unbounded monotone ℓ^2_f is the space of all sequences $x = (x_\nu)_{\nu\in\mathbb{N}}$ with $\sum_{\nu=1}^\infty f(\nu) x_\nu^2 < \infty$. The space ℓ^2_f becomes a Hilbert space if we endow it with the inner product

$$\langle x, y\rangle_f = \sum_{\nu=1}^\infty f(\nu) x_\nu y_\nu, \qquad \|x\|_f := \sqrt{\langle x, x\rangle_f}.$$

Therefore we can define for every real number r a Hilbert space $H_r := \ell^2_{f^r}$. For every $s < r$ the inclusion $H_r \subset H_s$ is compact and dense; see Section 2.

Definition 1.1 (Weak Hessian)**.** The topological[1] **space of weak Hessians**

$$\mathcal{H}_h \subset \mathcal{L}(H_1, H_0) \cap \mathcal{L}(H_2, H_1)$$

[1] $\mathcal{L}(H_1, H_0) \cap \mathcal{L}(H_2, H_1)$ is a Banach space under the norm $\max\{\|\cdot\|_{\mathcal{L}(H_1,H_0)}, \|\cdot\|_{\mathcal{L}(H_2,H_1)}\}$.

consists of all bounded linear maps $A\colon H_1 \to H_0$ which are Fredholm[2] of index zero and $\boldsymbol{H_0}$**-symmetric**, that is

$$\langle Ax, y\rangle_0 = \langle x, Ay\rangle_0 \tag{1.1}$$

for all $x, y \in H_1$. Furthermore, it is required that A restricts to a bounded linear map $H_2 \to H_1$, notation A_1, which is Fredholm of index zero as well.

Since the inclusion $H_1 = \ell^2_h \hookrightarrow H_0 = \ell^2$ is compact, the resolvent $R\colon \ell^2 \to \ell^2$ of a weak Hessian A is a compact operator. Hence there exists an orthonormal basis $\mathcal{V}$ of ℓ^2 composed of eigenvectors $v_\nu \in \ell^2_h$ of $A\colon \ell^2_h \to \ell^2$ and the spectrum of A consists of discrete real eigenvalues a_ν of finite multiplicity each. Arrange the squares of eigenvalues of A with multiplicity in an increasing sequence $\{a_\nu^2\}_{\nu\in\mathbb{N}}$ and order the orthonormal eigenvector basis $\mathcal{V} = (v_\nu)_{\nu\in\mathbb{N}} \subset \ell^2_h$ of ℓ^2 accordingly. Assume that A is invertible. Then the squares of all eigenvalues of A are positive and we define a positive monotone increasing function, called **total growth of a weak Hessian** $\boldsymbol{A} \in \mathcal{H}_h$, by

$$f_A\colon \mathbb{N} \to (0,\infty), \quad f_A(\nu) := a_\nu^2. \tag{1.2}$$

Lemma 1.2. *Any invertible weak Hessian $A \in \mathcal{H}_h$ has a total growth function f_A of the same* ***growth type*** *as h, notation $f_A \sim h$.*[3]

Proof. By Theorem A.8 we need to find an isomorphism of Hilbert space pairs

$$T\colon (\ell^2, \ell^2_{f_A}) \to (\ell^2, \ell^2_h).$$

By assumption $A\colon \ell^2_h \to \ell^2$ is an isomorphism. Hence the pull-back inner product $A^* \langle \cdot,\cdot\rangle_{\ell^2}$ is on ℓ^2_h an inner product equivalent to the inner product $\langle \cdot,\cdot\rangle_h$. Therefore the identity gives rise to an isomorphism

$$I\colon (\ell^2, (\ell^2_h, A^*\langle \cdot,\cdot\rangle_{\ell^2})) \to (\ell^2, \ell^2_h)$$

of Hilbert pairs.

Let $(e_\nu)_{\nu\in\mathbb{N}}$ be the standard basis of ℓ^2 and $(v_\nu)_{\nu\in\mathbb{N}}$ the orthonormal eigenvector basis introduced above ($Av_\nu = a_\nu v_\nu$). Consider the isometry given by

$$\Psi\colon \ell^2 \to \ell^2, \quad e_\nu \mapsto v_\nu.$$

For $\mu, \nu \in \mathbb{N}$ we compute

$$\langle A\Psi e_\mu, A\Psi e_\nu\rangle_{\ell^2} = \langle Av_\mu, Av_\nu\rangle_{\ell^2} = a_\mu a_\nu \delta_{\mu\nu} = a_\mu^2 \delta_{\mu\nu} \overset{\substack{(1.2)\\(2.7)}}{=} \langle e_\nu, e_\mu\rangle_{f_A}.$$

This shows that $\Psi^* A^* \langle , \rangle_{\ell^2} = \langle \cdot,\cdot\rangle_{f_A}$. Therefore the restriction

$$\Psi|_{\ell^2_{f_A}} \colon \ell^2_{f_A} \to (\ell^2_h, A^*\langle \cdot,\cdot\rangle_{\ell^2})$$

[2] Closed image and finite dimensional kernel and cokernel, index := dim ker − dim coker.
[3] That is, there exists a constant $c > 0$ such that $\frac{1}{c} \cdot h(\nu) \le f_A(\nu) \le c \cdot h(\nu)$, $\forall \nu \in \mathbb{N}$.

is an isometry. To summarize we have an isometry of pairs

$$\Psi\colon (\ell^2, \ell^2_{f_A}) \to \left(\ell^2, (\ell^2_h, A^* \langle \cdot, \cdot \rangle_{\ell^2})\right).$$

Hence $T := I\Psi\colon (\ell^2, \ell^2_{f_A}) \to (\ell^2, \ell^2_h)$ is an isomorphism of Hilbert pairs. □

According to the lemma the total growth type of A is completely determined by the ambient Hilbert spaces. Hence it cannot be used to distinguish different connected components of the space $\mathcal{H}_h$.
In order to get some information on the topology of the space $\mathcal{H}_h$, in particular its connected components, we are taking into account as well the signs of the eigenvalues. We distinguish the following three **types**.

1. **Morse.** Finitely many negative, infinitely many positive eigenvalues.
2. **Co-Morse.** Finitely many positive, infinitely many negative eigenvalues.
3. **Floer.** Infinitely many negative and positive eigenvalues.

An application of spectral flow [RS95] shows the following.

Lemma 1.3. *Suppose that A^0 and A^1 are connected by a continuous path in $\mathcal{H}_h$. Then A^0 and A^1 are either both Morse, both Co-Morse, or both Floer.*

According to the above three types the set of weak Hessians $\mathcal{H}_h$ decomposes into three open and closed subsets denoted by

$$\mathcal{H}_h = \mathcal{H}_h^{\mathrm{Morse}} \cup \mathcal{H}_h^{\text{co-Morse}} \cup \mathcal{H}_h^{\mathrm{Floer}}.$$

In this article we are interested in the Floer Hessians. Therefore we abbreviate

$$\mathcal{F}_h := \mathcal{H}_h^{\mathrm{Floer}}, \qquad \mathcal{F}_h^* := \{A \in \mathcal{F}_h \text{ invertible}\}.$$

In the Floer case, if $A \in \mathcal{F}_h$ is invertible, i.e. zero is not an eigenvalue, we look separately at the **positive** and **negative growth functions** f_A^+ and f_A^-—given by ordering the positive eigenvalues of the weak Hessian A, with multiplicities, respectively the *absolute values* of the negative eigenvalues.

Definition 1.4. A **growth function** is a monotone unbounded function $f\colon \mathbb{N} \to (0, \infty)$. Having the same growth type, symbol $f \sim g$, see previous footnote, defines an equivalence relation on the set $\mathcal{G}$ of all growth functions. The elements $\mathfrak{f} = [f]$ of the quotient $\mathfrak{G} := \mathcal{G}/\sim$ are called **growth types**.

We are interested in weak Hessians of given negative and positive growth types. That is, given $\mathfrak{a}, \mathfrak{b} \in \mathfrak{G}$, we are interested in the sets

$$(\mathcal{F}_h^{\mathfrak{a}\mathfrak{b}})^* := \{A \in \mathcal{F}_h^* \mid [f_A^-] = \mathfrak{a},\ [f_A^+] = \mathfrak{b}\}. \tag{1.3}$$

Given a growth function $f\colon \mathbb{N} \to (0, \infty)$, we define the shifted growth function $f_1(\nu) := f(\nu + 1)$. We say that the growth type is **shift invariant** if $f \sim f_1$. We denote the **set of shift-invariant growth types** by $\mathfrak{G}_1 \subset \mathfrak{G}$.

For $\lambda \in \mathbb{R}$ consider the translation

$$T_\lambda \colon \mathcal{F}_h \to \mathcal{F}_h, \quad A \mapsto A - \lambda\iota =: \mathbb{A}_\lambda \tag{1.4}$$

which is a homeomorphism with inverse $T_{-\lambda}$. Observe that $\mathbb{A}_\lambda \colon H_1 \to H_0$ is in fact a weak Floer Hessian operator. Indeed the inclusion $\iota \colon H_1 \hookrightarrow H_0$ is compact and H_0-symmetric and so is $-\lambda\iota$. But the sum of a Fredholm operator A and a compact operator $-\lambda\iota$ is Fredholm of the same index. Moreover, the sum of H_0-symmetric operators is H_0-symmetric. The inclusion restricts to a map $\iota| \colon H_2 \hookrightarrow H_1$ and the same arguments show that $A - \lambda\iota \colon H_2 \to H_1$ is Fredholm of index zero and H_0-symmetric. This shows that $A - \lambda\iota \in \mathcal{F}_h$.

Proposition 1.5. *Let $\mathfrak{a}, \mathfrak{b} \in \mathfrak{G}_1$ be shift-invariant growth types. If A is a weak Hessian in $(\mathcal{F}_h^{\mathfrak{ab}})^*$, then $\forall \lambda \in \mathbb{R} \setminus \operatorname{spec} A$ so is the shifted operator $\mathbb{A}_\lambda \in (\mathcal{F}_h^{\mathfrak{ab}})^*$.*

Proof. Section 3.2. □

If $\mathfrak{a}, \mathfrak{b} \in \mathfrak{G}_1$ are *shift-invariant* growth types we define

$$\mathcal{F}_h^{\mathfrak{ab}} := \{A \in \mathcal{F}_h \mid \forall \lambda \in \mathbb{R} \setminus \operatorname{spec} A \colon [f^+_{\mathbb{A}_\lambda}] = \mathfrak{a} \wedge [f^-_{\mathbb{A}_\lambda}] = \mathfrak{b}\}.$$

Note that the resolvent set $\mathbb{R} \setminus \operatorname{spec} A \neq \emptyset$ is non-empty by Remark 3.3. Observe that, by Proposition 1.5, for $A \in \mathcal{F}_h$ to lie in $\mathcal{F}_h^{\mathfrak{ab}}$ it suffices that for one $\lambda \in \mathbb{R} \setminus \operatorname{spec} A$ the positive and negative growth types are $[f^+_{\mathbb{A}_\lambda}] = \mathfrak{a}$ and $[f^-_{\mathbb{A}_\lambda}] = \mathfrak{b}$. Therefore, alternatively, we can define $\mathcal{F}_h^{\mathfrak{ab}}$ for $\mathfrak{a}, \mathfrak{b} \in \mathfrak{G}_1$ as well in the form

$$\mathcal{F}_h^{\mathfrak{ab}} := \{A \in \mathcal{F}_h \mid \exists \lambda_0 \in \mathbb{R} \setminus \operatorname{spec} A \colon [f^+_{\mathbb{A}_{\lambda_0}}] = \mathfrak{a} \wedge [f^-_{\mathbb{A}_{\lambda_0}}] = \mathfrak{b}\}. \tag{1.5}$$

Note that $(\mathcal{F}_h^{\mathfrak{ab}})^* \subset \mathcal{F}_h^{\mathfrak{ab}}$ by choosing $\lambda_0 = 0$ in which case $\mathbb{A}_{\lambda_0} = A$.

Observe that $\mathcal{F}_h^{\mathfrak{ab}}$ is translation invariant in the following sense

$$A \in \mathcal{F}_h^{\mathfrak{ab}} \qquad \Leftrightarrow \qquad \mathbb{A}_\lambda \in \mathcal{F}_h^{\mathfrak{ab}} \quad \forall \lambda \in \mathbb{R}. \tag{1.6}$$

Indeed '$\Leftarrow$' is obvious. For '$\Rightarrow$' assume that $A \in \mathcal{F}_h^{\mathfrak{ab}}$. This means that there exists $\lambda_0 \in \mathbb{R} \setminus \operatorname{spec} A$ such that $\mathbb{A}_{\lambda_0} \in (\mathcal{F}_h^{\mathfrak{ab}})^*$. Given $\lambda \in \mathbb{R}$, note that $T_{\lambda_0 - \lambda} T_\lambda A = T_{\lambda_0} A = \mathbb{A}_{\lambda_0} \in (\mathcal{F}_h^{\mathfrak{ab}})^*$, thus $T_\lambda A \in \mathcal{F}_h^{\mathfrak{ab}}$ by (1.5) with A and λ_0 replaced by $T_\lambda A$ and $\lambda_0 - \lambda$, respectively.

Theorem A. *Suppose that $\mathfrak{a}, \mathfrak{b} \in \mathfrak{G}_1$ are shift-invariant growth types. Then $\mathcal{F}_h^{\mathfrak{ab}}$ is open and closed in $\mathcal{F}_h$.*

The proof of Theorem A is based on Proposition 1.5 and the following result.

Theorem B. *Let $A \in \mathcal{F}_h^*$. Then there exists an open neighborhood $\mathcal{U}_A$ of A in $\mathcal{F}_h^*$ such that the spectral projection selector maps*

$$\Pi_\pm \colon \mathcal{U}_A \to \mathcal{L}(H_{\frac{1}{2}}) \cap \mathcal{L}(H_{\frac{3}{2}}), \quad B \mapsto \Pi_\pm(B)$$

defined by (6.26) are continuous.

In the next theorem we discuss the question when $\mathcal{F}_h^{\mathfrak{ab}}$ is non-empty.

Theorem C. *Let $h = h(H_0, H_1)$ be a pair growth function.*

a) *If $\mathfrak{a}, \mathfrak{b} \in \mathfrak{G}_1$ are shift-invariant growth types and $\mathcal{F}_h^{\mathfrak{ab}} \neq \emptyset$, then the pair growth type $[h] = [h]_1$ is shift-invariant.*

If the pair growth type $[h] = [h]_1$ is shift-invariant, then the following are true:

b) *If a growth type $\mathfrak{a} \in \mathfrak{G}$ satisfies $(\mathcal{F}_h^{\mathfrak{aa}})^* \neq \emptyset$, then $\mathfrak{a} = \mathfrak{a}_1$ is shift-invariant.*

c) *There are infinitely many shift-invariant pairs $\mathfrak{a}, \mathfrak{b} \in \mathfrak{G}_1$ with $\mathcal{F}_h^{\mathfrak{ab}} \neq \emptyset$.*

Corollary 1.6. *If the pair growth type $[h] = [h]_1$ is shift invariant, then the set $\mathcal{F}_h$ of Floer type weak Hessians has infinitely many connected components.*

Proof. By Theorem A and Theorem C c): $(\mathfrak{a}, \mathfrak{b}) \neq (\mathfrak{a}', \mathfrak{b}') \Rightarrow \mathcal{F}_h^{\mathfrak{ab}} \cap \mathcal{F}_h^{\mathfrak{a}'\mathfrak{b}'} = \emptyset$. □

It is interesting to contrast the corollary with the result of Atiyah and Singer [AS69] for the case of real self-adjoint Fredholm operators from separable infinite dimensional Hilbert space into itself. There, in the corollary on p. 307, they show that the complement of the essentially positive and essentially negative Fredholm operators is a classifying space for the real K-theory functor $\mathrm{K}\mathbb{R}^{-7}$, see also [Phi96].

Question 1.7. Given shift-invariant growth types $\mathfrak{a}, \mathfrak{b} \in \mathfrak{G}_1$, is the space $\mathcal{F}_h^{\mathfrak{ab}}$ connected?

This question is related to the question if there is a scale version of the Atiyah-Jänich theorem, [Jän65, Satz 2] and [Ati67, Thm. A1 p. 154].
The Atiyah-Jänich theorem also plays an important role in the result of Atiyah-Singer mentioned above. Both results are based on the result of Kuiper [Kui65] on the contractibility of the infinite dimensional orthogonal group.
A scale version of the Kuiper theorem can be found in the book by Kronheimer and Mrowka [KM07, Prop. 33.1.5].

Question 1.8. Is the assumption that our Fredholm operators restrict to operators from H_2 to H_1 necessary?

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1.2 Motivation and general perspective

Nowadays we have many powerful examples of Floer theories, although we do not have a good notion what a Floer theory actually is. This issue especially gets important in the extension to Hamiltonian delay equations.

Hamiltonian delay equations have a long history. One of their first occurrences was a paper by Carl Neumann in 1868 [Neu68], English translation [Ass21], in which Carl Neumann deduces Wilhelm Weber's force law in electrodynamics from a retarded Coulomb potential; see [FW19] for a modern treatment. Quite recently Hamiltonian delay equations found surprising new applications due to an interesting paper by Barutello, Ortega, and Verzini [BOV21] where they discovered a new non-local regularization for two-body collisions.

For autonomous Hamiltonian systems there are various techniques to regularize two-body collisions by blowing up the energy hypersurface; see [FvK18] for a discussion of such techniques from the point of view of symplectic and contact topology. For non-autonomous Hamiltonian systems there is no preserved energy and therefore Barutello, Ortega, and Verzini don't blow up an energy surface, but the loop space by reparametrizing every loop individually. The new equation they obtain is a Hamiltonian delay equation which for non-collisional orbits can be transformed back to the original Hamiltonian ODE. This has the advantage that it also covers collision orbits. The resulting action functionals have fascinating properties [FW21a, CFV23]. How Floer theory can be used to prove existence for some Hamiltonian delay equations was already discussed in [AFS20]. With the new techniques discovered by Barutello, Ortega, and Verzini the question of Floer homologies for Hamiltonian delay equations becomes now a topic of major concern.

In view of the growing interest in Hamiltonian delay equations the question becomes pressing what a Floer theory actually is. When Floer introduced his celebrated homology to prove the Arnol′d conjecture [Flo88, Flo89] he considered the gradient flow equation with respect to a very weak metric. This was in sharp contrast to previous work by Conley and Zehnder [CZ83, CZ84] where they considered an interpolation metric that led to a gradient flow equation which can be interpreted as an ODE on a Hilbert manifold. Different from that the Hessian in Floer's theory is not a self-adjoint operator from a fixed Hilbert space into itself, but a symmetric operator on a scale of Hilbert spaces. This in recent years led to the discovery of a new smooth structure in infinite dimensions by Hofer, Wysocki, and Zehnder [HWZ07, HWZ21] referred to as scale-smoothness.

For Hamiltonian delay equations the structure of the space of such Hessians is now a topic of major interest [AFS20, FW22, CFV23]. In particular, the topology of the space of Fredholm operators plays a major role in orientation questions. As we show in the present article this space differs drastically from the case of self-adjoint operators on a fixed Hilbert space: While in the scale case, as established in this article, the space has infinitely many different connected components, it is connected in the later case [Ati67, Phi96].

The proof of our main result, the space of Floer Hessians has infinitely many connected components, uses interpolation theory and Schur multipliers. That Floer theory is related to interpolation theory was used before in several contexts [RS95, AF13, Sim14, HWZ21]. The connection between Floer theory and Schur multipliers seems to be completely new.

1.3 Outline

This article is organized as follows.

In Section 2 "Growth functions and growth types" we recall in § 2.1 the pointwise $f \cdot g$ and the Kang, or union, product $f * g$ on the set $\mathcal{G}$ of growth functions. The latter is the growth function $\mathbb{N} \to (0, \infty)$ whose values are the union of values $f(\mathbb{N}) \cup g(\mathbb{N})$ enumerated increasingly with multiplicities. The two products are compatible in the sense that $(f * g)^2 = f^2 * g^2$.
In § 2.2 we endow the set of growth types $\mathfrak{G} := [\mathcal{G}] := \mathcal{G}/\sim$ with a partial order "$\le$". Both, pointwise and Kang product, descend to $[\mathcal{G}]$ as commutative associative products, same notation. In § 2.2.1 we look at shift invariance of growth types and their Kang and pointwise products. In § 2.2.2 we introduce scale invariance of growth types an example of which are known Floer growth types. § 2.2.3 establishes the existence of infinitely many pairs of shift invariant growth types $[f_i]$ representing a given shift invariant growth type $[f] = [f_i * f_j]$.

Section 3 "Weak Hessians" provides properties of general weak Hessians $A \in \mathcal{H}_h$ such as "regularity and spectrum" in § 3.1 which also deals with properties inherited from $A \colon H_1 \to H_0$ by the restriction $A \colon H_2 \to H_1$ such as invertibility, eigenspaces, and discrete, finite multiplicity, pure eigenvalue spectrum. A crucial side fact is that $\mathbb{R} \setminus \operatorname{spec} A$ is non-empty.
§ 3.2 "Proof of Proposition 1.5" shows that the set $(\mathcal{F}_h^{\mathfrak{a}\mathfrak{b}})^*$ of invertible Floer Hessians A with prescribed negative and positive growth types $\mathfrak{a}$ and $\mathfrak{b}$ is preserved under translation $T_\lambda A := A - \lambda\iota$ by the elements λ of $\mathbb{R} \setminus \operatorname{spec} A$.

In Section 4 "Proof of Theorem A" we show using Theorem B (continuity of $B \mapsto \Pi_+^B$) that the restriction of the projection Π_+^B to the image of the projection Π_+^A gives rise to a scale isomorphism

$$\Pi_+^B|_A \colon (\Pi_+^A H_{\frac{1}{2}}, \Pi_+^A H_{\frac{3}{2}}) \to (\Pi_+^B H_{\frac{1}{2}}, \Pi_+^B H_{\frac{3}{2}})$$

whenever B is close enough to A. Then Theorem A follows from the result that a scale isomorphism preserves the growth type of the pair, as proved in the appendix in Theorem A.8.

Section 5 "Schur multipliers" deals with the Schur product of two matrices which is defined by taking the entry-wise product. A Schur multiplier is a matrix which, via Schur multiplication, keeps the space of matrices corresponding to bounded linear maps $\ell^2 \to \ell^2$ invariant. In this section we explain a criterium of Schur to show that a given matrix is a Schur multiplier and mention a result by Grothendieck which shows that the criterium of Schur is not just sufficient but also necessary. Via the result of Grothendieck we discuss a criterium explained to us by Gilles Pisier to deduce that a given matrix is not a Schur multiplier.

The positive results of Schur are one of the major ingredients to prove Theorem B. While the obstruction is the reason that our proof of Theorem B does not extend from $H_{\frac{1}{2}}$ to H_0. This obstruction also the main reason why for Theorem A we need that our weak Hessians restrict to Fredholm operators of index zero from H_2 to H_1.

Section 6 "Proof of Theorem B" is the analytical heart of the present article. To prove Theorem B, inspired by Kato's perturbation theory for linear operators, we write the projection Π_+^A for A invertible as an operator-valued integral in the complex plane where the contour γ_+ encircles all positive eigenvalues of A^{-1}, but no negative eigenvalue of A^{-1}, as illustrated by Figure 1.

The spectrum of A^{-1} has the origin as its unique accumulation point. Therefore we decompose our contour $\gamma_+ = \beta \# \alpha_+$ into two paths, where the dangerous part β of the path is going through the origin and the harmless part α_+ stays at bounded distance of the spectrum of A^{-1}. Using restriction and inclusion, we interpret the projection as a linear operator from H_1 to H_0, see Figure 2, and explicitly compute its differential with respect to variation Δ of A.

The main ingredient to prove Theorem B is to show that the images of this operator are actually bounded linear maps from $H_{\frac{1}{2}}$ to $H_{\frac{1}{2}}$. Along the harmless part α_+ this follows by interpolation theory. While along the dangerous part β we need the theory of Schur multipliers.

In Appendix A "Hilbert space pairs", following [Fra09], we define the growth type $[h]$ of a pair of Hilbert spaces (H_0, H_1) and show that up to isomorphism a Hilbert space pair is uniquely characterized by its growth type

In Appendix B "Interpolation" we explain a special case of an interpolation theorem by Stein going back to a method discovered by Thorin.

2 Growth functions and growth types

A **growth function** is a monotone unbounded function

$$f \colon \mathbb{N} \to (0, \infty).$$

We abbreviate by $\ell^2_f = \ell^2_f(\mathbb{N})$ the space of all sequences $x = (x_\nu)_{\nu \in \mathbb{N}}$ satisfying

$$\sum_{\nu=1}^{\infty} f(\nu) x_\nu^2 < \infty.$$

The space ℓ^2_f becomes a Hilbert space if we endow it with the inner product

$$\langle x, y \rangle_f = \sum_{\nu=1}^{\infty} f(\nu) x_\nu y_\nu. \tag{2.7}$$

Let $\|x\|_f := \sqrt{\langle x, x \rangle_f}$ be the associated norm. For $r \in \mathbb{R}$ we abbreviate

$$H_r := \ell^2_{f^r}, \qquad \langle \cdot, \cdot \rangle_r := \langle \cdot, \cdot \rangle_{f^r}, \qquad \|\cdot\|_r := \|\cdot\|_{f^r}. \tag{2.8}$$

Because the function f is monotone increasing, it follows that whenever $s \leq r$ there is an inclusion

$$H_r \hookrightarrow H_s$$

of Hilbert spaces and the corresponding linear inclusion operator is bounded, a bound being $1/f(1)$. For strict inequality $s < r$, by unboundedness of f, the inclusion operator is compact. Moreover, its image is dense. For integers $r, s \in \mathbb{N}_0$ these facts are proved in [FW21b, Thm. 8.1]. In the real case the same arguments prove density and compactness of the inclusion $\ell^2_{f^r} \hookrightarrow \ell^2$ for $r > 0$ and $\ell^2 \hookrightarrow \ell^2_{f^r}$ for $r < 0$. To deal with the case $s < r$ use the isometric isomorphisms ϕ_r below. In particular,

$$H = H_0 \supset H_1 \supset H_2 \supset \dots$$

defines an sc-structure, also called a Banach scale, in the sense of Hofer, Wysocki and Zehnder, see [HWZ07,HWZ21]. As all levels are Hilbert spaces, one speaks of a Hilbert scale. For $r \in \mathbb{R}$ we abbreviate by H^r the Hilbert scale with levels

$$H^r_m := H_{r+m}, \quad m \in \mathbb{N}_0.$$

For $r \in \mathbb{N}$ these are the shifted scale spaces introduced in [HWZ07, §2.1]. In the set-up of fractal Hilbert scales, introduced in [FW21b], the shift works more generally for every $r \in \mathbb{R}$. Observe that the sc-Hilbert spaces H^r and H are isomorphic via the levelwise isometries

$$\phi_r \colon H \to H^r, \quad (x_\nu)_\nu \mapsto \left(f(\nu)^{-r/2} x_\nu\right)_\nu .$$

2.1 Growth functions

Let $\mathcal{G}$ be the **set of growth functions**. The **shift** f_1 of $f \in \mathcal{G}$ is defined by $f_1(\nu) := f(\nu + 1)$. On $\mathcal{G}$ there are two products. The **pointwise product**

$$\cdot \colon \mathcal{G} \times \mathcal{G} \to \mathcal{G}, \quad (f, g) \mapsto f \cdot g, \qquad (f \cdot g)(\nu) := f(\nu) \cdot g(\nu), \qquad f^2 := f \cdot f,$$

and the **Kang product** [Kan19, (3.1)]

$$* \colon \mathcal{G} \times \mathcal{G} \to \mathcal{G}, \quad (f, g) \mapsto f * g.$$

The product $f * g$ is the monotone unbounded function $f * g \colon \mathbb{N} \to (0, \infty)$ whose value at ν is the ν^{th} smallest number, counted with multiplicities, among the members of the two lists $(f) = (f(1), f(2), \dots)$ and $(g) = (g(1), g(2), \dots)$.[4]
For example the lists

$$(f) = (1, 1, 2, 3, 5, \dots), \quad (g) = (1, 2, 2, 3, 4, 5, \dots),$$

have Kang product

$$(f * g) = (1, 1, 1, 2, 2, 2, 3, 3, 4, 5, 5, \dots).$$

[4] If f and g are strictly monotone and their images $f(\mathbb{N}) \cap g(\mathbb{N}) = \emptyset$ are disjoint, then the Kang product is given by the formula $(f * g)(\nu) = \min \left\{ (f(\mathbb{N}) \cup g(\mathbb{N})) \setminus \bigcup_{\mu=1}^{\mu-1} (f * g)(\mu) \right\}$.

Recall that for a real x **ceiling** $\lceil x \rceil$ and **floor** $\lfloor x \rfloor$ denote the integer next to x, larger or equal x, respectively less or equal x, in symbols

$$\lceil x \rceil := \min\{n \in \mathbb{Z} \mid n \geq x\}, \ \ \lfloor x \rfloor := \max\{n \in \mathbb{Z} \mid n \leq x\}, \ \ \lfloor \tfrac{x}{2} \rfloor + 1 = \lceil \tfrac{x}{2} + \tfrac{1}{2} \rceil.$$

The Kang square and its shift satisfy

$$(f * f)(\nu) = f(\lceil \tfrac{\nu}{2} \rceil), \qquad (f * f)_1(\nu) = f(\lfloor \tfrac{\nu}{2} \rfloor + 1) = f_1(\lfloor \tfrac{\nu}{2} \rfloor).$$

In particular, it holds

$$(f * f)(2\nu) = f(\nu), \qquad (f * f)_1(2\nu) = f(\nu + 1) = f_1(\nu). \tag{2.9}$$

Kang product and pointwise square are compatible in the sense that

$$(f * g)^2 = f^2 * g^2. \tag{2.10}$$

This holds since squaring preserves the ordering.

2.2 Growth types

On $\mathcal{G}$ there is the equivalence relation of having the **same growth type**

$$f \sim f' \quad :\Leftrightarrow \quad \exists c > 0 \colon \frac{1}{c} f(\nu) \leq f'(\nu) \leq c f(\nu), \ \forall \nu \in \mathbb{N}. \tag{2.11}$$

An equivalence class $\mathfrak{f} := [f]$ is referred to as a **growth type**. The set of equivalence classes $\mathfrak{G} := [\mathcal{G}] := \mathcal{G}/\sim$ is the **set of growth types**.

Simple examples of growth functions of the same growth type are growth functions which just differ by an additive constant as we show next.

Lemma 2.1. *Let f and g be growth functions. Assume that there exists $\lambda \in \mathbb{R}$ such that $f(\nu) = g(\nu) + \lambda$ for every $\nu \in \mathbb{N}$. Then f and g have the same growth type, in symbols $f \sim g$.*

Proof. We can assume without loss of generality that λ is positive. Indeed if $\lambda = 0$ the assertion is trivial and if $\lambda < 0$ we can switch the role of f and g.
Since $g > 0$ we have $0 < g(1) = f(1) - \lambda$ and therefore $\lambda \in (0, f(1))$.
The function $x \mapsto \frac{x}{x-\lambda}$ for $x > \lambda$ is monotone decreasing. Indeed its derivative is given by $-\lambda/(x-\lambda)^2 < 0$. As f is monotone increasing we have $\frac{f(\nu)}{f(\nu)-\lambda} \leq \frac{f(1)}{f(1)-\lambda}$. Hence $f(\nu) = \frac{f(\nu)}{f(\nu)-\lambda}(f(\nu) - \lambda) \leq \frac{f(1)}{f(1)-\lambda}(f(\nu) - \lambda) = \frac{f(1)}{f(1)-\lambda} g(\nu)$. Therefore the growth function f is bounded from above by the growth function g up to a multiplicative constant.
Since $\lambda > 0$ we have $g(\nu) \leq f(\nu)$ and therefore the two growth functions have the same growth type. $\square$

Definition 2.2 (Partial order)**.** On the set $[\mathcal{G}]$ of growth types we define a partial order as follows. Assume $[f]$ and $[g]$ are growth types. We write $[f] \leq [g]$ if there exists a constant $d > 0$ such that $f(\nu) \leq d g(\nu)$ for every $\nu \in \mathbb{N}$. We write $[f] < [g]$ if $[f] \leq [g]$ and $[f] \neq [g]$.

Pointwise and Kang product both descend to $[\mathcal{G}]$ as commutative associative products, still denoted by

$$\cdot, *\colon [\mathcal{G}] \times [\mathcal{G}] \to [\mathcal{G}], \quad [f]\cdot[g] := [f\cdot g], \quad [f]*[g] := [f*g].$$

For the pointwise product this is easy,[5] for the Kang product see [Kan19, § 3].

Lemma 2.3 (Kang [Kan19])**.** *The Kang product is compatible with the partial order on the set of growth types $\mathfrak{G}$ as follows.*

(i) $[g_1] \le [g_2] \Rightarrow [f*g_1] \le [f*g_2]$;

(ii) $f*g \le f$ *pointwise at any* $\nu \in \mathbb{N}$ *and for all growth functions* f *and* g.

2.2.1 Shift invariance

Note that if two growth functions f and g have the same growth type, then their shifts have the same growth type as well, in symbols

$$f \sim g \quad \Rightarrow \quad f_1 \sim g_1.$$

Thus the shift descends to growth type, so the following is well defined

$$[f]_1 := [f_1].$$

Definition 2.4. A growth type $[f]$ is called **shift invariant** if $[f]_1 = [f]$.

Lemma 2.5. *For growth functions* $f, g\colon \mathbb{N} \to (0,\infty)$ *the following is true.*

(i) $(f\cdot g)_1 = f_1 \cdot g_1$.

(ii) $(f*g)_1 = \begin{cases} f_1 * g & \text{, if } f(1) \le g(1), \\ f * g_1 & \text{, if } g(1) \le f(1). \end{cases}$

(iii) *If the growth types* $[f]$ *and* $[g]$ *are shift invariant, then, by (ii), their pointwise product and their Kang product are shift invariant, too, in symbols*

$$[f] = [f]_1 \;\wedge\; [g] = [g]_1 \qquad \Rightarrow \qquad [f\cdot g]_1 = [f\cdot g] \;\wedge\; [f*g]_1 = [f*g].$$

(iv) *A Kang square* $[f*f]$ *is shift invariant iff the growth type* $[f]$ *itself is, in symbols*

$$[f*f]_1 = [f*f] \quad \Leftrightarrow \quad [f]_1 = [f].$$

(v) *A pointwise square* $[f\cdot f]$ *is shift invariant iff the growth type* $[f]$ *itself is, in symbols*

$$[f\cdot f]_1 = [f\cdot f] \quad \Leftrightarrow \quad [f]_1 = [f].$$

Proof. (i) and (ii) follow directly by definition. (iii) follows from (i) and (ii). We prove (iv). '$\Rightarrow$' Suppose $f*f \sim (f*f)_1$, that is there is a constant $c>0$ such that $\frac{1}{c}(f*f)(\nu) \le (f*f)_1(\nu) \le c(f*f)(\nu)$ for every $\nu \in \mathbb{N}$. In particular, $\frac{1}{c}(f*f)(2\nu) \le (f*f)_1(2\nu) \le c(f*f)(2\nu)$ for every $\nu \in \mathbb{N}$. Hence, by (2.9), $\frac{1}{c}f(\nu) \le f_1(\nu) \le cf(\nu)$ for every $\nu \in \mathbb{N}$. Thus $f \sim f_1$. '$\Leftarrow$' Choose $g = f$ in (iii). (v) This follows by taking roots. $\square$

[5] In (2.11), if $f \sim f'$ with constant c and $g \sim g'$ with constant d, then $f\cdot g \sim f'\cdot g'$ with cd.

2.2.2 Scale invariance

Definition 2.6. A growth type $[f]$ is called **scale invariant** if there exists a constant c such that $f(2\nu) \le cf(\nu)$ for every $\nu \in \mathbb{N}$.[6]

Lemma 2.7. *Suppose that $[f]$ is a scale invariant growth type. Then $[f]$ is (i) shift-invariant and (ii) Kang idempotent.*

Proof. (i) It holds $f_1(\nu) := f(\nu+1) \le f(2\nu) \le cf(\nu)$ where inequality one uses monotonicity of f. (ii) It holds $(f*f)(2\nu) \le f(2\nu) \le cf(\nu) = c(f*f)(2\nu)$. □

Example 2.8 (Floer growth types)**.** Given a real parameter r, the growth function $f_r(\nu) = \nu^r$ for $\nu \in \mathbb{N}$ is scale invariant with $c = 2^r$. The growth type of f_r satisfies $[f_r] = [f_r * f_s]$ whenever $s \ge r$. The growth types of Floer theories studied so far are of this type, see [FW21b, p. 378].

Example 2.9. The growth function $f(\nu) = e^\nu$ is not scale invariant.

Lemma 2.10. *Assume $[f]$ is a scale invariant growth type. Then there exists a strictly larger scale invariant growth type $[g]$, in symbols $[g] > [f]$.*

Proof. Pick $\kappa > 0$. Define $g(\nu) := \nu^\kappa f(\nu)$. Then $[f] \le [g]$ with constant $d = 1$. On the other hand $[f] \ne [g]$, since $\nu \mapsto \nu^\kappa$ is not bounded by any constant c. For scale invariance note that $g(2\nu) = (2\nu)^\kappa f(2\nu) \le 2^\kappa \nu^\kappa c f(\nu) = 2^\kappa c g(\nu)$. □

2.2.3 Kang product representations

Theorem 2.11. *Assume $[f] \in \mathfrak{G}_1$ is a shift invariant growth type. Then there exist infinitely many pairs $([f_1], [f_2]) \in \mathfrak{G}_1 \times \mathfrak{G}_1$ such that $[f] = [f_1 * f_2]$.*

Proof. **Case 1 ($[f]$ scale invariant).** By Lemma 2.10 there exists an infinite sequence of scale invariant growth types $[f] = [g_1] < [g_2] < [g_3] < \dots$.
We claim that $[f] = [f * g_k]$ whenever $k \in \mathbb{N}$: Since $[f]$ is scale invariant it is Kang idempotent by Lemma 2.7. So the assertion is true for $k = 1$. Hence, using parts (i) and (ii) of Lemma 2.3, we have $[f] = [f*f] \le [f*g_k] \le [f]$. This proves that $[f] = [f * g_k]$ for every $k \in \mathbb{N}$. Since scale invariant growth types are shift invariant, by Lemma 2.7 (i), Case 1 is proved.

Case 2 ($[f]$ not scale invariant). Pick an integer $k \ge 2$. We decompose the image of f into two parts as follows. For $\nu \in \mathbb{N}$ we define $f_k(\nu) := f(k\nu)$ and

$$g_k(\nu) := \begin{cases} f(\nu) & , \nu \in \{1, \dots, k-1\} \\ f(\nu+1) & , \nu \in \{k, \dots, 2k-2\} \\ f(\nu+2) & , \nu \in \{2k-1, \dots, 3k-3\} \\ f(\nu+3) & , \nu \in \{3k-2, \dots, 4k-4\} \\ \quad\vdots & \qquad\vdots \end{cases}$$

[6] Well defined: In (2.11), if $f \sim f'$ with constant d, then $f'(2\nu) \le cd^2 f'(\nu)$.

Observe that $g_k(\nu) = f(\nu + \ell)$ whenever $\ell k - (\ell - 1) \le \nu \le (\ell + 1)k - (\ell + 1)$ for some $\ell \in \mathbb{N}_0$. Then $f_k * g_k = f$. Moreover, shift invariance of f implies that f_k and g_k are shift invariant.[7]

Claim. Let $k_2 > k_1 \ge 2$. Then the growth types of f_{k_1} and f_{k_2} are different.

Proof of Claim. We argue by contradiction and assume that there exists a constant $c > 0$ such that

$$f(k_2\nu) = f_{k_2}(\nu) \le c f_{k_1}(\nu) = cf(k_1\nu) \tag{2.12}$$

for every $\nu \in \mathbb{N}$. To derive a contradiction we show that this implies that there exists a constant $C > 0$ such that

$$f(2\nu) \le Cf(\nu) \tag{2.13}$$

for every $\nu \in \mathbb{N}$ which contradicts the assumption that f is not scale invariant.

Subclaim. Suppose $\nu_1 \in \mathbb{N}$ satisfies $\nu_1 \ge k_1\sqrt{k_2}/(\sqrt{k_2} - \sqrt{k_1})$. Then there exists $\nu_2 \in \mathbb{N}$ such that a) $f(\nu_1) \ge \frac{1}{c} f(\nu_2)$ and b) $\nu_2 \ge \sqrt{k_2/k_1}\nu_1$.

Proof of Subclaim. There exist unique $m \in \mathbb{N}_0$ and $r \in \{0, \dots, k_1 - 1\}$ such that $\nu_1 = k_1 m + r$. We set $\nu_2 := mk_2$.
a) We estimate $f(\nu_1) \ge f(mk_1) \ge \frac{1}{c} f(mk_2)$ where the first step is by monotonicity of f and the second step is by (2.12) for $\nu = m$. b) We estimate $\frac{k_1\sqrt{k_2}}{\sqrt{k_2}-\sqrt{k_1}} \le \nu_1 = k_1 m + r < (m + 1)k_1$. Resolving for m we get

$$m \ge \frac{\sqrt{k_2}}{\sqrt{k_2} - \sqrt{k_1}} - 1 = \frac{\sqrt{k_1}}{\sqrt{k_2} - \sqrt{k_1}} =: m_0. \tag{2.14}$$

The function $\rho\colon (-1, \infty) \to \mathbb{R}$, $x \mapsto x/(1 + x)$ is strictly monotone increasing, indeed its derivative is $\rho'(x) = 1/(1 + x)^2 > 0$. Therefore

$$\frac{m}{m+1} = \rho(m) \ge \rho(m_0) = \sqrt{k_1/k_2}.$$

We use this estimate in the last step of what follows to get

$$\nu_2 := mk_2 = \frac{mk_2}{\nu_1}\nu_1 \ge \frac{m}{m+1}\frac{k_2}{k_1}\nu_1 \ge \sqrt{\frac{k_2}{k_1}}\nu_1$$

where we used $\nu_1 < (m+1)k_1$ in the first inequality. This proves the Subclaim.

Since $\sqrt{k_2/k_1} > 1$ there exists $N \in \mathbb{N}$ such that $\sqrt{k_2/k_1}^N \ge 2$. Now suppose that $\nu \ge k_1\sqrt{k_2}/(\sqrt{k_2} - \sqrt{k_1})$. Then applying the Subclaim iteratively N times, we obtain the scale invariance criterium $f(\nu) \ge \frac{1}{c^N} f(2\nu)$, but only for large ν. But along finitely many ν's the function f is bounded. So (2.13) follows with

$$C := \max\left\{c^N, \frac{f(2\lceil k_1\sqrt{k_2}/(\sqrt{k_2} - \sqrt{k_1})\rceil)}{f(1)}\right\}.$$

This proves the Claim and Theorem 2.11. □

[7] e.g. $f_k(\nu + 1) = f(k\nu + k) \sim f(k\nu) = f_k(\nu)$

2.3 Proof of Theorem C

a) Since $\mathcal{F}_h^{\mathfrak{ab}}$ is non-empty, there exists $A \in \mathcal{F}_h^{\mathfrak{ab}}$. Without loss of generality we may assume that A is invertible.[8] By definition of f_A (order the eigenvalue squares) there is the first identity in

$$f_A = (f_A^+)^2 * (f_A^-)^2 \overset{(2.10)}{=} (f_A^+ * f_A^-)^2. \tag{2.15}$$

Hence, by Lemma 2.5 (iii), the growth type of f_A is shift-invariant and therefore the same is true for the pair growth function h by Lemma 1.2. This proves a).

b) By Lemma 1.2 it holds $f_A \sim h$. Therefore, by the assumption $[h] = [h]_1$, the growth type of f_A is shift invariant, in symbols $[f_A]_1 = [f_A]$.
By assumption $f_A^- \sim f_A^+$. Thus, by (2.15) and Lemma 2.5 (v), it follows that the growth type of $f_A^+ * f_A^+$ is shift invariant, in symbols $[f_A^+ * f_A^+]_1 = [f_A^+ * f_A^+]$. Hence by Lemma 2.5 (iv) it follows that the growth type of f_A^+ itself is shift invariant, in symbols $[f_A^+]_1 = [f_A^+]$.

c) Theorem 2.11.

3 Weak Hessians

3.1 Regularity and spectrum

Lemma 3.1 (Regularity)**.** *Let A be a weak Hessian, in symbols $A \in \mathcal{H}_h$. If ξ and $A\xi$ both lie in H_1, then ξ actually lies in H_2.*

Proof. Since H_2 is separable the operator A, as a map $H_2 \to H_1$, can only have countably many eigenvalues. Hence there exists $\lambda \in \mathbb{R}$ which is not an eigenvalue. We claim that $A - \lambda\iota \colon H_2 \to H_1$ is an isomorphism.

Since λ is not an eigenvalue of A there is no eigenvector, in symbols $\ker(A - \lambda\iota) = \{0\}$. Since $\iota \colon H_2 \to H_1$ is compact and $A \colon H_2 \to H_1$ is Fredholm of index zero, by stability of the Fredholm index under compact perturbation, $A - \lambda\iota \colon H_2 \to H_1$ is as well Fredholm of index zero. Therefore, since the kernel of $A - \lambda\iota$ is trivial so must be the cokernel. In particular $A - \lambda\iota \colon H_2 \to H_1$ is bijective, hence by the open mapping theorem the inverse is continuous, hence $A - \lambda\iota \colon H_2 \to H_1$ is an isomorphism.

Now the proof of the lemma is straightforward, indeed

$$\xi = (A - \lambda\iota)^{-1} (\underbrace{A\xi - \lambda\xi}_{\in H_1}) \in H_2.$$

$\square$

Corollary 3.2 (Kernel and invertibility)**.** *Let $A \in \mathcal{H}_h$. Then* $\ker(A \colon H_1 \to H_0) = \ker(A \colon H_2 \to H_1)$. *In particular $A \colon H_1 \to H_0$ is invertible iff $A \colon H_2 \to H_1$ is invertible.*

[8] Otherwise, consider $\mathbb{A}_\lambda := A - \lambda\iota$ for $\lambda \in \mathbb{R} \setminus \operatorname{spec} A$.

Proof. '$\supset$' is obvious. '$\subset$' Suppose that $\xi \in H_1$ satisfies $A\xi = 0 \in H_1$. Thus $\xi \in H_2$ by Lemma 3.1. Concerning invertibility, since A is Fredholm of index zero, as a map $H_1 \to H_0$ and as a map $H_2 \to H_1$, invertibility is equivalent to triviality of the kernel; see the second paragraph in the proof of Lemma 3.1. $\square$

Remark 3.3 (Resolvent set non-empty)**.** The second paragraph of the proof of Lemma 3.1 shows that the spectrum[9] of A only consists of eigenvalues.[10] Moreover, in view of the first paragraph of the proof of Lemma 3.1 the resolvent set $\mathbb{R} \setminus \operatorname{spec} A \neq \emptyset$ is non-empty.

Corollary 3.4 (Spectrum)**.** *Let $A \in \mathcal{H}_h$. Then* $\operatorname{spec}(A\colon H_2 \to H_1)$ *is discrete, consists of eigenvalues of finite multiplicity, and is equal to* $\operatorname{spec}(A\colon H_1 \to H_0)$*. Moreover, the eigenspaces of $A\colon H_1 \to H_0$ and $A\colon H_2 \to H_1$ are equal, in particular multiplicities are equal.*

Proof. Choose $\lambda \in \mathbb{R} \setminus \operatorname{spec} A$. This is possible since the resolvent set of $A\colon H_1 \to H_0$ is non-empty by the previous remark. Since A is H_0-symmetric the resolvent operator $R_\lambda := \iota \circ (A - \lambda\iota)^{-1}\colon H_0 \to H_1 \to H_0$ is symmetric as well and it is compact since ι is. Therefore, by the spectral theory for symmetric compact operators, the spectrum of R_λ consists of real eigenvalues μ of finite multiplicity which only accumulate at the origin. Observe the equivalence

$$R_\lambda \xi := (A - \lambda\iota)^{-1}\xi = \mu\xi \quad \Leftrightarrow \quad A\xi = \tfrac{1+\lambda\mu}{\mu}\xi.$$

Hence the spectrum of $A\colon H_1 \to H_0$ consists of discrete eigenvalues a_ℓ of finite multiplicities which accumulate at $\pm\infty$.

For any $\lambda \in \mathbb{R}$, applying Corollary 3.2 to $A - \lambda\iota$, we see that the kernels of $A - \lambda\iota\colon H_1 \to H_0$ and of $A - \lambda\iota\colon H_2 \to H_1$ coincide. In particular, the eigenvalues, together with their multiplicities, of $A\colon H_1 \to H_0$ and of $A\colon H_2 \to H_1$ are equal. $\square$

3.2 Proof of Proposition 1.5

Suppose $\mathfrak{a}, \mathfrak{b} \in \mathfrak{G}_1$ are shift-invariant and $A \in (\mathcal{F}_h^{\mathfrak{a}\mathfrak{b}})^*$. For any non-eigenvalue $\lambda \in \mathbb{R} \setminus \operatorname{spec} A$ the shifted operator $\mathbb{A}_\lambda := A - \lambda\iota$ is invertible.

Claim +. $f^+_{\mathbb{A}_\lambda} \sim f^+_A$ equivalently $[f^+_{\mathbb{A}_\lambda}] = [f^+_A] = \mathfrak{b}$

Proof of Claim +. Enumerate for $\ell \in \mathbb{Z}$ the eigenvalue $a_\ell \in \mathbb{R}$ of A in the form

$$\cdots \le a_{-2} \le a_{-1} < 0 < a_1 \le a_2 \le \dots, \qquad \mathbb{Z}^* := \mathbb{Z} \setminus \{0\}. \tag{3.16}$$

Since λ is not an eigenvalue, either

$$a_{-1} < \lambda < a_1 \qquad \text{or} \qquad a_{\ell_\lambda} < \lambda < a_{\ell_\lambda + 1}$$

[9] $\operatorname{spec} A := \{\lambda \in \mathbb{R} \mid A - \lambda\iota\colon H_2 \to H_1 \text{ is not an isomorphism}\}$

[10] Eigenvalues are those $\lambda \in \operatorname{spec} A$ for which $A - \lambda\iota\colon H_2 \to H_1$ is not injective.

for some $\ell_\lambda \in \mathbb{Z}^* \setminus \{-1\} = \mathbb{Z} \setminus \{-1, 0\}$. Hence, either

$$a_{-1} - \lambda < 0 < a_1 - \lambda = f^+_{\mathbb{A}_\lambda}(1) \quad \text{or} \quad a_{\ell_\lambda} - \lambda < 0 < a_{\ell_\lambda+1} - \lambda = f^+_{\mathbb{A}_\lambda}(1).$$

So the growth function of the *positive* eigenvalues of $\mathbb{A}_\lambda = A - \iota\lambda$ is either

$$f^+_{\mathbb{A}_\lambda}(\nu) = a_\nu - \lambda \qquad \text{or} \qquad f^+_{\mathbb{A}_\lambda}(\nu) = a_{\ell_\lambda+\nu} - \lambda \tag{3.17}$$

where $\nu \in \mathbb{N}$. Because $[f^+_A]_1 = [f^+_A]$ by Step 1, there exists $c > 0$ such that for each $k \geq 1$ we have $a_{k+1} \leq c a_k$, hence

$$a_{k+i} \leq c^i a_k \tag{3.18}$$

for every $i \in \mathbb{N}_0$.

Case 1 ($\ell_\lambda \geq 1$). So by our enumeration $0 < a_{\ell_\lambda}$ ($< \lambda$). Therefore we estimate

$$f^+_{\mathbb{A}_\lambda}(\nu) = a_{\ell_\lambda+\nu} - \lambda \leq c^{\ell_\lambda} a_\nu - \lambda \leq c^{\ell_\lambda} a_\nu = c^{\ell_\lambda} f^+_A(\nu). \tag{3.19}$$

Since $\ell_\lambda \geq 1$ we get the first inequality in

$$f^+_A(\nu) = a_\nu \leq a_{\ell_\lambda+\nu} \leq \frac{a_{1+\ell_\lambda}}{a_{1+\ell_\lambda} - \lambda}(a_{\ell_\lambda+\nu} - \lambda) = \frac{a_{1+\ell_\lambda}}{a_{1+\ell_\lambda} - \lambda} f^+_{\mathbb{A}_\lambda}(\nu) \tag{3.20}$$

and the second inequality is obtained by applying Lemma 2.1 to the by ℓ_λ shifted positive growth function, namely to $(f^+_A)_{\ell_\lambda}$.

The two inequalities (3.19) and (3.20) show that $f^+_{\mathbb{A}_\lambda}$ and f^+_A are of the same growth type. This proves the + assertion of Proposition 1.5 in case $\ell_\lambda \geq 1$.

Case 2 ($\ell_\lambda \leq -2$). In this case $\lambda < a_{\ell_\lambda+1} \leq a_{-1} < 0$ is negative and

$$f^+_{\mathbb{A}_\lambda}(\nu) = a_{\ell_\lambda+\nu} - \lambda \leq a_\nu - \lambda = \tfrac{a_\nu - \lambda}{a_\nu}\, a_\nu \leq \tfrac{a_1 - \lambda}{a_1}\, a_\nu = \tfrac{a_1 - \lambda}{a_1}\, f^+_A(\nu)$$

for each $\nu \in \mathbb{N}$. For the second inequality we use that for $\mu = -\lambda > 0$ the function $x \mapsto \frac{x+\mu}{x}$ is monotone decreasing. Indeed its derivative is $\mu/x^2 > 0$.
For $\nu \geq 1 - \ell_\lambda$ we obtain by (3.18) the first inequality in what follows

$$f^+_A(\nu) = a_\nu \leq c^{-\ell_\lambda} a_{\ell_\lambda+\nu} < c^{-\ell_\lambda}(a_{\ell_\lambda+\nu} - \lambda) \overset{(3.17)}{=} c^{-\ell_\lambda} f^+_{\mathbb{A}_\lambda}(\nu)$$

where the second inequality holds since $-\lambda > 0$.
Since this inequality is true for all $\nu \in \mathbb{N}$, except finitely many exceptions, there exists a constant $C > 0$ such that $f^+_A(\nu) \leq C f^+_{\mathbb{A}_\lambda}(\nu)\ \forall \nu \in \mathbb{N}$.

The above two inequalities show that $f^+_{\mathbb{A}_\lambda}$ and f^+_A are of the same growth type. This proves the + assertion of Proposition 1.5 in case $\ell_\lambda \leq -2$.

Case 3 ($a_{-1} < \lambda < a_1$). In this case $f^+_{\mathbb{A}_\lambda}(\nu) = a_\nu - \lambda = f^+_A(\nu) - \lambda$. Hence the two growth functions just differ by a constant and therefore, by Lemma 2.1, they have the same growth type.

This proves Claim +, so Proposition 1.5 in the (+) case, namely $[f^+_{\mathbb{A}_\lambda}] = \mathfrak{b}$. $\square$

The $(-)$ case of Proposition 1.5 follows from the $(+)$ case by replacing A by $-A$. The proof of Proposition 1.5 is complete.

4 Proof of Theorem A

We prove Theorem A in three steps.

Step 1. Given growth types $\mathfrak{a}, \mathfrak{b} \in \mathfrak{G}$, shift invariant or not, and $A \in (\mathcal{F}_h^{\mathfrak{a}\mathfrak{b}})^*$, in particular A is invertible. Then there exists an open neighborhood $\mathcal{V}_A$ of A in $\mathcal{F}_h^*$ which lies in $(\mathcal{F}_h^{\mathfrak{a}\mathfrak{b}})^*$, in symbols $\mathcal{V}_A \subset (\mathcal{F}_h^{\mathfrak{a}\mathfrak{b}})^* \subset \mathcal{F}_h$.

Proof of Step 1. In Case 1 shift invariance is not required. By Theorem B, there is an open neighborhood $\mathcal{U}_A$ of A in $\mathcal{F}_h^*$ such that both maps

$$\Pi_\pm \colon \mathcal{U}_A \to \mathcal{L}(H_{\frac{1}{2}}) \cap \mathcal{L}(H_{\frac{3}{2}}), \quad B \mapsto \Pi_\pm(B) =: \Pi_\pm^B$$

are continuous. Hence, given $\varepsilon > 0$, there exists an open neighborhood $\mathcal{U}_\varepsilon^+(A) \subset \mathcal{U}_A$ of A such that

$$\left\|\Pi_+^B - \Pi_+^A\right\|_{\mathcal{L}(H_{\frac{1}{2}})\cap\mathcal{L}(H_{\frac{3}{2}})} \le \varepsilon$$

for every $B \in \mathcal{U}_+^\varepsilon(A)$. We fix $\varepsilon_+ := \min\{\frac{1}{2}, 1/4\|\Pi_+^A\|_{\mathcal{L}(H_{\frac{1}{2}})\cap\mathcal{L}(H_{\frac{3}{2}})}\}$ and set

$$\mathcal{V}_A^+ := \mathcal{U}_{\varepsilon_+}^+(A).$$

Claim +. For every $B \in \mathcal{V}_A^+$ it holds that $f_B^+ \sim f_A^+$.

Proof of Claim +. Pick $B \in \mathcal{V}_A^+$. We restrict the projections $\Pi_+^B, \Pi_+^A \colon H_{\frac{1}{2}} \to H_{\frac{1}{2}}$ and consider them as maps between the following subspaces

$$\Pi_+^B|_A := \Pi_+^B|_{\Pi_+^A H_{\frac{1}{2}}} : \Pi_+^A H_{\frac{1}{2}} \to \Pi_+^B H_{\frac{1}{2}}$$

and

$$\Pi_+^A|_B := \Pi_+^A|_{\Pi_+^B H_{\frac{1}{2}}} : \Pi_+^B H_{\frac{1}{2}} \to \Pi_+^A H_{\frac{1}{2}}.$$

We prove that the following compositions are isomorphisms

$$\text{a) } \Pi_+^A|_B \circ \Pi_+^B|_A \in \mathcal{L}(\Pi_+^A H_{\frac{1}{2}}), \qquad \text{b) } \Pi_+^B|_A \circ \Pi_+^A|_B \in \mathcal{L}(\Pi_+^B H_{\frac{1}{2}}). \tag{4.21}$$

This is true for $B = A$ since the image of the projection Π_+^A is its fixed point set, in symbols $\Pi_+^A H_{\frac{1}{2}} = \operatorname{Fix} \Pi_+^A$. Hence $\Pi_+^A|_A = \mathrm{Id}$ on $\Pi_+^A H_{\frac{1}{2}}$.

To see that the above two compositions are invertible we show that they differ in norm from the identity by less than 1. Indeed

$$\begin{aligned}
&\left\|\Pi_+^A|_B \circ \Pi_+^B|_A - \mathrm{Id}_{\Pi_+^A H_{\frac{1}{2}}}\right\|_{\mathcal{L}(H_{\frac{1}{2}})\cap\mathcal{L}(H_{\frac{3}{2}})} \\
&= \left\|\Pi_+^A|_B \circ \Pi_+^B|_A - \Pi_+^A|_A \circ \Pi_+^A|_A\right\|_{\mathcal{L}(H_{\frac{1}{2}})\cap\mathcal{L}(H_{\frac{3}{2}})} \\
&= \left\|\Pi_+^A \left(\Pi_+^B|_A - \Pi_+^A|_A\right)\right\|_{\mathcal{L}(H_{\frac{1}{2}})\cap\mathcal{L}(H_{\frac{3}{2}})} \\
&\le \left\|\Pi_+^A\right\|_{\mathcal{L}(H_{\frac{1}{2}})\cap\mathcal{L}(H_{\frac{3}{2}})} \left\|\Pi_+^B|_A - \Pi_+^A|_A\right\|_{\mathcal{L}(H_{\frac{1}{2}})\cap\mathcal{L}(H_{\frac{3}{2}})} \\
&\le \left\|\Pi_+^A\right\|_{\mathcal{L}(H_{\frac{1}{2}})\cap\mathcal{L}(H_{\frac{3}{2}})} \left\|\Pi_+^B - \Pi_+^A\right\|_{\mathcal{L}(H_{\frac{1}{2}})\cap\mathcal{L}(H_{\frac{3}{2}})} \\
&\le \left\|\Pi_+^A\right\|_{\mathcal{L}(H_{\frac{1}{2}})\cap\mathcal{L}(H_{\frac{3}{2}})} \cdot \varepsilon_+ \le \tfrac{1}{4}
\end{aligned}$$

and

$$
\begin{aligned}
&\left\| \Pi_+^B|_A \circ \Pi_+^A|_B - \mathrm{Id}_{\Pi_+^B H_{\frac{1}{2}}} \right\|_{\mathcal{L}(H_{\frac{1}{2}})\cap\mathcal{L}(H_{\frac{3}{2}})} \\
&= \left\|\Pi_+^B|_A \circ \Pi_+^A|_B - \Pi_+^B|_B \circ \Pi_+^B|_B\right\|_{\mathcal{L}(H_{\frac{1}{2}})\cap\mathcal{L}(H_{\frac{3}{2}})} \\
&= \left\|\Pi_+^B \left(\Pi_+^A|_B - \Pi_+^B|_B\right)\right\|_{\mathcal{L}(H_{\frac{1}{2}})\cap\mathcal{L}(H_{\frac{3}{2}})} \\
&\le \left\|\Pi_+^B\right\|_{\mathcal{L}(H_{\frac{1}{2}})\cap\mathcal{L}(H_{\frac{3}{2}})} \left\|\Pi_+^A|_B - \Pi_+^B|_B\right\|_{\mathcal{L}(H_{\frac{1}{2}})\cap\mathcal{L}(H_{\frac{3}{2}})} \\
&\le \left\|\Pi_+^B\right\|_{\mathcal{L}(H_{\frac{1}{2}})\cap\mathcal{L}(H_{\frac{3}{2}})} \left\|\Pi_+^A - \Pi_+^B\right\|_{\mathcal{L}(H_{\frac{1}{2}})\cap\mathcal{L}(H_{\frac{3}{2}})} \\
&\le \left\|\Pi_+^B - \Pi_+^A + \Pi_+^A\right\|_{\mathcal{L}(H_{\frac{1}{2}})\cap\mathcal{L}(H_{\frac{3}{2}})} \cdot \varepsilon_+ \\
&\le \varepsilon_+^2 + \varepsilon_+ \left\|\Pi_+^A\right\|_{\mathcal{L}(H_{\frac{1}{2}})\cap\mathcal{L}(H_{\frac{3}{2}})} \le \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}.
\end{aligned}
$$

This proves that both compositions a) and b) in (4.21) are invertible. Therefore $\Pi_+^B|_A \colon \Pi_+^A H_{\frac{1}{2}} \to \Pi_+^B H_{\frac{1}{2}}$ is a) injective and b) surjective, hence an isomorphism. By the same argument $\Pi_+^B|_A \colon \Pi_+^A H_{\frac{3}{2}} \to \Pi_+^B H_{\frac{3}{2}}$ is an isomorphism.

Conclusion. For any B in the open neighborhood $\mathcal{V}_A^+$ of A in $\mathcal{F}_h$ the restriction

$$\Pi_+^B|_A \colon (\Pi_+^A H_{\frac{1}{2}}, \Pi_+^A H_{\frac{3}{2}}) \to (\Pi_+^B H_{\frac{1}{2}}, \Pi_+^B H_{\frac{3}{2}})$$

is an isomorphism of Hilbert pairs.
Hence, by Theorem A.8, the two Hilbert pairs have the same growth type. The growth type of the pair $(\Pi_+^A H_0, \Pi_+^A H_1)$ is $[(f_A^+)^2]$ and by shift-invariance of scales, see Example A.3, the same is true for the growth type of $(\Pi_+^A H_{\frac{1}{2}}, \Pi_+^A H_{\frac{3}{2}})$. By the same reasoning the growth type of $(\Pi_+^B H_{\frac{1}{2}}, \Pi_+^B H_{\frac{3}{2}})$ is $[(f_B^+)^2]$.

Hence $(f_A^+)^2 \sim (f_B^+)^2$ and therefore $f_A^+ \sim f_B^+$. This proves Claim $+$.

Claim $-$. There exists an open neighborhood $\mathcal{V}_A^-$ of A in $\mathcal{F}_h$ such that for every $B \in \mathcal{V}_A^-$ it holds that $f_A^- \sim f_B^-$.

Proof of Claim $-$. Same argument as in Claim $+$.

Define $\mathcal{V}_A := \mathcal{V}_A^+ \cap \mathcal{V}_A^-$. Then $\mathcal{V}_A$ is an open neighborhood of A in $\mathcal{F}_h^*$ and for every $B \in \mathcal{V}_A$ we have $[f_B^+] = [f_A^+] = \mathfrak{b}$ and $[f_B^-] = [f_A^-] = \mathfrak{a}$. Thus B belongs to $(\mathcal{F}_h^{\mathfrak{ab}})^*$. This proves Step 1. □

Step 2. Given shift invariant growth types $\mathfrak{a}, \mathfrak{b} \in \mathfrak{G}_1$, then $\mathcal{F}_h^{\mathfrak{ab}}$ is open in $\mathcal{F}_h$.

Proof of Step 2. Pick any non-eigenvalue $\lambda \in \mathbb{R} \setminus \operatorname{spec} A$; see Remark 3.3. Then the (invertible) operator $\mathbb{A}_\lambda := A - \lambda\iota$ lies in $(\mathcal{F}_h^{\mathfrak{ab}})^*$, by Proposition 1.5. Hence Case 2 follows by applying Case 1 to $\mathbb{A}_\lambda$ as we explain next. By Case 1 we get an open neighborhood $\mathcal{V}_{\mathbb{A}_\lambda} \subset \mathcal{F}_h^{\mathfrak{ab}}$ of $\mathbb{A}_\lambda$ in $\mathcal{F}_h$. The translated set $\mathcal{V}_A := \mathcal{V}_{\mathbb{A}_\lambda} + \lambda\iota$ is still open and contains A. By translation invariance (1.6) of $\mathcal{F}_h^{\mathfrak{ab}}$ the set $\mathcal{V}_A$ is an open neighborhood of A in $\mathcal{F}_h^{\mathfrak{ab}}$. This proves Step 2. □

Before we continue the proof to show closedness we introduce some notation.

Definition 4.1 (Wild Floer Hessians)**.** A Floer Hessian $A \in \mathcal{F}_h$ is called **wild** if there exist $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \operatorname{spec} A$ such that

$$\left([f^+_{\mathbb{A}_{\lambda_1}}], [f^-_{\mathbb{A}_{\lambda_1}}]\right) \neq \left([f^+_{\mathbb{A}_{\lambda_2}}], [f^-_{\mathbb{A}_{\lambda_2}}]\right) \in \mathfrak{G} \times \mathfrak{G}. \tag{4.22}$$

Equivalently, by Proposition 1.5, a Floer Hessian A is wild if for one, hence for every, $\lambda \in \mathbb{R} \setminus \operatorname{spec} A$ at least one of the growth types $[f^+_{\mathbb{A}_\lambda}]$ or $[f^-_{\mathbb{A}_\lambda}]$ is not shift invariant. We call $A \in \mathcal{F}_h$ **tame** if it is not wild.

Therefore we have the disjoint union in wild and tame Floer Hessians

$$\mathcal{F}_h = \mathcal{W}_h \cup \mathcal{T}_h$$

where $\mathcal{W}_h = \{A \in \mathcal{F}_h \mid A \text{ wild}\}$ and $\mathcal{T}_h = \{A \in \mathcal{F}_h \mid A \text{ tame}\}$. Note that A is tame iff there exist shift-invariant growth types $\mathfrak{a}$ and $\mathfrak{b}$ such that $A \in \mathcal{F}_h^{\mathfrak{a}\mathfrak{b}}$.

Lemma 4.2. *The set of wild Floer Hessians $\mathcal{W}_h$ is open in $\mathcal{F}_h$.*

Proof. Suppose that $A \in \mathcal{W}_h$. Then there exist $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \operatorname{spec} A$ such that (4.22) holds true. By definition (1.3) we have that

$$\mathbb{A}_{\lambda_1} \in \left(\mathcal{F}_f^{[f^-_{\mathbb{A}_{\lambda_1}}]\,[f^+_{\mathbb{A}_{\lambda_1}}]}\right)^*$$

and so Step 1 applies: there is an open neighborhood $\mathcal{V}_{\mathbb{A}_{\lambda_1}}$ of $\mathbb{A}_{\lambda_1}$ in $\mathcal{F}_h$ with

$$\mathcal{V}_{\mathbb{A}_{\lambda_1}} \subset \left(\mathcal{F}_f^{[f^-_{\mathbb{A}_{\lambda_1}}]\,[f^+_{\mathbb{A}_{\lambda_1}}]}\right)^*.$$

Similarly there exists an open neighborhood $\mathcal{V}_{\mathbb{A}_{\lambda_2}}$ of $\mathbb{A}_{\lambda_2}$ in $\mathcal{F}_h$ with

$$\mathcal{V}_{\mathbb{A}_{\lambda_2}} \subset \left(\mathcal{F}_f^{[f^-_{\mathbb{A}_{\lambda_2}}]\,[f^+_{\mathbb{A}_{\lambda_2}}]}\right)^*.$$

For $\lambda \in \mathbb{R}$ consider the translation

$$T_\lambda \colon \mathcal{F}_h \to \mathcal{F}_h, \quad B \mapsto B - \lambda\iota =: \mathbb{B}_\lambda.$$

Note that $A = T_{-\lambda_1}\mathbb{A}_{\lambda_1}$ and $A = T_{-\lambda_2}\mathbb{A}_{\lambda_2}$. So $A \in T_{-\lambda_1}\mathcal{V}_{\mathbb{A}_{\lambda_1}} \cap T_{-\lambda_2}\mathcal{V}_{\mathbb{A}_{\lambda_2}} =: \mathcal{V}_A$ and $\mathcal{V}_A$ is open in $\mathcal{F}_h$.

It remains to show that $\mathcal{V}_A$ is contained in $\mathcal{W}_h$. To see this pick $B \in \mathcal{V}_A$. Then $T_{\lambda_1}B \in \mathcal{V}_{\mathbb{A}_{\lambda_1}} \subset \left(\mathcal{F}_f^{[f^-_{\mathbb{A}_{\lambda_1}}]\,[f^+_{\mathbb{A}_{\lambda_1}}]}\right)^*$. So $([f^-_{T_{\lambda_1}B}], [f^+_{T_{\lambda_1}B}]) = ([f^-_{\mathbb{A}_{\lambda_1}}], [f^+_{\mathbb{A}_{\lambda_1}}])$. Similarly for λ_2. Therefore, together with (4.22), it follows that $([f^-_{T_{\lambda_1}B}], [f^+_{T_{\lambda_1}B}]) \neq ([f^-_{T_{\lambda_2}B}], [f^+_{T_{\lambda_2}B}])$ which means that B is wild. This proves Lemma 4.2. $\square$

Corollary 4.3. *The set of tame Floer Hessians $\mathcal{T}_h$ is closed in $\mathcal{F}_h$.*

We are now in position to finish the proof of the theorem.

Step 3. Given shift invariant growth types $\mathfrak{a}, \mathfrak{b} \in \mathfrak{G}_1$, then $\mathcal{F}_h^{\mathfrak{a}\mathfrak{b}}$ is closed in $\mathcal{F}_h$.

Proof of Step 3. Since $\mathcal{T}_h$ is closed in $\mathcal{F}_h$, it suffices to show that $\mathcal{F}_h^{\mathfrak{a}\mathfrak{b}}$ is closed in $\mathcal{T}_h$. We write

$$\mathcal{T}_h = \bigcup_{\substack{(\mathfrak{c},\mathfrak{d})\in \\ \mathfrak{G}_1\times\mathfrak{G}_1}} \mathcal{F}_h^{\mathfrak{c}\mathfrak{d}}.$$

Since for every pair $(\mathfrak{c}, \mathfrak{d}) \in \mathfrak{G}_1 \times \mathfrak{G}_1$ the subset $\mathcal{F}_h^{\mathfrak{c}\mathfrak{d}}$ is open in $\mathcal{T}_h$ by Step 2, it follows that the complement of $\mathcal{F}_h^{\mathfrak{a}\mathfrak{b}}$, namely

$$\mathcal{T}_h \setminus \mathcal{F}_h^{\mathfrak{a}\mathfrak{b}} = \bigcup_{\substack{(\mathfrak{c},\mathfrak{d})\neq(\mathfrak{a},\mathfrak{b})\in \\ \mathfrak{G}_1\times\mathfrak{G}_1}} \mathcal{F}_h^{\mathfrak{c}\mathfrak{d}}.$$

is open in $\mathcal{T}_h$ and therefore $\mathcal{F}_h^{\mathfrak{a}\mathfrak{b}}$ is closed in $\mathcal{T}_h$. This proves Step 3. $\square$

This concludes the proof of Theorem A.

5 Schur multipliers

Consider the Hilbert space $\ell^2 = \ell^2_{\mathbb{N}}(\mathbb{R})$ of square summable real sequences enumerated by the positive integers. Let $\{e_\nu\}_{\nu\in\mathbb{N}}$ be the standard basis of ℓ^2. Let $\mathcal{L}(\ell^2)$ be the vector space of bounded linear operators on ℓ^2. Let $\ell^\infty_{\mathbb{N}\times\mathbb{N}} = \ell^\infty_{\mathbb{N}\times\mathbb{N}}(\mathbb{R})$ the vector space of (infinite) real matrices.

Definition 5.1. A bounded linear operator $\mathbf{a} \in \mathcal{L}(\ell^2)$ we identify with its (infinite) matrix $(a_{\mu\nu}) = (a_{\mu\nu})_{\mu,\nu}$ with entries $a_{\mu\nu} := \langle e_\mu, \mathbf{a} e_\nu\rangle$ for $\mu, \nu \in \mathbb{N}$. Thus we write $\mathbf{a} = (a_{\mu\nu})$. A matrix $\mathbf{b} = (b_{\mu\nu})$ is called a **Schur multiplier** if it gives rise to a continuous (bounded) linear operator on $\mathcal{L}(\ell^2)$ defined by

$$S_{\mathbf{b}} \colon \mathcal{L}(\ell^2) \to \mathcal{L}(\ell^2), \quad (a_{\mu\nu}) \mapsto (b_{\mu\nu} a_{\mu\nu}) =: \mathbf{b} \odot \mathbf{a}.$$

The operator $S_{\mathbf{b}} \in \mathcal{L}(\mathcal{L}(\ell^2))$ is called a **Schur operator**.

The set $\mathcal{S}$ of all Schur operators $S_{\mathbf{b}}$ is a vector subspace of $\mathcal{L}(\mathcal{L}(\ell^2))$. It is endowed with the commutative[11] product

$$\begin{aligned} &\mathcal{S} \times \mathcal{S} \to \mathcal{S} \\ &(S_{\mathbf{b}}, S_{\tilde{\mathbf{b}}}) \mapsto S_{\mathbf{b}} \circ S_{\tilde{\mathbf{b}}} = S_{\mathbf{b}\odot\tilde{\mathbf{b}}}. \end{aligned}$$

The following lemma is due to Schur.

[11] Since multiplication in $\mathbb{R}$ is associative it holds $S_{\mathbf{b}} \circ S_{\tilde{\mathbf{b}}} = S_{\mathbf{b}\odot\tilde{\mathbf{b}}}$ and since it is commutative it holds $\mathbf{b} \odot \tilde{\mathbf{b}} = \tilde{\mathbf{b}} \odot \mathbf{b}$.

Lemma 5.2 (Schur [Sch11, Thm. VI]). *Let $I \subset \mathbb{R}$ be an interval and κ a constant. Suppose $f_\nu, g_\mu \in L^2(I, \mathbb{R})$ are two uniformly bounded function sequences*

$$\|f_\nu\|_{L^2(I)}^2 \le \kappa, \qquad \|g_\mu\|_{L^2(I)}^2 \le \kappa, \tag{5.23}$$

for all $\mu, \nu \in \mathbb{N}$. Then the matrix entries

$$b_{\mu\nu} := \int_I g_\mu(s) f_\nu(s)\, ds$$

define a Schur multiplier $\mathbf{b} = (b_{\mu\nu})$ whose Schur operator satisfies

$$\|S_{\mathbf{b}}\|_{\mathcal{L}(\mathcal{L}(\ell^2))} \le \kappa.$$

Proof. We follow the proof of Schur [Sch11, p. 13]. Let $\mathbf{a} \in \mathcal{L}(\ell^2)$. For unit vectors $x, y \in \ell^2$ we get

$$\begin{aligned}
|\langle y, S_{\mathbf{b}}\mathbf{a}x\rangle| &= \left|\sum_{\mu,\nu} y_\mu b_{\mu\nu} a_{\mu\nu} x_\nu\right| \\
&\overset{1}{\le} \int_I \left|\sum_{\mu,\nu} y_\mu g_\mu(s) a_{\mu\nu} x_\nu f_\nu(s)\right| ds \\
&\overset{2}{\le} \int_I \|\mathbf{a}\|_{\mathcal{L}(\ell^2)} \sqrt{\sum_\mu y_\mu^2 g_\mu(s)^2} \sqrt{\sum_\nu x_\nu^2 f_\nu(s)^2}\, ds \\
&\overset{3}{\le} \frac{1}{2}\|\mathbf{a}\|_{\mathcal{L}(\ell^2)} \int_I \left(\sum_\mu y_\mu^2 g_\mu(s)^2 + \sum_\mu x_\mu^2 f_\mu(s)^2\right) ds \\
&= \frac{1}{2}\|\mathbf{a}\|_{\mathcal{L}(\ell^2)} \left(\sum_\mu y_\mu^2 \int_I g_\mu(s)^2\, ds + \sum_\mu x_\mu^2 \int_I f_\mu(s)^2\, ds\right) \\
&\overset{4}{\le} \frac{\kappa}{2}\|\mathbf{a}\|_{\mathcal{L}(\ell^2)} \left(\sum_\mu y_\mu^2 + \sum_\mu x_\mu^2\right) \\
&= \kappa \|\mathbf{a}\|_{\mathcal{L}(\ell^2)}.
\end{aligned}$$

Here inequality 1 is by definition of $b_{\mu\nu}$. Inequality 2 is by definition of the operator norm of $\mathbf{a}$ applied to the ℓ^2 elements $(y_\mu g_\mu(s))$ and $(x_\nu f_\nu(s))$. Inequality 3 uses that $rs \le (r^2 + s^2)/2$. Inequality 4 uses hypothesis (5.23). The final identity holds since $x, y \in \ell^2$ are unit vectors.

The above estimate implies that $\|S_{\mathbf{b}}\mathbf{a}\|_{\mathcal{L}(\ell^2)} \le \kappa\|\mathbf{a}\|_{\mathcal{L}(\ell^2)}$ and therefore we obtain $\|S_{\mathbf{b}}\|_{\mathcal{L}(\mathcal{L}(\ell^2))} \le \kappa$. ∎

The following corollary essentially follows the argument of Schur [Sch11, p. 14] when he showed that the Hilbert matrix $\mathbf{b} = (\frac{1}{\mu+\nu})$ is a Schur multiplier.

Corollary 5.3. *Suppose that a_μ, b_ν are sequences of positive reals, then*

$$b_{\mu\nu} := \frac{\sqrt{a_\mu b_\nu}}{a_\mu + b_\nu}$$

defines a Schur multiplier $\mathbf{b} = (b_{\mu\nu})$ such that $\|S_\mathbf{b}\|_{\mathcal{L}(\mathcal{L}(\ell^2))} \le \frac{1}{2}$.

Proof. Consider the interval $I = [0, \infty)$ and the functions $f_\nu(s) := \sqrt{b_\nu} e^{-b_\nu s}$ and $g_\mu(s) := \sqrt{a_\mu} e^{-a_\mu s}$. Then calculation shows that

$$\|f_\nu\|^2_{L^2(I)} = \tfrac{1}{2}, \quad \|g_\mu\|^2_{L^2(I)} = \tfrac{1}{2}, \quad b_{\mu\nu} := \int_I f_\nu(s) g_\mu(s)\, ds = \tfrac{\sqrt{a_\mu b_\nu}}{a_\mu + b_\nu}.$$

Now the Schur Lemma 5.2 applies with $\kappa = \frac{1}{2}$. $\square$

Corollary 5.4. *Suppose there exists a separable Hilbert space H, sequences of vectors $f_\nu, g_\mu \in H$, and a constant κ such that*

$$\|f_\nu\|^2_H \le \kappa, \qquad \|g_\mu\|^2_H \le \kappa,$$

for all $\nu, \mu \in \mathbb{N}$. Then the matrix entries

$$b_{\mu\nu} := \langle g_\mu, f_\nu \rangle_H$$

define a Schur multiplier $\mathbf{b} = (b_{\mu\nu})$ whose Schur operator satisfies

$$\|S_\mathbf{b}\|_{\mathcal{L}(\mathcal{L}(\ell^2))} \le \kappa.$$

Proof. An infinite dimensional separable Hilbert space H is isometric to $L^2(I, \mathbb{R})$ for any interval $I = [a, b]$ with $a < b$. Now Lemma 5.2 applies. If H has finite dimension, embed it into an infinite dimensional separable Hilbert space. $\square$

Corollary 5.5. *Suppose there exists a separable Hilbert space H, sequences of vectors $f_\nu, g_\mu \in H$, and constants κ_1 and κ_2 such that*

$$\|f_\nu\|_H \le \kappa_1, \qquad \|g_\mu\|_H \le \kappa_2,$$

for all $\nu, \mu \in \mathbb{N}$. Then the matrix entries

$$b_{\mu\nu} := \langle g_\mu, f_\nu \rangle_H$$

define a Schur multiplier $\mathbf{b} = (b_{\mu\nu})$ whose Schur operator satisfies

$$\|S_\mathbf{b}\|_{\mathcal{L}(\mathcal{L}(\ell^2))} \le \kappa_1 \kappa_2.$$

Proof. The rescaled sequences $F_\nu := \sqrt{\kappa_2/\kappa_1} f_\nu$ and $G_\mu := \sqrt{\kappa_1/\kappa_2} g_\mu$ satisfy

$$\|F_\nu\|^2_H \le \kappa_1 \kappa_2, \qquad \|g_\mu\|^2_H \le \kappa_1 \kappa_2,$$

and the matrix entries are the same

$$B_{\mu\nu} := \langle G_\mu, F_\nu \rangle_H = \langle g_\mu, f_\nu \rangle_H =: b_{\mu\nu}.$$

Now apply Corollary 5.4 to the sequences F_ν and G_μ. $\square$

Example 5.6. Let a_μ and b_ν be $\mathbb{N}$-sequences of positive reals. Let $H = \mathbb{R}$.

I. For $f_\nu := 1$ and $g_\mu := \arctan a_\mu$ we have $|f_\nu|_\mathbb{R} = 1$ and $|g_\mu|_\mathbb{R} \le \pi/2$. Thus, by Corollary 5.5, the matrix entries $b^{\mathrm{I}}_{\mu\nu} := g_\mu f_\nu = \arctan a_\mu$ define a Schur multiplier $\mathbf{b}^{\mathrm{I}} = (b^{\mathrm{I}}_{\mu\nu})$ whose Schur operator satisfies $\|S_{\mathbf{b}^{\mathrm{I}}}\|_{\mathcal{L}(\mathcal{L}(\ell^2))} \le \pi/2$.

II. For $f_\nu := \arctan b_\nu$ and $g_\mu := 1$ we have $|f_\nu|_\mathbb{R} \le \pi/2$ and $|g_\mu|_\mathbb{R} = 1$. Thus, by Corollary 5.5, the matrix entries $b^{\mathrm{I}}_{\mu\nu} := g_\mu f_\nu = \arctan b_\nu$ define a Schur multiplier $\mathbf{b}^{\mathrm{II}} = (b^{\mathrm{II}}_{\mu\nu})$ whose Schur operator satisfies $\|S_{\mathbf{b}^{\mathrm{II}}}\|_{\mathcal{L}(\mathcal{L}(\ell^2))} \le \pi/2$.

III. Combining I and II with Corollary 5.3 and using that Schur multipliers form an algebra we conclude that the matrix entries

$$\tilde{b}_{\mu\nu} := \frac{\sqrt{a_\mu b_\nu}}{a_\mu + b_\nu} \left(\arctan a_\mu + \arctan b_\nu\right)$$

define a Schur multiplier $\tilde{\mathbf{b}} = (\tilde{b}_{\mu\nu})$ and $\|S_{\tilde{\mathbf{b}}}\|_{\mathcal{L}(\mathcal{L}(\ell^2))} \le \pi/2$.

For the proofs of Theorem A and Theorem B Corollary 5.5 is sufficient. The reason that we cannot, with the method of Schur multipliers, improve Theorem B from $H_{\frac{1}{2}}$ to H_0 lies in the fact that there is a converse to Corollary 5.5 which then gives obstructions for matrices to be a Schur multiplier.

Indeed Grothendieck [Gro56] proved, see also [Pis01, Thm. 5.1], the converse

Theorem 5.7. *A matrix $\mathbf{b} = (b_{\mu\nu})$ is a Schur multiplier whose Schur operator satisfies*

$$\|S_{\mathbf{b}}\|_{\mathcal{L}(\mathcal{L}(\ell^2))} \le \kappa$$

if and only if there exists a Hilbert space H and sequences $f_\nu, g_\mu \in H$ satisfying

$$\|f_\nu\|_H \le \kappa_1, \qquad \|g_\mu\|_H \le \kappa_2, \qquad \forall \nu, \mu \in \mathbb{N},$$

such that $b_{\mu\nu} = \langle g_\mu, f_\nu \rangle_H$ for all $\nu, \mu \in \mathbb{N}$ and $\kappa = \kappa_1 \kappa_2$.

The following corollary of Theorem 5.7 was explained to us by Gilles Pisier.

Corollary 5.8. *Let $\mathbf{b} = (b_{\mu\nu})$ be a Schur multiplier such that both limits exist*

$$\lim_{\mu\to\infty} \lim_{\nu\to\infty} b_{\mu\nu}, \qquad \lim_{\nu\to\infty} \lim_{\mu\to\infty} b_{\mu\nu}.$$

Then these two limits are equal.

Proof. By Theorem 5.7 there exist a Hilbert space H and bounded sequences $f_\nu, g_\mu \in H$ such that $b_{\mu\nu} = \langle g_\mu, f_\nu \rangle_H$ for all $\nu, \mu \in \mathbb{N}$.
By the Banach-Alaoglu Theorem bounded sequences in Hilbert space are weakly compact. Hence there exist $f, g \in H$ and subsequences f_{ν_j} and g_{μ_j} such that

$$\lim_{j\to\infty} \left\langle f_{\nu_j}, h \right\rangle = \langle f, h \rangle, \qquad \lim_{j\to\infty} \left\langle g_{\mu_j}, h \right\rangle = \langle g, h \rangle,$$

for every $h \in H$. Computing both limits

$$\lim_{\mu\to\infty} \lim_{\nu\to\infty} b_{\mu\nu} = \lim_{i\to\infty} \lim_{j\to\infty} b_{\mu_i \nu_j} = \lim_{i\to\infty} \underbrace{\lim_{j\to\infty} \left\langle g_{\mu_i}, f_{\nu_j} \right\rangle}_{\overset{h=g_{\mu_i}}{=} \left\langle g_{\mu_i}, f \right\rangle} \overset{h=f}{=} \langle g, f \rangle$$

and

$$\lim_{\nu\to\infty}\lim_{\mu\to\infty} b_{\mu\nu} = \lim_{j\to\infty}\lim_{i\to\infty} b_{\mu_i\nu_j} = \lim_{j\to\infty}\underbrace{\lim_{i\to\infty}\left\langle g_{\mu_i}, f_{\nu_j}\right\rangle}_{\stackrel{h=f_{\nu_j}}{=}\left\langle g,f_{\nu_j}\right\rangle} \stackrel{h=g}{=} \langle g,f\rangle$$

shows that they are equal. □

Corollary 5.9. *The matrix* $\mathbf{b}=(\frac{\mu}{\mu+\nu})$ *is not a Schur multiplier.*

Proof. The limits

$$\lim_{\mu\to\infty}\lim_{\nu\to\infty}\tfrac{\mu}{\mu+\nu} = \lim_{\mu\to\infty} 0 = 0, \qquad \lim_{\nu\to\infty}\lim_{\mu\to\infty}\tfrac{\mu}{\mu+\nu} = \lim_{\nu\to\infty} 1 = 1$$

are different. So $\mathbf{b}$ it is not a Schur multiplier by Corollary 5.8. □

For the significance of Corollary 5.9 see Remark 6.18.

6 Proof of Theorem B

6.1 Spectral projections

In the present Section 6 we make the following assumptions. By $A\colon H_1\to H_0$ we denote an H_0-symmetric isomorphism whose spectrum $\operatorname{spec} A$ consists of infinitely many positive and infinitely many negative eigenvalues, of finite multiplicity each. Let the eigenvalues a_ℓ of A, where $\ell\in\mathbb{Z}^*$, be enumerated as in (3.16), so a_1 is the smallest positive eigenvalue of A and a_{-1} is the largest negative one. Let $\sigma=\sigma(A):=\min\{a_1,-a_{-1}\}>0$ be the **spectral gap** of A.

Furthermore, in Section 6.1 – to avoid annoying constants – we endow the space H_1 with the (since A is an isomorphism) equivalent inner product

$$\langle\cdot,\cdot\rangle_A := \langle A\cdot, A\cdot\rangle_0 \sim \langle\cdot,\cdot\rangle_1 . \tag{6.24}$$

Because the inclusion $H_1\hookrightarrow H_0$ is compact, the resolvent $R\colon H_0\to H_0$ of A is compact and symmetric. Hence there exists an orthonormal basis of H_0 composed of eigenvectors of $A\colon H_1\to H_0$, in symbols

$$\mathcal{V}=\mathcal{V}(A)=(v_\ell)_{\ell\in\mathbb{Z}^*}\text{ ONB of }H_0,\quad v_\ell\in H_1,\quad Av_\ell=a_\ell v_\ell. \tag{6.25}$$

Since $|v_\ell|_1^2=\langle v_\ell,v_\ell\rangle_1\stackrel{(6.24)}{=}\langle Av_\ell,Av_\ell\rangle_0=a_\ell^2\langle v_\ell,v_\ell\rangle_0=a_\ell^2$ we obtain

$$|v_\ell|_1=|a_\ell|=\begin{cases}a_\ell & \text{, if } \ell>0,\\ -a_\ell & \text{, if } \ell<0.\end{cases}$$

Definition 6.1 ($\gamma_\pm$)**.** Let $\gamma_+\colon[0,1]\to\mathbb{C}\setminus(\operatorname{spec}A^{-1})$ be a continuous loop such that the winding number of γ_+ with respect to $1/a_\ell$ is 1 and with respect to

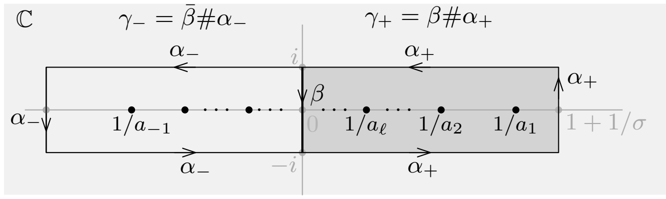

Figure 1: Loops $\gamma_\pm$ enclosing the positive/negative spectrum $(1/a_{\pm\nu})_{\nu\in\mathbb{N}}$ of A^{-1}

$1/a_{-\ell}$ is 0 for all $\ell \in \mathbb{N}$. Since the positive eigenvalues $1/a_\ell$, as well as the negative eigenvalues $1/a_{-\ell}$, of A^{-1} accumulate at the origin, the path γ_+ has to run through the origin. For definiteness and later use we choose the concatenation $\gamma_+ = \beta\#\alpha_+$ as indicated in Figure 1 (which also shows an analogously defined path $\gamma_- = \bar{\beta}\#\alpha_-$ where $\bar{\beta}$ denotes β traversed in the opposite direction).

The following map selects for each A the projection $\Pi^A_\pm \in \mathcal{L}(H_0)$ to the positive/negative eigenspaces of A, namely

$$\Pi^A_\pm := \Pi_\pm(A) = \frac{i}{2\pi}\int_{\gamma_\pm} \underbrace{A\,(\mathrm{Id}-\zeta A)^{-1}}_{=:K_\zeta(A)} d\zeta = \Pi_\pm(A^{-1}). \tag{6.26}$$

This is formula [Kat95, (5.22) for the resolvent of A^{-1} instead of A] where we use the identity in the footnote[12] and the fact that the positive eigenspaces of A and A^{-1} are equal.

Remark 6.2. Let $\mathcal{L}_{sym_0}(H_1, H_0) \subset \mathcal{L}(H_1, H_0)$ be the subspace of H_0-symmetric elements, and $\mathcal{L}^*_{sym_0} \subset \mathcal{L}_{sym_0}$ the open subset of isomorphisms.

a) For $A \in \mathcal{L}^*_{sym_0}(H_1, H_0)$ the map $\Pi_\pm$ takes values in $\mathcal{L}(H_0) \cap \mathcal{L}(H_1)$. Indeed $\Pi_\pm(A)$ is the projection in H_0 to the positive eigenspaces of A and each of them lies in H_1, so $\Pi_\pm(A)$ is also a projection in H_1.

b) The intersection $\mathcal{L}(H_0) \cap \mathcal{L}(H_1)$ is a subset of $\mathcal{L}(H_{\frac{1}{2}})$, by Appendix B.

c) The intersection $\mathcal{L}(H_0) \cap \mathcal{L}(H_1)$ is a subset of $\mathcal{L}(H_1, H_0)$, by composing a map $H_1 \to H_1$ with the inclusion $\iota\colon H_1 \hookrightarrow H_0$.

So the projection selectors $\Pi_\pm$ are maps between the spaces shown in Figure 2.

There is a natural embedding $\mathcal{I}\colon \mathcal{L}(H_{\frac{1}{2}}) \hookrightarrow \mathcal{L}(H_1, H_0)$ via the compact dense inclusions $H_1 \subset H_{\frac{1}{2}} \subset H_0$. More precisely, there is the commuting diagram

$$\begin{tikzcd} H_1 \arrow[r, dashed, "\mathcal{I}(B)"] \arrow[d, hook, "\iota_1"'] & H_0 \\ H_{\frac{1}{2}} \arrow[r, "B"] & H_{\frac{1}{2}} \arrow[u, hook, "\iota_0"'] \end{tikzcd} \tag{6.27}$$

where the operator defined by $\mathcal{I}(B) := \iota_0 \circ B \circ \iota_1$ is compact.

[12] $R^{A^{-1}}(\zeta) = \left(A^{-1} - \zeta\mathrm{Id}\right)^{-1} = \left((\mathrm{Id} - \zeta A)A^{-1}\right)^{-1} = A\,(\mathrm{Id} - \zeta A)^{-1}$

$$
\begin{tikzcd}
 & & \mathcal{L}(H_{\frac{1}{2}}) \arrow[dd, "\mathcal{I}\ (6.27)"] \\
\Pi_\pm \colon \mathcal{L}^*_{sym_0}(H_1, H_0) \arrow[r, "(6.26)", "\text{Rmk. 6.2 a)}"'] \arrow[urr, dashed] \arrow[drr] & \mathcal{L}(H_0) \cap \mathcal{L}(H_1) \arrow[ur, phantom, "\subset_{\text{b)}}"] \arrow[dr, phantom, "\subset^{\text{c)}}"] & \\
 & & \mathcal{L}(H_1, H_0)
\end{tikzcd}
$$

Figure 2: The spectral projection selectors $\Pi_\pm \colon A \mapsto \Pi^A_\pm$

6.2 Differentiability of the projection selectors $\Pi_\pm$

Proposition 6.3. *The maps $\Pi_\pm$ defined by (6.26), viewed as maps*

$$\Pi_\pm \colon \mathcal{L}^*_{sym_0}(H_1, H_0) \to \mathcal{L}(H_1, H_0),$$

are differentiable and the derivative is given by

$$\begin{aligned} d\Pi_\pm(A) \colon \mathcal{L}_{sym_0}(H_1, H_0) &\to \mathcal{L}(H_1, H_0) \\ \Delta &\mapsto \frac{i}{2\pi} \int_{\gamma_\pm} \underbrace{(\mathrm{Id} - \zeta A)^{-1} \Delta (\mathrm{Id} - \zeta A)^{-1}}_{=: \mathcal{D}^A_{1,\zeta}(\Delta)} d\zeta. \end{aligned} \tag{6.28}$$

That (6.28) is the derivative of $\Pi_\pm$ can formally be seen as follows. Write $B = B(A) := \mathrm{Id} - \zeta A$. Then $dB(A)\Delta = -\zeta\Delta$, so we get that $dB^{-1}(A)\Delta = -B^{-1}(dB(A)\Delta)B^{-1} = B^{-1}\zeta\Delta B^{-1}$. Clearly $AB = BA$, conjugation with B^{-1} shows that $AB^{-1} = B^{-1}A$. Put these facts together to obtain

$$\begin{aligned} d\left(AB^{-1}\right)\Delta &= \Delta B^{-1} + A\, dB^{-1}(A)\Delta \\ &= B^{-1}B\Delta B^{-1} + AB^{-1}\zeta\Delta B^{-1} \\ &= B^{-1}(\mathrm{Id}\underline{-\zeta A})\Delta B^{-1} + B^{-1}\underline{A\zeta}\Delta B^{-1} \\ &= B^{-1}\Delta B^{-1}. \end{aligned}$$

In order to make this rigorous, for $A \in \mathcal{L}^*_{sym_0}(H_1, H_0)$ and $\zeta \in \mathbb{C}$ we abbreviate

$$K_\zeta(A) := A(\mathrm{Id} - \zeta A)^{-1}.$$

For $k \in \mathbb{N}$ and $\Delta \in \mathcal{L}_{sym_0}(H_1, H_0)$ we abbreviate further

$$\mathcal{D}^A_{k,\zeta}(\Delta) := \zeta^{k-1}(\mathrm{Id} - \zeta A)^{-1} \left(\Delta (\mathrm{Id} - \zeta A)^{-1}\right)^k. \tag{6.29}$$

The map $\Delta \mapsto \mathcal{D}^A_{k,\zeta}(\Delta)$ is k-homogenous; $k = 1$ is the integrand in (6.28), i.e.

$$\mathcal{D}^A_{1,\zeta}(\Delta) = (\mathrm{Id} - \zeta A)^{-1} \Delta (\mathrm{Id} - \zeta A)^{-1}. \tag{6.30}$$

Lemma 6.4. *Let $A \in \mathcal{L}^*_{sym_0}(H_1, H_0)$. If $\Delta \in \mathcal{L}_{sym_0}(H_1, H_0)$ and $\zeta \in \mathbb{C}$ satisfy*

$$\left\|\zeta\Delta(\mathrm{Id} - \zeta A)^{-1}\right\|_{\mathcal{L}(H_0)} < 1, \tag{6.31}$$

then

$$K_\zeta(A + \Delta) - K_\zeta(A) = \sum_{k=1}^{\infty} \mathcal{D}^A_{k,\zeta}(\Delta).$$

Proof. We consider the operator difference

$$\mathrm{Id} - \zeta(A + \Delta) = \mathrm{Id} - \zeta A - \zeta\Delta = \left(\mathrm{Id} - \zeta\Delta(\mathrm{Id} - \zeta A)^{-1}\right)(\mathrm{Id} - \zeta A).$$

In view of (6.31) this difference is invertible and its inverse can be constructed with the help of the geometric series as follows

$$\begin{aligned}(\mathrm{Id} - \zeta(A + \Delta))^{-1} &= (\mathrm{Id} - \zeta A)^{-1}\left(\mathrm{Id} - \zeta\Delta(\mathrm{Id} - \zeta A)^{-1}\right)^{-1}\\ &= (\mathrm{Id} - \zeta A)^{-1}\sum_{k=0}^{\infty}\left(\zeta\Delta\,(\mathrm{Id} - \zeta A)^{-1}\right)^{k}.\end{aligned}$$

In what follows we use the definition of K_ζ and add zero to obtain equality 1, then we use the above formula to get equality 2

$$\begin{aligned}
&K_\zeta(A + \Delta) - K_\zeta(A)\\
&\overset{1}{=} (A + \Delta)\,(\mathrm{Id} - \zeta(A + \Delta))^{-1} - (A + \Delta)(\mathrm{Id} - \zeta A)^{-1} + (A + \Delta)(\mathrm{Id} - \zeta A)^{-1}\\
&\quad - A(\mathrm{Id} - \zeta A)^{-1}\\
&\overset{2}{=} (A + \Delta)(\mathrm{Id} - \zeta A)^{-1}\sum_{k=1}^{\infty}\left(\zeta\Delta\,(\mathrm{Id} - \zeta A)^{-1}\right)^{k} + \Delta(\mathrm{Id} - \zeta A)^{-1}\\
&\overset{3}{=} \sum_{k=2}^{\infty}\zeta^k(A + \Delta)(\mathrm{Id} - \zeta A)^{-1}\left(\Delta\,(\mathrm{Id} - \zeta A)^{-1}\right)^{k}\\
&\quad + (\underline{\underline{A}} + \underline{\Delta})(\mathrm{Id} - \zeta A)^{-1}\zeta\Delta\,(\mathrm{Id} - \zeta A)^{-1}\\
&\quad + (\mathrm{Id} - \zeta A)^{-1}(\mathrm{Id} - \zeta A)\,\Delta(\mathrm{Id} - \zeta A)^{-1}\\
&\overset{4}{=} \sum_{k=2}^{\infty}\zeta^{k-1}(\underbrace{\zeta A - \mathrm{Id}}_{sum_1}\underbrace{+\mathrm{Id}}_{sum_2} + \underbrace{\zeta\Delta}_{sum_3})(\mathrm{Id} - \zeta A)^{-1}\left(\Delta\,(\mathrm{Id} - \zeta A)^{-1}\right)^{k}\\
&\quad + \zeta\underline{\Delta}(\mathrm{Id} - \zeta A)^{-1}\Delta(\mathrm{Id} - \zeta A)^{-1} + (\mathrm{Id} - \zeta A)^{-1}\zeta\underline{\underline{A}}\Delta\,(\mathrm{Id} - \zeta A)^{-1}\\
&\quad + (\mathrm{Id} - \zeta A)^{-1}(\Delta - \zeta A\Delta)(\mathrm{Id} - \zeta A)^{-1}
\end{aligned}$$

$$
\begin{aligned}
&\overset{5}{=} \zeta\left(\Delta(\mathrm{Id}-\zeta A)^{-1}\right)^2 + (\mathrm{Id}-\zeta A)^{-1}\,\Delta(\mathrm{Id}-\zeta A)^{-1} \\
&\quad -\sum_{k=2}^{\infty}\zeta^{k-1}\left(\Delta\left(\mathrm{Id}-\zeta A\right)^{-1}\right)^k \left(=-\sum_{k=1}^{\infty}\zeta^{k}\left(\Delta\left(\mathrm{Id}-\zeta A\right)^{-1}\right)^{k+1}\right) \\
&\quad +\sum_{k=2}^{\infty}\zeta^{k-1}(\mathrm{Id}-\zeta A)^{-1}\left(\Delta\left(\mathrm{Id}-\zeta A\right)^{-1}\right)^k \\
&\quad +\sum_{k=2}^{\infty}\zeta^{k}\left(\Delta\left(\mathrm{Id}-\zeta A\right)^{-1}\right)^{k+1} \\
&\overset{6}{=} 0+\sum_{k=1}^{\infty}\underbrace{\zeta^{k-1}(\mathrm{Id}-\zeta A)^{-1}\left(\Delta\left(\mathrm{Id}-\zeta A\right)^{-1}\right)^k}_{=:\mathcal{D}^A_{k,\zeta}(\Delta)}.
\end{aligned}
$$

In equality 3 we write the summand for $k=1$ separately and multiply by the identity. Equality 4 uses that A commutes with $(\mathrm{Id}-\zeta A)^{-1}$. Equality 5 uses that the two terms involving $\zeta A\Delta$ cancel, moreover, we split the sum into three sums. In equality 6 three terms cancel. □

Proof of Proposition 6.3. The proof for Π_+ is in four steps.

Step 1. Condition (6.31) holds for ζ along the image of the path $\gamma_+=\beta\#\alpha_+$; see Figure 1.

Along $\boldsymbol{\alpha_+}$. Since the path α_+ is uniformly bounded away from spec A^{-1}, there is a constant $c_1>0$ with $\|(\mathrm{Id}-\zeta A)^{-1}\|_{\mathcal{L}(H_0,H_1)}\le c_1$ for any ζ on the image of the path α_+. Because the image of α_+ is compact, there is a constant $c_2>0$ such that $|\zeta|\le c_2$ for any $\zeta\in\operatorname{Im}\alpha_+$. Therefore, if $\|\Delta\|_{\mathcal{L}(H_1,H_0)}<1/c_1c_2$, then condition (6.31) is satisfied for any $\zeta\in\operatorname{Im}\alpha_+$.

Along $\boldsymbol{\beta}$. In this case $\zeta=it$ for $t\in[-1,1]$. Then there exists a constant $c_3>0$ such that $\|(\mathrm{Id}-itA)^{-1}\|_{\mathcal{L}(H_0,H_1)}\le c_3/|t|$. If $\|\Delta\|_{\mathcal{L}(H_1,H_0)}<1/c_3$, then

$$
\|it\Delta(\mathrm{Id}-itA)^{-1}\|_{\mathcal{L}(H_0)}\le |t|\|\Delta\|_{\mathcal{L}(H_1,H_0)}\|(\mathrm{Id}-itA)^{-1}\|_{\mathcal{L}(H_0,H_1)}<1.
$$

So, from now on we suppose that $\|\Delta\|_{\mathcal{L}(H_1,H_0)}<\min\{\frac{1}{c_1c_2},\frac{1}{c_3}\}$. By the above this guarantees that (6.31) is satisfied for any ζ in the image of the path γ_+.

By definition (6.26) of $\Pi_\pm(A+\Delta)$ and $\Pi_\pm(A)$, together with formula (6.28) for the candidate for the derivative $d\Pi_\pm(A)\Delta$, we obtain the first equality in

$$
\begin{aligned}
&\Pi_\pm(A+\Delta)-\Pi_\pm(A)-\tfrac{i}{2\pi}\int_{\gamma_\pm}\mathcal{D}^A_{1,\zeta}(\Delta)\,d\zeta \\
&=\frac{i}{2\pi}\int_{\gamma_\pm}K_\zeta(A+\Delta)-K_\zeta(A)-\mathcal{D}^A_{1,\zeta}(\Delta)\,d\zeta \\
&\overset{2}{=}\frac{i}{2\pi}\int_{\gamma_\pm}\sum_{k=2}^{\infty}\mathcal{D}^A_{k,\zeta}(\Delta)\,d\zeta
\end{aligned}
\tag{6.32}
$$

where equality 2 is by Lemma 6.4.

STEP 2. We investigate the norm of the terms $\mathcal{D}^A_{k,\zeta}(\Delta)$ for $k \geq 2$ and for ζ along the path $\gamma_+ = \beta \# \alpha_+$.

Along $\boldsymbol{\alpha_+}$. By the continuous inclusion $H_1 \hookrightarrow H_0$ and by definition (6.29) of $\mathcal{D}^A_{k,\zeta}$ as a composition and c_1, c_2 in Step 1 we get, respectively, inequalities one and two

$$\|\mathcal{D}^A_{k,\zeta}(\Delta)\|_{\mathcal{L}(H_1,H_0)} \leq \|\mathcal{D}^A_{k,\zeta}(\Delta)\|_{\mathcal{L}(H_0,H_1)} \leq c_2^{k-1} c_1^{k+1} \|\Delta\|^k_{\mathcal{L}(H_1,H_0)}.$$

Along $\boldsymbol{\beta}$. There is a constant $c_4 > 0$ such that

$$\|(\mathrm{Id} - itA)^{-1}\|_{\mathcal{L}(H_0)} \leq c_4, \qquad \|(\mathrm{Id} - itA)^{-1}\|_{\mathcal{L}(H_1)} \leq c_4,$$

whenever $t \in [-1, 1]$. Therefore, for $\zeta = it$ and using the fact that (6.29) is a composition of operators $(\mathrm{Id} - itA)^{-1}\Delta \colon H_1 \to H_0 \to H_1$, we get

$$\|\mathcal{D}^A_{k,\zeta}(\Delta)\|_{\mathcal{L}(H_1,H_0)} \leq |t|^{k-1} c_4^2 \left(\tfrac{c_3}{|t|}\right)^{k-1} \|\Delta\|^k_{\mathcal{L}(H_1,H_0)} = c_4^2 c_3^{k-1} \|\Delta\|^k_{\mathcal{L}(H_1,H_0)}.$$

STEP 3. Fix $\varepsilon \in (0, 1)$. Let $C := \max\{1, c_1c_2, c_1^2c_2, c_3, c_3c_4\}$. If $\|\Delta\|_{\mathcal{L}(H_1,H_0)} = \frac{\varepsilon}{2C}$, then

$$\|\mathcal{D}^A_{k,\zeta}(\Delta)\|_{\mathcal{L}(H_1,H_0)} \leq \tfrac{\varepsilon^k}{2^k} \tag{6.33}$$

for any ζ along the path γ_+. This is true by Step 2.

STEP 4. We prove the proposition. By (6.32) we get equality 1 in what follows

$$\begin{aligned}
&\left\| \Pi_\pm(A + \Delta) - \Pi_\pm(A) - \tfrac{i}{2\pi} \int_{\gamma_\pm} \mathcal{D}^A_{1,\zeta}(\Delta)\, d\zeta \right\|_{\mathcal{L}(H_1,H_0)} \\
&= \left\| \tfrac{1}{2\pi} \sum_{k=2}^{\infty} \int_{\gamma_\pm} \mathcal{D}^A_{k,\zeta}(\Delta)\, d\zeta \right\|_{\mathcal{L}(H_1,H_0)} \\
&\overset{1}{\leq} \tfrac{1}{2\pi} \sum_{k=2}^{\infty} \int_{\gamma_\pm} \|\mathcal{D}^A_{k,\zeta}(\Delta)\|_{\mathcal{L}(H_1,H_0)}\, d|\zeta| \\
&\overset{2}{\leq} \tfrac{1}{2\pi} \sum_{k=2}^{\infty} \int_{\gamma_\pm} \left(\tfrac{\varepsilon}{2}\right)^k d|\zeta| \\
&= \tfrac{\mathrm{length}(\gamma_+)}{2\pi} \sum_{k=2}^{\infty} \left(\tfrac{\varepsilon}{2}\right)^k \\
&= \tfrac{\mathrm{length}(\gamma_+)}{2\pi} \left(\tfrac{\varepsilon}{2}\right)^2 \frac{1}{1 - \frac{\varepsilon}{2}} \\
&\leq \tfrac{\mathrm{length}(\gamma_+)}{4\pi} \varepsilon^2 \\
&= \tfrac{\mathrm{length}(\gamma_+)}{\pi} C^2 \|\Delta\|^2_{\mathcal{L}(H_1,H_0)}.
\end{aligned}$$

Here inequality 2 is by estimate (6.33). The final equality holds by Step 3.

This proves Proposition 6.3 for Π_+. The proof for Π_- is analogous. □

6.3 Directional derivative of $\Pi_\pm$ extends from H_1 to $H_{\frac{1}{2}}$

By Proposition 6.3 the derivative of $\Pi_\pm$ in (6.26) and Figure 2, viewed as map

$$\Pi_\pm\colon \mathcal{L}^*_{sym_0}(H_1, H_0) \to \mathcal{L}(H_1, H_0),$$

exists at $A \in \mathcal{L}^*_{sym_0}$ in direction $\Delta \in \mathcal{L}_{sym_0}(H_1, H_0)$ and is a bounded linear map $S_\pm := d\Pi_\pm(A)\Delta\colon H_1 \to H_0$. The theorem below shows, in particular, that $S_\pm$ extends to a bounded linear map $\tilde{S}_\pm$ on $H_{\frac{1}{2}}$, more precisely the diagram

$$\begin{tikzcd}
H_{\frac{1}{2}} \arrow[r, dashed, "\tilde{S}_\pm"] & H_{\frac{1}{2}} \arrow[d, hook, "\iota_0"] \\
H_1 \arrow[u, hook, "\iota_1"] \arrow[r, "S_\pm = d\Pi_\pm(A)\Delta"] & H_0
\end{tikzcd}$$

commutes, in symbols $S_\pm = \iota_0 \circ \tilde{S}_\pm \circ \iota_1$.

Theorem 6.5. *Let $A \in \mathcal{L}^*_{sym_0}(H_1, H_0)$. Then the derivative $d\Pi_\pm(A)$ from Proposition 6.3 factorizes through $\mathcal{L}(H_{\frac{1}{2}})$ via a map $T^\pm_A$, namely*

$$\begin{tikzcd}
 & \mathcal{L}(H_{\frac{1}{2}}) \arrow[d, hook, "\mathcal{I}\text{ cf. (6.27)}"] \\
\mathcal{L}_{sym_0}(H_1, H_0) \arrow[ur, dashed, "T^\pm_A"] \arrow[r, "d\Pi_\pm(A)"] & \mathcal{L}(H_1, H_0)
\end{tikzcd}$$

still given by formula (6.28), namely

$$T^\pm_A \Delta = d\Pi_\pm(A)\Delta := \frac{i}{2\pi} \int_{\gamma_\pm} (\mathrm{Id} - \zeta A)^{-1} \Delta (\mathrm{Id} - \zeta A)^{-1} \, d\zeta. \tag{6.34}$$

Moreover, there is the bound

$$\left\| T^\pm_A \right\|_{\mathcal{L}(\mathcal{L}_{sym_0}(H_1,H_0), \mathcal{L}(H_{\frac{1}{2}}))} \le \pi + \frac{4\sigma + 2}{\pi\sigma^2}.$$

As preparation for the proof of Theorem 6.5 we use additivity of the integral along the domain of integration $\gamma_\pm = \beta \# \alpha_\pm$ (Figure 1) to decompose $d\Pi_\pm(A)$ into the sum of operators

$$d\Pi_+(A) = T^\beta_A + T^{\alpha_+}_A, \qquad d\Pi_-(A) = T^{\bar{\beta}}_A + T^{\alpha_-}_A = -T^\beta_A + T^{\alpha_-}_A, \tag{6.35}$$

where the operators

$$T^*_A\colon \mathcal{L}_{sym_0}(H_1, H_0) \to \mathcal{L}(H_1, H_0)$$

are for $* \in \{\beta, \alpha_+, \alpha_-\}$ defined by

$$T^*_A \Delta := \frac{i}{2\pi} \int_* (\mathrm{Id} - \zeta A)^{-1} \Delta (\mathrm{Id} - \zeta A)^{-1} \, d\zeta. \tag{6.36}$$

We show that all these three operators factorize through $\mathcal{L}(H_{\frac{1}{2}})$ and by abuse of notation we keep on using the same notation.

6.3.1 Block decomposing $d\Pi^A_\pm$ via positive/negative subspaces

Suppose that $A \in \mathcal{L}^*_{sym_0}(H_1, H_0)$ and pick an ONB $\mathcal{V} = \mathcal{V}(A)$ of H_0 as in (6.25) consisting of eigenvectors $v_\ell \in H_1$ of A where $\ell \in \mathbb{Z}^*$ associated to eigenvalues $a_\nu > 0$ for and $a_{-\nu} < 0$ for $\nu \in \mathbb{N}$. For $\ell \in \mathbb{Z}^*$ let

$$E_\ell = \ker(a_\ell \,\mathrm{Id} - A) \subset H_1 \subset H_0$$

be the eigenspace for the eigenvalue a_ℓ. For $r \in \mathbb{R}$ the **positive/negative subspaces** of H_r, see (2.8), are defined by taking the closure in the H_r norm as follows

$$H_r^+ := \overline{\bigoplus_{\nu\in\mathbb{N}} E_\nu}^{H_r}, \qquad H_r^- := \overline{\bigoplus_{\nu\in\mathbb{N}} E_{-\nu}}^{H_r}.$$

Thus there are the H_r orthogonal decompositions

$$H_r = H_r^+ \oplus H_r^-, \quad r \in \mathbb{R}.$$

The maps defined by (6.26) are projections $\Pi^A_\pm = (\Pi^A_\pm)^2$ in H_i for $i = 0, 1$, same notation. These maps satisfy $\Pi^A_+ + \Pi^A_- = \mathrm{Id}$ in H_i and

$$H_i^+ = \ker \Pi^A_- = \mathrm{Im}\,\Pi^A_+ = \mathrm{Fix}\,\Pi^A_+, \qquad H_i^- = \ker \Pi^A_+ = \mathrm{Im}\,\Pi^A_- = \mathrm{Fix}\,\Pi^A_-.$$

For $\flat, \sharp \in \{-, +\}$ we abbreviate

$$\begin{aligned} (d\Pi^A_\pm)^{\flat\sharp} \colon \mathcal{L}_{sym_0}(H_1, H_0) &\to \mathcal{L}_{sym_0}(H_1^\flat, H_0^\sharp) \\ \Delta &\mapsto \Pi^A_\sharp \circ d\Pi_\pm(A)\Delta \circ \Pi^A_\flat. \end{aligned}$$

Lemma 6.6. *For $A \in \mathcal{L}^*_{sym_0}(H_1, H_0)$ and $\Delta \in \mathcal{L}_{sym_0}(H_1, H_0)$ the composition*

$$0\colon H_1 \xrightarrow{\Pi^A_\pm} H_1 \xrightarrow{d\Pi_\pm(A)\Delta} H_0 \xrightarrow{\Pi^A_\pm} H_0$$

vanishes, in symbols $\Pi^A_\pm \circ d\Pi_\pm(A)\Delta \circ \Pi^A_\pm = 0$.

Proof. Since $p = p(A) := \Pi^A_\pm \in \mathcal{L}(H_0) \cap \mathcal{L}(H_1)$ is a projection the identity $p^2 = p$ holds. Differentiating this identity we get $p \circ dp + dp \circ p = dp$. Multiplying this identity from the right by p yields $p \circ dp \circ p + dp \circ p^2 = dp \circ p$ which, since $p^2 = p$, is equivalent to $p \circ dp \circ p = 0$. □

Corollary 6.7. *For any $A \in \mathcal{L}^*_{sym_0}(H_1, H_0)$ the diagonal blocks vanish*

$$(d\Pi^A_\pm)^{++} = 0, \qquad (d\Pi^A_\pm)^{--} = 0.$$

Proof. Since $\Pi^A_+ + \Pi^A_- = \mathrm{Id}$ it follows that $d\Pi_+(A) = -d\Pi_-(A)$. Hence the corollary follows from Lemma 6.6. □

For $A \in \mathcal{L}^*_{sym_0}(H_1, H_0)$ and subpaths $* \in \{\beta, \alpha_+, \alpha_-\}$ and block indices $\flat, \sharp \in \{-, +\}$ we abbreviate

$$\begin{aligned} (T^*_A)^{\flat\sharp} \colon \mathcal{L}_{sym_0}(H_1, H_0) &\to \mathcal{L}_{sym_0}(H_1^\flat, H_0^\sharp) \\ \Delta &\mapsto \Pi^A_\sharp \circ T^*_A \Delta \circ \Pi^A_\flat. \end{aligned} \tag{6.37}$$

Corollary 6.8. *For any $A \in \mathcal{L}^*_{sym_0}(H_1, H_0)$ there are the block identities*

$$(T^\beta_A)^{++} = -(T^{\alpha+}_A)^{++}, \qquad (T^\beta_A)^{--} = -(T^{\alpha+}_A)^{--}.$$

Proof. Corollary 6.7 for $d\Pi^A_+$ and the first identity in (6.35). □

Lemma 6.9. *Let H be any Hilbert space with an orthogonal decomposition $H = H^+ \oplus H^-$. Any operator $B \in \mathcal{L}(H)$ block decomposes into four operators*

$$B = \begin{pmatrix} B^{++} & B^{-+} \\ B^{+-} & B^{--} \end{pmatrix} : H^+ \oplus H^- \to H^+ \oplus H^- \tag{6.38}$$

and the operator norms are related by

$$\|B\| \le \|B^{++}\| + \|B^{-+}\| + \|B^{+-}\| + \|B^{--}\|.$$

Proof. Due to the decomposition $H = H^+ \oplus H^-$ any $v \in H$ is a sum $v = v_+ + v_-$ for unique elements $v_\pm \in H^\pm$. By orthogonality identity one (Pythagoras) holds

$$\begin{aligned} |Bv|^2 &= |(Bv)_+|^2 + |(Bv)_-|^2 \\ &\le \big(\underbrace{|B^{++}v_+| + |B^{-+}v_-|}_{a}\big)^2 + \big(\underbrace{|B^{+-}v_+| + |B^{--}v_-|}_{b}\big)^2 \le (a+b)^2 \\ &\le \big(|B^{++}v_+| + |B^{-+}v_-| + |B^{+-}v_+| + |B^{--}v_-|\big)^2. \end{aligned}$$

Here inequality one holds since $(Bv)_+ = B^{++}v_+ + B^{-+}v_-$, by (6.38), together with the triangle inequality. By definition of the operator norm and the previous estimate we get

$$\begin{aligned} \|B\| :&= \sup_{|v|=1} |Bv| \\ &\le \sup_{\substack{1=|v| \\ =|v_+|+|v_-|}} \big(|B^{++}v_+| + |B^{-+}v_-| + |B^{+-}v_+| + |B^{--}v_-|\big) \\ &\le \|B^{++}\| + \|B^{-+}\| + \|B^{+-}\| + \|B^{--}\|. \end{aligned}$$

where we used that $|B^{++}v_+| \le \|B^{++}\| \cdot 1$ and so on. This proves Lemma 6.9. □

The operator $T^{\alpha\pm}_A$

6.3.2 Estimating the operator $T^{\alpha\pm}_A$

Proposition 6.10. $\|T^{\alpha\pm}_A\|_{\mathcal{L}(\mathcal{L}_{sym_0}(H_1,H_0),\mathcal{L}(H_{\frac{1}{2}}))} \le \frac{2\sigma+1}{\pi\sigma^2}$.

Lemma 6.11. *For any $\zeta \in \operatorname{im} \alpha_\pm$ it holds* $\|(\mathrm{Id} - \zeta A)^{-1}\|_{\mathcal{L}(H_0,H_1)} \le 1$.

Proof. The proof is in two steps.

Step 1. It holds $\|(\mathrm{Id} - \zeta A)^{-1}\|_{\mathcal{L}(H_0,H_1)} = \max_{\ell\in\mathbb{Z}^*} \frac{|a_\ell|}{|1-\zeta a_\ell|}$.

Proof. As $|(\mathrm{Id}-\zeta A)^{-1}x|_{H_1} = |A(\mathrm{Id}-\zeta A)^{-1}x|_{H_0}$, by (6.24), we get equality 2

$$\begin{aligned}\|(\mathrm{Id}-\zeta A)^{-1}\|_{\mathcal{L}(H_0,H_1)} &= \sup_{\|x\|_0=1} |(\mathrm{Id}-\zeta A)^{-1}x|_{H_1} \\ &\stackrel{2}{=} \sup_{\|x\|_0=1} |A(\mathrm{Id}-\zeta A)^{-1}x|_{H_0} \\ &= \|A(\mathrm{Id}-\zeta A)^{-1}\|_{\mathcal{L}(H_0)} \\ &= \max_{\ell\in\mathbb{Z}^*} \frac{|a_\ell|}{|1-\zeta a_\ell|}\end{aligned}$$

where the last equality uses that the orthonormal basis $\mathcal{V}$ of H_0, see (6.25), consists of eigenvectors v_ℓ such that operator A with eigenvalues a_ℓ and therefore $A(\mathrm{Id}-\zeta A)^{-1}v_\ell = a_\ell(1-\zeta a_\ell)^{-1}v_\ell$.

Step 2. For all $\ell \in \mathbb{Z}^*$ and $\zeta \in \operatorname{im}\alpha_\pm$ it holds $|a_\ell/(1-\zeta a_\ell)| \le 1$.

Proof. We distinguish four cases (the four edges of the rectangle in Figure 1).

Case 1. Let $\zeta = i + t$ where $t \in [-1-1/\sigma, 1+1/\sigma]$. We estimate for the denominator $|1-\zeta a_\ell| = |1 - ta_\ell - ia_\ell| \ge |a_\ell|$. Hence $|a_\ell/(1-\zeta a_\ell)| \le 1$.

Case 2. Let $\zeta = 1+1/\sigma+it$ where $t \in [-1-1]$. We estimate for the denominator $|1-\zeta a_\ell| = |1 - a_\ell - a_\ell/\sigma - ita_\ell| \ge |1 - a_\ell - a_\ell/\sigma|$.
Case $\ell > 0$. Then $a_\ell \ge a_1 > 0$, hence $a_\ell/\sigma \ge a_\ell/a_1 \ge 1$. Thus $|1-a_\ell-a_\ell/\sigma| = |a_\ell + a_\ell/\sigma - 1| \ge |a_\ell|$. Hence $|a_\ell/(1-\zeta a_\ell)| \le 1$.
Case $\ell < 0$. Then $1-a_\ell-a_\ell/\sigma = 1+|a_\ell|+|a_\ell|/\sigma \ge |a_\ell|$. Hence $|a_\ell/(1-\zeta a_\ell)| \le 1$.

Case 3. Let $\zeta = -1-1/\sigma+it$ where $t \in [-1-1]$. Analogue to Case 2.

Case 4. Let $\zeta = -i+t$ where $t \in [-1-1/\sigma, 1+1/\sigma]$. Analogue to Case 1. □

Corollary 6.12. *For all $\Delta \in \mathcal{L}_{sym_0}(H_1,H_0)$ and $\zeta \in \operatorname{im}\alpha_\pm$ it holds*

$$\|(\mathrm{Id}-\zeta A)^{-1}\Delta(\mathrm{Id}-\zeta A)^{-1}\|_{\mathcal{L}(H_i)} \le \tfrac{1}{\sigma}\|\Delta\|_{\mathcal{L}(H_1,H_0)} \tag{6.39}$$

whenever $i \in \{0, \frac{1}{2}, 1\}$.

Proof. The operator norm of the inclusion $\iota\colon H_1 \hookrightarrow H_0$ is bounded by

$$\|\iota\|_{\mathcal{L}(H_1,H_0)} = \sup_{|x|_1=1} |x|_0 = \tfrac{1}{\sigma}\sup_{|x|_1=1} \sigma\,|x|_0 \le \tfrac{1}{\sigma}$$

where we used that $\sigma\,|x|_0 \le |Ax|_0 = |x|_1 = 1$. In case $i=0$ we get

$$\begin{aligned}&\|\iota(\mathrm{Id}-\zeta A)^{-1}\Delta(\mathrm{Id}-\zeta A)^{-1}\|_{\mathcal{L}(H_0)} \\ &\le \|\iota\|_{\mathcal{L}(H_1,H_0)}\|(\mathrm{Id}-\zeta A)^{-1}\|_{\mathcal{L}(H_0,H_1)}\|\Delta\|_{\mathcal{L}(H_1,H_0)}\|(\mathrm{Id}-\zeta A)^{-1}\|_{\mathcal{L}(H_0,H_1)} \\ &\le \tfrac{1}{\sigma}\|\Delta\|_{\mathcal{L}(H_1,H_0)}\end{aligned}$$

and in case $i=1$ we get

$$\begin{aligned}&\|(\mathrm{Id}-\zeta A)^{-1}\Delta(\mathrm{Id}-\zeta A)^{-1}\iota\|_{\mathcal{L}(H_1)} \\ &\le \|(\mathrm{Id}-\zeta A)^{-1}\|_{\mathcal{L}(H_0,H_1)}\|\Delta\|_{\mathcal{L}(H_1,H_0)}\|(\mathrm{Id}-\zeta A)^{-1}\|_{\mathcal{L}(H_0,H_1)}\|\iota\|_{\mathcal{L}(H_1,H_0)} \\ &\le \tfrac{1}{\sigma}\|\Delta\|_{\mathcal{L}(H_1,H_0)}.\end{aligned}$$

The case $i=\frac{1}{2}$ follows from $i=0$ and $i=1$ by interpolation (Prop. B.1). □

Proof of Proposition 6.10. Pick $\Delta \in \mathcal{L}_{sym_0}(H_1, H_0)$ and estimate

$$\begin{aligned}\|T_A^{\alpha_\pm}\Delta\|_{\mathcal{L}(H_{\frac{1}{2}})} &\overset{(6.36)}{\leq} \frac{1}{2\pi}\int_{\alpha_\pm} \|(\mathrm{Id}-\zeta A)^{-1}\Delta(\mathrm{Id}-\zeta A)^{-1}\|_{\mathcal{L}(H_{\frac{1}{2}})} d|\zeta| \\ &\overset{(6.39)}{\leq} \frac{1}{2\pi\sigma}\cdot\|\Delta\|_{\mathcal{L}(H_1,H_0)}\cdot\underbrace{\mathrm{length}(\alpha_\pm)}_{2+2(1+1/\sigma)}\end{aligned}$$

where the length calculation is illustrated by Figure 1. □

The operator T_A^{β}

6.3.3 Matrix representation and Hadamard product

If $\Delta \in \mathcal{L}_{sym_0}(H_1, H_0)$ we identify Δ with its matrix representation

$$\mathbf{d} := [\Delta]_{\mathcal{V}} \in \mathbb{R}^{\mathbb{Z}^*\times\mathbb{Z}^*} = \mathrm{Map}(\mathbb{Z}^*\times\mathbb{Z}^*, \mathbb{R})$$

whose entries are given by

$$d_{n\ell} := \langle v_n, \Delta v_\ell\rangle_0 \tag{6.40}$$

where $\mathcal{V} = (v_\ell)_{\ell\in\mathbb{Z}^*}$ is the ONB of H_0 composed of eigenvectors of A; see (6.25). Hence

$$\Delta v_\ell = \sum_{n\in\mathbb{Z}^*} d_{n\ell} v_n. \tag{6.41}$$

Definition 6.13. Let $\mathbf{b}$ be the (infinite) matrix whose entries are defined by

$$b_{\ell m} := \begin{cases} \frac{\arctan a_m - \arctan a_\ell}{a_m - a_\ell} & \text{, if } \ell \neq m, \\ \frac{1}{a_\ell^2+1} & \text{, if } \ell = m\ . \end{cases} \tag{6.42}$$

Then the **Hadamard**, i.e. entry-wise, **product** of the matrizes $\mathbf{b}$ and $\mathbf{d}$ is the matrix $\mathbf{c}$ defined by

$$\mathbf{c} = \mathbf{b}\odot\mathbf{d}, \qquad c_{\ell m} := b_{\ell m} d_{\ell m}, \text{ for } \ell, m \in \mathbb{Z}^*.$$

6.3.4 Matrix block decomposition

Since the spectrum of A has a positive and a negative part, and since we use the convention $\ell^2_{\mathbb{Z}^*} = \ell^2_{\mathbb{N}} \oplus \ell^2_{-\mathbb{N}}$, our matrices decompose into four blocks

$$\begin{aligned}\mathbf{d} = (d_{mn})_{m,n\in\mathbb{Z}^*} &= \begin{pmatrix} (d_{\mu\nu}) & (d_{\mu(-\nu)}) \\ (d_{(-\mu)\nu}) & (d_{(-\mu)(-\nu)}) \end{pmatrix} \\ &= \begin{pmatrix} (d^{++}_{\mu\nu}) & (d^{-+}_{\mu\nu}) \\ (d^{+-}_{\mu\nu}) & (d^{--}_{\mu\nu}) \end{pmatrix} \in \mathcal{L}(\ell^2_{+\mathbb{N}} \oplus \ell^2_{-\mathbb{N}})\end{aligned}$$

where $d^{++} = (d^{++}_{\mu\nu})$ abbreviates $(d^{++}_{\mu\nu})_{\mu,\nu\in\mathbb{N}}$ and so on. Analogously for

$$\mathbf{b} = (b_{mn})_{m,n\in\mathbb{Z}^*} = \begin{pmatrix} \mathbf{b}^{++} & \mathbf{b}^{-+} \\ \mathbf{b}^{+-} & \mathbf{b}^{--} \end{pmatrix} : \ell^2_{+\mathbb{N}} \oplus \ell^2_{-\mathbb{N}} \to \ell^2_{+\mathbb{N}} \oplus \ell^2_{-\mathbb{N}}.$$

Being entry-wise, the **Hadamard multiplication operator** defined by

$$H_{\mathbf{b}} \colon \mathcal{L}_{sym_0}(H_1, H_0) \to \mathcal{L}_{sym_0}(H_0), \quad \mathbf{d} \mapsto \mathbf{b} \odot \mathbf{d} =: \mathbf{c}$$

respects the four block decomposition $++$, $+-$, $-+$, $--$.

6.3.5 Representing T_A^{β} as Hadamard multiplication operator $H_{\mathbf{b}}$

Proposition 6.14. *Applying the operator T_A^β to $\Delta \in \mathcal{L}_{sym_0}(H_1, H_0)$, in symbols $\Delta \mapsto T_A^\beta \Delta$, corresponds to Hadamard multiplying $\mathbf{d}$ by $\mathbf{b}$, in symbols $\mathbf{d} \mapsto \mathbf{b} \odot \mathbf{d}$.*

Proof. Pick elements v_ℓ, v_m of the orthonormal basis $\mathcal{V} = \{v_\ell\}_{\ell \in \mathbb{Z}^*} \subset H_1$ of H_0. Recall that $Av_\ell = a_\ell v_\ell$ for $\ell \in \mathbb{Z}^*$ where $a_\ell \in \mathbb{R} \setminus \{0\}$. For $\ell \in \mathbb{Z}^*$ we have

$$(\mathrm{Id} - \zeta A)^{-1} v_\ell = \frac{1}{1 - \zeta a_\ell} v_\ell.$$

As in (6.40) we denote the matrix elements of Δ by $d_{n\ell} := \langle v_n, \Delta v_\ell \rangle_0$ for $\ell, m \in \mathbb{Z}^*$. Hence $\Delta v_\ell = \sum_{n \in \mathbb{Z}^*} d_{n\ell} v_n$. By definition of $\mathcal{D}^a_{1,\zeta}(\Delta)$, by symmetry of A, and since the inner product now being complex (say complex anti-linear in the second variable), we get equality one

$$\begin{aligned}
c_{\ell m} &:= \left\langle \left(\frac{i}{2\pi} \int_{\gamma_1} \mathcal{D}^A_{1,\zeta}(\Delta)\, d\zeta \right) v_\ell, v_m \right\rangle_0 \\
&\overset{1}{=} \frac{i}{2\pi} \int_{\gamma_1} \left\langle \Delta\, (\mathrm{Id} - \zeta A)^{-1} v_\ell, \left(\mathrm{Id} - \bar{\zeta} A\right)^{-1} v_m \right\rangle_0 d\zeta \\
&= \frac{i \langle \Delta v_\ell, v_m \rangle_0}{2\pi} \int_{\gamma_1} \frac{1}{(1 - \zeta a_\ell)(1 - \zeta a_m)} d\zeta \\
&\overset{3}{=} \frac{i d_{\ell m}}{2\pi} \int_{-1}^{1} \frac{1}{(1 + ita_\ell)(1 + ita_m)} (-i\, dt) \\
&\overset{4}{=} \frac{d_{\ell m}}{2\pi} \int_{-1}^{1} \frac{(1 - ita_\ell)(1 - ita_m)}{(1 + t^2 a_\ell^2)(1 + t^2 a_m^2)} dt \\
&= \frac{d_{\ell m}}{2\pi} \int_{-1}^{1} \frac{1 - it(a_\ell + a_m) - t^2 a_\ell a_m}{(1 + t^2 a_\ell^2)(1 + t^2 a_m^2)} dt \\
&\overset{6}{=} \frac{d_{\ell m}}{2\pi} \int_{-1}^{1} \frac{1 - t^2 a_\ell a_m}{(1 + t^2 a_\ell^2)(1 + t^2 a_m^2)} dt \\
&= \frac{d_{\ell m}}{\pi} \begin{cases} \frac{\arctan(a_m) - \arctan(a_\ell)}{a_m - a_\ell} & \text{, if } \ell \neq m, \\ \arctan'(a_\ell) = \frac{1}{a_\ell^2 + 1} & \text{, if } \ell = m\ . \end{cases}
\end{aligned}$$

In equality 3 we parametrize the path γ_1 by $[-1, 1] \ni t \mapsto -it \in [i, -i] \subset \mathbb{C}$. In equality 4 we expand the fraction by the complex conjugate of the denominator. In equality 6 the imaginary part vanishes during integration by symmetry. Finally integration was carried out by integralrechner.de. ∎

6.3.6 Estimating the off-diagonal block $(T_A^\beta)^{-+}$ alias $H_{\mathbf{b}^{-+}}$

The main result of this subsection is

Proposition 6.15. *Hadamard multiplication*

$$H_{\mathbf{b}^{-+}}\colon \mathcal{L}(H_1^-,H_0^+)\to\mathcal{L}(H_{\frac{1}{2}}^-,H_{\frac{1}{2}}^+),\quad \mathbf{d}^{-+}\mapsto\mathbf{b}^{-+}\odot\mathbf{d}^{-+}$$

is a continuous linear map with bound $\|H_{\mathbf{b}^{-+}}\|_{\mathcal{L}(\mathcal{L}(H_1^-,H_0^+),\mathcal{L}(H_{\frac{1}{2}}^-),H_{\frac{1}{2}}^+))}\le\pi/2$. *The same estimate holds true for the operator* $H_{\mathbf{b}^{+-}}$.

Note that the constant in Proposition 6.15 is universal and does not depend on the growth types f_A^+ and f_A^- of the positive and negative eigenvalues of A.

We prove Proposition 6.15 for $H_{\mathbf{b}^{-+}}$, the case $H_{\mathbf{b}^{+-}}$ being analogous by interchanging the roles of a_μ and b_ν: Recall that for $\mu\in\mathbb{N}$ we have $a_\mu>0$. We define $b_\nu:=-a_{-\nu}>0$ for $\nu\in\mathbb{N}$. We further define

$$b_{\mu\nu}^{-+}:=b_{\mu(-\nu)}=\frac{\arctan a_\mu+\arctan b_\nu}{a_\mu+b_\nu}\tag{6.43}$$

for all $\mu,\nu\in\mathbb{N}$.

To prove the proposition we need the following lemma and isometries.

Lemma 6.16. *The matrix* $\tilde{\mathbf{b}}$ *whose entries for* $\mu,\nu\in\mathbb{N}$ *are defined by*

$$\tilde{b}_{\mu\nu}:=\sqrt{a_\mu b_\nu}\cdot b_{\mu\nu}^{-+}=\frac{\sqrt{a_\mu b_\nu}}{a_\mu+b_\nu}\left(\arctan a_\mu+\arctan b_\nu\right)\tag{6.44}$$

is a Schur multiplier whose Schur operator satisfies $\|S_{\tilde{\mathbf{b}}}\|_{\mathcal{L}(\mathcal{L}(\ell^2))}\le\frac{\pi}{2}$.

Proof. Example 5.6 part III. ☐

Definition 6.17 (Isometries)**.** Given the standard orthonormal bases $(e_\nu)_{\nu\in\mathbb{N}}$ of $\ell^2=\ell^2(\mathbb{N})$ and $\mathcal{V}=(v_\ell)_{\ell\in\mathbb{Z}^*}$ of H_0, see (6.25), we introduce for $r\in\mathbb{R}$ the isometries

$$\Psi_r^+\colon\ell^2\to H_r^+,\quad e_\mu\mapsto\tfrac{1}{{a_\mu}^r}\,v_\mu$$

and

$$\Psi_r^-\colon\ell^2\to H_r^-,\quad e_\nu\mapsto\tfrac{1}{{b_\mu}^r}\,v_{-\nu}.$$

Proof of Proposition 6.15. Given $\mathbf{d}^{-+}\in\mathcal{L}(H_1^-,H_0^+)$, we use the isometries Ψ to define a map

$$\mathbf{D}:=(\Psi_0^+)^{-1}\mathbf{d}^{-+}\Psi_1^-\colon\ell^2\xrightarrow{\Psi_1^-}H_1^-\xrightarrow{\mathbf{d}^{-+}}H_0^+\xrightarrow{(\Psi_0^+)^{-1}}\ell^2.$$

By the isometry property of the Ψ's the operator norms are equal

$$\|\mathbf{D}\|_{\mathcal{L}(\ell^2)}=\|\mathbf{d}^{-+}\|_{\mathcal{L}(H_1^-,H_0^+)}.$$

The matrix elements of $\mathbf{D}$ and the entries $d^{-+}_{\mu\nu}$ defined by

$$D_{\mu\nu} := \langle e_\mu, De_\nu \rangle_{\ell^2_{\mathbb{N}}} , \qquad d^{-+}_{\mu\nu} := \left\langle v_\mu, \mathbf{d}^{-+} v_{-\nu} \right\rangle_{\ell^2_{\mathbb{Z}^*}} ,$$

for $\mu, \nu \in \mathbb{N}$ are related by

$$\begin{aligned} D_{\mu\nu} &= \left\langle e_\mu, (\Psi^+_0)^{-1} \mathbf{d}^{-+} \Psi^-_1 e_\nu \right\rangle_{\ell^2_{\mathbb{N}}} \\ &\overset{2}{=} \left\langle \Psi^+_0 e_\mu, \mathbf{d}^{-+} \Psi^-_1 e_\nu \right\rangle_{\ell^2_{\mathbb{Z}^*}} \\ &\overset{3}{=} \left\langle \tfrac{1}{{a_\mu}^0}\, v_\mu, \mathbf{d}^{-+} \tfrac{1}{{b_\nu}^1}\, v_\nu \right\rangle_{\ell^2_{\mathbb{Z}^*}} \\ &= \tfrac{1}{b_\nu} d^{-+}_{\mu\nu}. \end{aligned} \tag{6.45}$$

Here identity 2 is by the isometry property of Ψ^+_0 and identity 3 is by Definition 6.17 of Ψ^+_0 and Ψ^-_1.

Since $\mathbf{D}$ is a bounded linear map on ℓ^2 and $\tilde{\mathbf{b}}$ is a Schur multiplier by Lemma 6.16, we obtain a bounded linear map $\mathbf{C}$ on ℓ^2 characterized by the condition that $\langle e_\mu, \mathbf{C} e_\nu \rangle_{\ell^2} = C_{\mu\nu} := D_{\mu\nu} \tilde{b}_{\mu\nu}$ for all $\mu, \nu \in \mathbb{N}$. Moreover, as by Lemma 6.16 the norm of $\tilde{\mathbf{b}}$ as a Schur multiplier is bounded above by $\frac{\pi}{2}$, we get

$$\|\mathbf{C}\|_{\mathcal{L}(\ell^2)} = \|S_{\tilde{\mathbf{b}}}(\mathbf{D})\|_{\mathcal{L}(\ell^2)} \le \tfrac{\pi}{2} \|\mathbf{D}\|_{\mathcal{L}(\ell^2)} = \tfrac{\pi}{2} \|\mathbf{d}^{-+}\|_{\mathcal{L}(H^-_1, H^+_0)}.$$

Observe that

$$C_{\mu\nu} := D_{\mu\nu} \tilde{b}_{\mu\nu} \overset{(6.45)}{=} \tfrac{1}{b_\nu} d^{-+}_{\mu\nu} \tilde{b}_{\mu\nu} \overset{(6.44)}{=} \sqrt{\tfrac{a_\mu}{b_\nu}}\, d^{-+}_{\mu\nu} b^{-+}_{\mu\nu}. \tag{6.46}$$

Here identity two is by the calculation above and identity three by definition of $\tilde{b}_{\mu\nu}$ in Lemma 6.16. We use again the isometries Ψ to define a linear map

$$\mathbf{C}^{-+} := \Psi^+_{\frac12} \mathbf{C} (\Psi^-_{\frac12})^{-1} \colon H^-_{\frac12} \xrightarrow{(\Psi^-_{\frac12})^{-1}} \ell^2 \xrightarrow{\mathbf{C}} \ell^2 \xrightarrow{\Psi^+_{\frac12}} H^+_{\frac12}.$$

By the isometry property of the Ψ's the operator norms are equal, hence

$$\|\mathbf{C}^{-+}\|_{\mathcal{L}(H^-_{\frac12}, H^+_{\frac12})} = \|\mathbf{C}\|_{\mathcal{L}(\ell^2)} \le \tfrac{\pi}{2} \|\mathbf{d}^{-+}\|_{\mathcal{L}(H^-_1, H^+_0)}. \tag{6.47}$$

The matrix entries of $\mathbf{C}^{-+}$ are for $\mu,\nu\in\mathbb{N}$ defined and given by

$$\begin{aligned}
C^{-+}_{\mu\nu} &:= \left\langle v_\mu, \mathbf{C}^{-+} v_{-\nu}\right\rangle_{\ell^2_{\mathbb{Z}^*}} \\
&\overset{2}{=} \left\langle v_\mu, \Psi^+_{1/2} \underbrace{\mathbf{C}\,(\Psi^-_{1/2})^{-1} v_{-\nu}}_{\mathbf{C}\sqrt{b_\nu}e_\nu=\sqrt{b_\nu}\sum_\lambda C_{\lambda\nu}e_\lambda} \right\rangle_{\ell^2_{\mathbb{Z}^*}} \\
&\overset{3}{=} \left\langle v_\mu, \sqrt{b_\nu}\sum_{\lambda\in\mathbb{N}} C_{\lambda\nu} \underbrace{\Psi^+_{1/2} e_\lambda}_{\frac{1}{\sqrt{a_\lambda}}v_\lambda} \right\rangle_{\ell^2_{\mathbb{Z}^*}} \\
&= \sqrt{b_\nu}\sum_{\lambda\in\mathbb{N}} \frac{C_{\lambda\nu}}{\sqrt{a_\lambda}} \underbrace{\left\langle v_\mu, v_\lambda\right\rangle_{\ell^2_{\mathbb{Z}^*}}}_{\delta_{\mu\lambda}} \\
&= \sqrt{b_\nu}\frac{C_{\mu\nu}}{\sqrt{a_\mu}} \\
&\overset{6}{=} d^{-+}_{\mu\nu} b^{-+}_{\mu\nu}.
\end{aligned}$$

Here identity 2 is by definition of $\mathbf{C}^{-+}$ and identity 3 is by Definition 6.17 of $\Psi^-_{1/2}$ and $\Psi^+_{1/2}$ and by expanding $\mathbf{C}e_\nu = \sum_{\lambda\in\mathbb{N}} C_{\lambda\nu}e_\lambda$. Identity 6 holds by (6.46).

Thus $\mathbf{C}^{-+}$ is the Hadamard product $\mathbf{C}^{-+} = \mathbf{b}^{-+}\odot\mathbf{d}^{-+} =: H_{\mathbf{b}^{-+}}(\mathbf{d}^{-+})$. Hence

$$\|H_{\mathbf{b}^{-+}}(\mathbf{d}^{-+})\|_{\mathcal{L}(H^-_{\frac{1}{2}},H^+_{\frac{1}{2}})} \overset{(6.47)}{\le} \tfrac{\pi}{2}\|\mathbf{d}^{-+}\|_{\mathcal{L}(H^-_1,H^+_0)}.$$

by (6.47). Therefore Hadamard multiplication by $\mathbf{b}^{-+}$ is a bounded linear map

$$H_{\mathbf{b}^{-+}}\colon \mathcal{L}(H^-_1,H^+_0)\to\mathcal{L}(H^-_{\frac{1}{2}},H^+_{\frac{1}{2}})$$

of norm

$$\|H_{\mathbf{b}^{-+}}\|_{\mathcal{L}(\mathcal{L}(H^-_1,H^+_0),\mathcal{L}(H^-_{\frac{1}{2}},H^+_{\frac{1}{2}}))} \le \tfrac{\pi}{2}.$$

This concludes the proof of Proposition 6.15. □

Remark 6.18. Because of Corollary 5.9 the proof of Proposition 6.15 cannot be strengthened to give an extension to $\mathcal{L}(H_0,H_0)$.

The operator sum $T^{\boldsymbol\beta}_{\boldsymbol A} + T^{\boldsymbol\alpha+}_{\boldsymbol A} = d\Pi_+(A)$

6.3.7 Proof of Theorem 6.5

Consider the operator $d\Pi_+(A) = T^\beta_A + T^{\alpha+}_A$ given by (6.35) and suppose that $\Delta\in\mathcal{L}_{sym_0}(H_1,H_0)$. By Corollary 6.7 the diagonal blocks vanish

$$d\Pi_+(A)\Delta \overset{(6.38)}{\underset{(6.37)}{=}} \begin{pmatrix} 0 & (T^\beta_A)^{-+}\Delta + (T^{\alpha+}_A)^{-+}\Delta \\ (T^\beta_A)^{+-}\Delta + (T^{\alpha+}_A)^{+-}\Delta & 0\end{pmatrix}.$$

We use Lemma 6.9 to estimate the operator norm by the norms of the four blocks, then we use the triangle inequality to get inequality two

$$
\begin{aligned}
&\|d\Pi_+(A)\Delta\|_{\mathcal{L}(H_{\frac{1}{2}})} \\
&\le \|(T_A^\beta)^{-+}\Delta + (T_A^{\alpha+})^{-+}\Delta\|_{\mathcal{L}(H^-_{\frac{1}{2}},H^+_{\frac{1}{2}})} \\
&\quad + \|(T_A^\beta)^{+-}\Delta + (T_A^{\alpha+})^{+-}\Delta\|_{\mathcal{L}(H^+_{\frac{1}{2}},H^-_{\frac{1}{2}})} \\
&\le \underbrace{\|(T_A^\beta)^{-+}\Delta\|_{\mathcal{L}(H^-_{\frac{1}{2}},H^+_{\frac{1}{2}})}}_{\le \frac{\pi}{2} \text{ by Prop. 6.15}} + \underbrace{\|(T_A^{\alpha+})^{-+}\Delta\|_{\mathcal{L}(H^-_{\frac{1}{2}},H^+_{\frac{1}{2}})}}_{\le \frac{2\sigma+1}{\pi\sigma^2} \text{ by Prop. 6.10}} \\
&\quad + \underbrace{\|(T_A^\beta)^{+-}\Delta\|_{\mathcal{L}(H^+_{\frac{1}{2}},H^-_{\frac{1}{2}})}}_{\le \frac{\pi}{2} \text{ by Prop. 6.15}} + \underbrace{\|(T_A^{\alpha+})^{+-}\Delta\|_{\mathcal{L}(H^+_{\frac{1}{2}},H^-_{\frac{1}{2}})}}_{\le \frac{2\sigma+1}{\pi\sigma^2} \text{ by Prop. 6.10}} \\
&\le \pi + \tfrac{4\sigma+2}{\pi\sigma^2}.
\end{aligned}
$$

Here we also used that the norm of each block is bounded from above by the norm of the total operator, in symbols $\|(T_A^{\alpha+})^{-+}\Delta\|_{\mathcal{L}(H^-_{\frac{1}{2}},H^+_{\frac{1}{2}})} \le \|T_A^{\alpha+}\Delta\|_{\mathcal{L}(H_{\frac{1}{2}})}$. This proves Theorem 6.5.

6.4 We prove Theorem B

Let A be in the open subset $\mathcal{F}_h^*$ of $\mathcal{F}_h$ consisting of invertible weak Hessians. Invertibility is an open condition. Since eigenvalues depend continuously on A we can choose an open convex neighborhood $\mathcal{U}$ of A in $\mathcal{F}_h^*$ with the property that there exists a **uniform spectral gap** $\sigma_0 > 0$ in the sense that the spectral gap of every $B \in \mathcal{U}$ is at least σ_0. Let $\Delta \in \mathcal{F}_h$ be such that $A + \Delta \in \mathcal{U}$. Since $\mathcal{U}$ is convex it follows that $A + t\Delta \in \mathcal{U}$ for every $t \in [0,1]$ and therefore the spectral gap of $A + t\Delta$ is at least σ_0.

Recall from (6.34) that T_A^+ denotes the extension of $d\Pi_+(A)$ from $\mathcal{L}(H_1, H_0)$ to $\mathcal{L}(H_{\frac{1}{2}})$. The fundamental theorem of calculus is used in the first inequality

$$
\begin{aligned}
&\|T^+_{A+\Delta} - T^+_A\|_{\mathcal{L}(H_{\frac{1}{2}})} \\
&\le \int_0^1 \|T^+_{A+t\Delta}\|_{\mathcal{L}(\mathcal{L}_{sym_0}(H_1,H_0),\mathcal{L}(H_{\frac{1}{2}}))} \|\Delta\|_{\mathcal{L}(H_1,H_0)}\, dt \\
&\le \left(\pi + \frac{4\sigma_0 + 2}{\pi\sigma_0^2}\right) \|\Delta\|_{\mathcal{L}(H_1,H_0)}
\end{aligned}
$$

and inequality two holds by Theorem 6.5. This proves continuity in $\mathcal{L}(H_{\frac{1}{2}})$. By scale shift invariance, as explained in Example A.3, continuity in $\mathcal{L}(H_{\frac{3}{2}})$ follows as well. This completes the proof of Theorem B.

A Hilbert space pairs

Definition A.1. A **Hilbert space pair** $H = (H_0, H_1)$ consists of *infinite* dimensional[13] Hilbert spaces such that H_1 is a dense subset of H_0 and the inclusion map is compact. An **isomorphism of Hilbert space pairs** $H = (H_0, H_1)$ and $W = (W_0, W_1)$ is a Hilbert space isomorphism $T\colon H_0 \to W_0$ whose restriction to H_1, notation T_1, takes values in W_1 and such that $T_1\colon H_1 \to W_1$ is a Hilbert space isomorphism. Two Hilbert space pairs are called **isomorphic** if there exists an isomorphism of Hilbert space pairs.

Definition A.2. An **isometry of Hilbert space pairs** is an isomorphism $\Phi\colon P \to Q$ of Hilbert space pairs with the additional property that $\Phi\colon H_0 \to W_0$ as well as its restriction $\Phi_1\colon H_1 \to W_1$ are Hilbert space isometries. Two Hilbert space pairs are called **isometric** if there exists an isometry of Hilbert space pairs.

Example A.3 (Scale shift invariance)**.** Let h be a growth function. Consider the Hilbert space pair $H = (H_0, H_1) = (\ell^2, \ell^2_h)$ and, for any real $r \in \mathbb{R}$, consider the Hilbert space pair $W = (W_0, W_1) = (\ell^2_{h^r}, \ell^2_{h^{r+1}})$. These two pairs are isometric Hilbert pairs. An isometry is given by the linear map determined on basis vectors by

$$\Phi\colon H_0 \to W_0, \quad e_\nu \mapsto h(\nu)^{-r/2} e_\nu.$$

Theorem A.4. *Given a Hilbert space pair $H = (H_0, H_1)$, there exists a growth function $h\colon \mathbb{N} \to (0, \infty)$ such that the pair (H_0, H_1) is isometric to (ℓ^2, ℓ^2_h).*

Corollary A.5. *In a Hilbert space pair (H_0, H_1) both Hilbert spaces H_0 and H_1 are separable.*

Remark A.6 (Uniqueness of pair growth)**.** The pair growth function $h = h(H_0, H_1)$ is unique up to pair iso<u>metry</u>. The growth type $[h]$ is unique up to pair iso<u>morphism</u> as we show in Theorem A.8 below.

Proof of Theorem A.4. On H_1 we have two inner products, namely $\langle \cdot, \cdot \rangle_1$ and the restriction to H_1 of the inner product of H_0, still denoted by $\langle \cdot, \cdot \rangle_0$. By the Theorem of Riesz there exists a bounded linear map $T\colon H_1 \to H_1$ such that

$$\langle \xi, \eta \rangle_0 = \langle \xi, T\eta \rangle_1 \tag{A.48}$$

for all $\xi, \eta \in H_1$. We need the following lemma.

Lemma A.7. *The operator $T \in \mathcal{L}(H_1)$ is compact, symmetric, positive definite.*

Proof of Lemma A.7. Symmetry, resp. positive definiteness, of the inner product $\langle \cdot, \cdot \rangle_1$ imply symmetry, resp. positive definiteness, of the operator T.

To prove compactness of T we pick a unit ball sequence $\xi_\nu \in H_1$, that is $\|\xi_\nu\|_1 \le 1$. By compactness of the inclusion $\iota\colon H_1 \hookrightarrow H_0$ there is a subsequence ξ_{ν_j} which converges in H_0.
We show that $T\xi_{\nu_j}$ converges in H_1: Since the sequence ξ_{ν_j} converges in H_0,

[13] In finite dimension the dense inclusion would imply that $H_1 = H_0$.

the sequence ξ_{ν_j} is Cauchy in H_0. Given $\varepsilon > 0$, there exists $j_0 = j_0(\varepsilon) \in \mathbb{N}$ such that for all $j, j' \geq j_0$ we have $\|\xi_{\nu_j} - \xi_{\nu_{j'}}\|_0 \leq \varepsilon / \sqrt{\|T\|}$ where $\|T\| = \|T\|_{\mathcal{L}(H_1)}$.

In the following we show that $\|T(\xi_{\nu_j} - \xi_{\nu_{j'}})\|_1 \leq \varepsilon$. For this purpose we abbreviate $v := \xi_{\nu_j} - \xi_{\nu_{j'}}$. Hence $\|v\|_0^2 \leq \varepsilon^2 / \|T\|$. We estimate

$$
\begin{aligned}
0 &\leq \left\| v - \|T\|^{-1} T v \right\|_0^2 \\
&= \left\langle v - \|T\|^{-1} T v, v - \|T\|^{-1} T v \right\rangle_0 \\
&= \|v\|_0^2 - \tfrac{2}{\|T\|} \langle v, Tv \rangle_0 + \tfrac{1}{\|T\|^2} \langle Tv, Tv \rangle_0 \\
&\overset{4}{=} \|v\|_0^2 - \tfrac{2}{\|T\|} \langle Tv, Tv \rangle_1 + \tfrac{1}{\|T\|^2} \langle Tv, TTv \rangle_1 \\
&\overset{5}{\leq} \tfrac{\varepsilon^2}{\|T\|} - \tfrac{2}{\|T\|} \|Tv\|_1^2 + \tfrac{1}{\|T\|^2} \|Tv\|_1 \underbrace{\|TTv\|_1}_{\leq \|T\| \|Tv\|_1} \\
&\leq \tfrac{\varepsilon^2}{\|T\|} - \tfrac{1}{\|T\|} \|Tv\|_1^2
\end{aligned}
$$

and this proves that $\|Tv\|_1 \leq \varepsilon$. In step 4 we used (A.48) and symmetry of T. In step 5 we used $\|v\|_0^2 \leq \varepsilon^2 / \|T\|$ and Cauchy-Schwarz. Thus the sequence $T\xi_{\nu_j}$ is Cauchy in H_1, but H_1 is complete. This proves Lemma A.7. □

By Lemma A.7 the spectral theorem applies and yields an orthonormal basis

$$
F := \{F_\nu\}_{\nu \in \mathbb{N}} \subset H_1, \qquad T F_\nu = \kappa_\nu F_\nu, \quad \kappa_\nu \searrow 0, \tag{A.49}
$$

of H_1 consisting of eigenvectors of $T \colon H_1 \to H_1$ whose eigenvalues are positive reals and form a monotone decreasing sequence converging to zero.

Claim. The appropriately rescaled F_ν's, namely

$$
E_\nu := \tfrac{1}{\sqrt{\kappa_\nu}} F_\nu \in H_1 \subset H_0, \tag{A.50}
$$

form an orthonormal basis of H_0.

Proof of the claim. Note that $TE_\nu = \kappa_\nu E_\nu$. Use (A.48) in the first step to get

$$
\begin{aligned}
\langle E_\nu, E_\mu \rangle_0 &= \langle E_\nu, T E_\mu \rangle_1 \\
&= \langle E_\nu, \kappa_\mu E_\mu \rangle_1 \\
&= \left\langle \tfrac{1}{\sqrt{\kappa_\nu}} F_\nu, \kappa_\mu \tfrac{1}{\sqrt{\kappa_\mu}} F_\mu \right\rangle_1 \\
&= \sqrt{\tfrac{\kappa_\mu}{\kappa_\nu}} \langle F_\nu, F_\mu \rangle_1 \\
&= \sqrt{\tfrac{\kappa_\mu}{\kappa_\nu}} \delta_{\nu\mu} \\
&= \delta_{\nu\mu}.
\end{aligned}
$$

This proves that E_ν and E_μ are orthonormal in H_0. Since the inclusion of H_1 in H_0 is dense and the F_ν form a basis of H_1 it follows that the E_ν form an orthonormal basis of H_0. This proves the claim. □

By the claim the following linear map is an isometry

$$\Phi\colon \ell^2 \to H_0, \quad (x_\nu) \mapsto \sum_{\nu=1}^{\infty} x_\nu E_\nu. \tag{A.51}$$

The **growth function associated to the Hilbert pair *H*** is defined by

$$\boxed{h(\nu) := \tfrac{1}{\kappa_\nu}.} \tag{A.52}$$

We calculate

$$\begin{aligned}
\langle E_\nu, E_\mu \rangle_1 &= \left\langle \tfrac{1}{\sqrt{\kappa_\nu}} F_\nu, \tfrac{1}{\sqrt{\kappa_\mu}} F_\mu \right\rangle_1 \\
&= \tfrac{1}{\sqrt{\kappa_\nu \kappa_\mu}} \langle F_\nu, F_\mu \rangle_1 \\
&= \tfrac{1}{\sqrt{\kappa_\nu \kappa_\mu}} \delta_{\nu\mu} \\
&= \tfrac{1}{\kappa_\nu} \delta_{\nu\mu} \\
&= h(\nu)\delta_{\nu\mu}.
\end{aligned}$$

Hence Φ restricts to an isometry $\Phi|\colon \ell^2_h \to H_1$. This finishes the proof of Theorem A.4. $\square$

Theorem A.8. *Two Hilbert space pairs (ℓ^2, ℓ^2_f) and (ℓ^2, ℓ^2_g) are isomorphic iff the growth functions f and g are equivalent.*

Proof. '$\Leftarrow$' Suppose $f \sim g$. Then the inner products (2.7) on ℓ^2_f and ℓ^2_g are equivalent. Then the identity map $\mathrm{Id}\colon \ell^2_f \to \ell^2_g$ is an isomorphism of Hilbert spaces. Hence $\mathrm{Id}\colon (\ell^2, \ell^2_f) \to (\ell^2, \ell^2_g)$, $x \mapsto x$, is a Hilbert space pair isomorphism.
'$\Rightarrow$' Suppose there is a Hilbert space pair isomorphism

$$T\colon (\ell^2, \ell^2_f) \to (\ell^2, \ell^2_g).$$

Since $T\colon \ell^2 \to \ell^2$ is an isomorphism, there is a constant c_0 such that

$$\frac{1}{c_0} \|x\|_{\ell^2} \le \|Tx\|_{\ell^2} \le c_0 \|x\|_{\ell^2}, \quad \forall x \in \ell^2. \tag{A.53}$$

Since the restriction $T_1 = T|\colon \ell^2_f \to \ell^2_g$ is an isomorphism and choosing c_0 larger, if necessary, it holds that

$$\frac{1}{c_0} \|x\|_{\ell^2_f} \le \|Tx\|_{\ell^2_g} \le c_0 \|x\|_{\ell^2_f}, \quad \forall x \in \ell^2_f. \tag{A.54}$$

The map $\Pi_n\colon \ell^2 \to \ell^2$ projects a sequence $x = (x_i)$ to its first n members and sets all others equal zero, in symbols $x \mapsto (x_1, \dots, x_n, 0, \dots)$. For natural numbers $n, m \in \mathbb{N}$ define the linear map

$$A^n_m := \Pi_n T^{-1} \Pi_{m-1} T\colon \Pi_n \ell^2 \to \Pi_n \ell^2.$$

Claim. If $f(n) < \frac{g(m)}{{c_0}^4}$ then A^n_m is bijective, hence a vector space isomorphism.

Proof of the claim. . Since $\Pi_n \ell^2$ is a finite dimensional vector space it suffices to show that A_m^n is injective. To show this we pick $\xi \in \ker A_m^n$. Therefore $\xi \in \ell^2$ satisfies the following equations $A_m^n \xi = 0$ and since ξ lies in the image of the projection $\Pi_n = \Pi_n^2$ we have in addition that $\xi = \Pi_n \xi$. Hence we obtain

$$\begin{aligned}\xi &= \Pi_n \xi \\ &= \Pi_n T^{-1} T \xi \\ &= \Pi_n T^{-1} T \xi - A_m^n \xi \\ &= \Pi_n T^{-1} T \xi - \Pi_n T^{-1} \Pi_{m-1} T \xi \\ &= \Pi_n T^{-1} \left(\mathrm{Id} - \Pi_{m-1}\right) T \xi.\end{aligned}$$

Now we estimate

$$\begin{aligned}\|\xi\|_{\ell^2} &= \left\|\Pi_n T^{-1} \left(\mathrm{Id} - \Pi_{m-1}\right) T \xi\right\|_{\ell^2} \\ &\le \|\Pi_n\|_{\mathcal{L}(\ell^2,\ell^2)} \left\|T^{-1}\right\|_{\mathcal{L}(\ell^2,\ell^2)} \left\|\left(\mathrm{Id} - \Pi_{m-1}\right) T \xi\right\|_{\ell^2}.\end{aligned}$$

Observe that $\|\Pi_n\|_{\mathcal{L}(\ell^2,\ell^2)} = 1$, by orthogonality of the projection Π_n, and $\|T^{-1}\|_{\mathcal{L}(\ell^2,\ell^2)} \le c_0$ by the first inequality in (A.53). Therefore

$$\|\xi\|_{\ell^2} \le c_0 \left\|\left(\mathrm{Id} - \Pi_{m-1}\right) T \xi\right\|_{\ell^2}. \tag{A.55}$$

We abbreviate

$$x := \left(\mathrm{Id} - \Pi_{m-1}\right) T \xi. \tag{A.56}$$

Since $\mathrm{Id} - \Pi_{m-1}$ is the orthogonal projection which makes the first $m-1$ entries zero, the element x is of the form

$$x = \left(0, \ldots, 0, x_m, x_{m+1}, \ldots\right).$$

By definition of the ℓ_g^2 norm, compare (2.7), and using that g is monotone increasing, we have

$$\begin{aligned}\|x\|_{\ell_g^2}^2 &= \sum_{i=m}^{\infty} g(i) x_i^2 \\ &\ge g(m) \sum_{i=m}^{\infty} x_i^2 \\ &= g(m) \|x\|_{\ell^2}^2.\end{aligned}$$

Therefore any x of the form $(0, \ldots, 0, x_m, x_{m+1}, \ldots)$ satisfies the inequality

$$\|x\|_{\ell^2} \le \frac{1}{\sqrt{g(m)}} \|x\|_{\ell_g^2}.$$

Using the particular form (A.56) of x we have

$$\begin{aligned}\left\|\left(\mathrm{Id} - \Pi_{m-1}\right) T \xi\right\|_{\ell^2} &\le \frac{1}{\sqrt{g(m)}} \left\|\left(\mathrm{Id} - \Pi_{m-1}\right) T \xi\right\|_{\ell_g^2} \\ &\le \frac{1}{\sqrt{g(m)}} \|\mathrm{Id} - \Pi_{m-1}\|_{\mathcal{L}(\ell_g^2,\ell_g^2)} \|T\|_{\mathcal{L}(\ell_f^2,\ell_g^2)} \|\xi\|_{\ell_f^2}\end{aligned}$$

Observe that $\|T\|_{\mathcal{L}(\ell^2_f,\ell^2_g)} \le c_0$, by the second inequality in (A.54), and that $\|\mathrm{Id} - \Pi_{m-1}\|_{\mathcal{L}(\ell^2_g,\ell^2_g)} \le 1$ since the projection $\mathrm{Id} - \Pi_{m-1}$ is also orthogonal in ℓ^2_g. Thus

$$\|(\mathrm{Id} - \Pi_{m-1})\, T\xi\|_{\ell^2} \le \frac{c_0}{\sqrt{g(m)}}\, \|\xi\|_{\ell^2_f}\,. \tag{A.57}$$

Using that ξ lies in the image of Π_n it is of the form $\xi = (\xi_1, \dots, \xi_n, 0, \dots)$.

$$\begin{aligned}\|\xi\|^2_{\ell^2_f} &= \sum_{i=1}^{n} f(i)\xi_i^2 \\ &\le f(n) \sum_{i=1}^{n} \xi_i^2 \\ &= f(n)\, \|\xi\|^2_{\ell^2}\,.\end{aligned}$$

Therefore

$$\|\xi\|_{\ell^2_f} \le \sqrt{f(n)}\, \|\xi\|_{\ell^2}\,. \tag{A.58}$$

Therefore we are now in position to estimate

$$\begin{aligned}\|\xi\|_{\ell^2} &\overset{(A.55)}{\le} c_0\, \|(\mathrm{Id} - \Pi_{m-1})\, T\xi\|_{\ell^2} \\ &\overset{(A.57)}{\le} \frac{c_0^2}{\sqrt{g(m)}}\, \|\xi\|_{\ell^2_f} \\ &\overset{(A.58)}{\le} a_{nm}\, \|\xi\|_{\ell^2}\,, \quad a_{nm} := c_0^2 \sqrt{\frac{f(n)}{g(m)}}.\end{aligned}$$

By assumption of the claim we have $a_{nm} < 1$. Therefore $\|\xi\|_{\ell^2} = 0$, hence $\xi = 0$ This means that A^n_m is injective and therefore a vector space isomorphism of the finite (n) dimensional vector space $\Pi_n \ell^2$ in itself. This proves the claim. □

Since, under the assumption of the claim, the composition $A^n_m = (\Pi_n T^{-1}) \circ (\Pi_{m-1} T)\colon \Pi_n \ell^2 \to \Pi_n \ell^2$ is bijective, it follows that the second part $B := \Pi_n T^{-1}|_{\Pi_{m-1}\ell^2}\colon \Pi_{m-1}\ell^2 \to \Pi_n \ell^2$ is surjective. By the dimension theorem for finite dimensional vector spaces

$$m - 1 = \dim \Pi_{m-1}\ell^2 = \dim \ker B + \dim \Pi_n \ell^2 \ge \dim \Pi_n \ell^2 = n.$$

Thus $n < m$.

So far we have shown the following implication

$$f(n) < \frac{g(m)}{{c_0}^4} \quad \Rightarrow \quad n < m.$$

Therefore at the same point $n = m$ both functions are related by

$$f(n) \ge \frac{g(n)}{{c_0}^4}.$$

Note that $T^{-1}\colon \ell^2 \to \ell^2$ is as well an isomorphism satisfying the inequalities (A.53) and (A.54) with f and g interchanged. Hence we obtain as well the inequality

$$g(n) \geq \frac{f(n)}{{c_0}^4}.$$

Now set $c := c_0^4$ to obtain

$$\frac{1}{c} f(n) \leq g(n) \leq c f(n).$$

This concludes the proof of Theorem A.8. □

B Interpolation

For convenience of the reader we include the proof of the Stein interpolation theorem [Ste46, Theorem 2] in our situation of the Hilbert spaces ℓ^2_f with inner product (2.7). The proof is based on the three line theorem whose usage in interpolation theory goes back to Thorin; we recommend the article [BGP08] on Thorin's life and work.

Proposition B.1. *Assume that T is a linear map with bounds*

$$\begin{aligned} T\colon \ell^2 \to \ell^2, &\qquad \|T\|_{\mathcal{L}(\ell^2,\ell^2)} \leq M_0, \\ T\colon \ell^2_f \to \ell^2_g, &\qquad \|T\|_{\mathcal{L}(\ell^2_f,\ell^2_g)} \leq M_1. \end{aligned}$$

Then T is also bounded as a linear map

$$T\colon \ell^2_{\sqrt{f}} \to \ell^2_{\sqrt{g}}, \qquad \|T\|_{\mathcal{L}(\ell^2_{\sqrt{f}},\ell^2_{\sqrt{g}})} \leq \sqrt{M_0 M_1}.$$

Proof. Pick two unit vectors $\xi \in \ell^2_{\sqrt{f}}$ and $\eta \in \ell^2_{\sqrt{g}}$. For $k \in \mathbb{N}$ let e_k be the sequence of reals whose members are all zero except member k which is 1, in symbols $e_k = (0,\dots,0,1,0,\dots)$. Write $\xi = \sum_k \xi_k e_k$ and $\xi = \sum_\ell \xi_\ell e_\ell$ and set $T_{k,l} := \langle T e_k, e_\ell\rangle_0$. On the strip $S = \{\theta + it \mid \theta \in [0,1], t \in \mathbb{R}\}$ consider the function $F\colon S \to \mathbb{C}$ defined by

$$F(\theta + it) := \sum_{k,\ell} T_{k,\ell} \xi_k f(k)^{\frac14 - \frac12\theta - \frac12 it} \eta_\ell g(\ell)^{\frac14 + \frac12\theta + \frac12 it}.$$

Claim. (i) $F(\frac12) = \langle T\xi, \eta\rangle_{\sqrt{g}}$. (ii) $|F(it)| \leq M_0$. (iii) $|F(1+it)| \leq M_1$.

Proof of the claim. (i) We have $F(\frac12) = \sum_{k,\ell} T_{k,\ell}\xi_k\eta_\ell\sqrt{g(\ell)} = \langle T\xi,\eta\rangle_{\sqrt{g}}$.
(ii) We have $F(it) = \sum_{k,\ell} T_{k,\ell}\xi_k f(k)^{\frac14 - \frac12 it}\eta_\ell g(\ell)^{\frac14+\frac12 it}$. The element

$$x := \sum_k \underbrace{\xi_k f(k)^{\frac14 - \frac12 it}}_{=:x_k} e_k \in \ell^2 \otimes \mathbb{C}$$

has unit norm square $\|x\|_{\ell^2}^2 = \sum_k |\xi_k f(k)^{\frac{1}{4}-\frac{1}{2}it}|^2 = \sum_k \xi_k^2 \sqrt{f(k)} = \|\xi\|_{\sqrt{f}}^2 = 1$. The element

$$y := \sum_\ell \underbrace{\eta_\ell g(\ell)^{\frac{1}{4}+\frac{1}{2}it}}_{=:y_\ell} e_\ell \in \ell^2 \otimes \mathbb{C}$$

has unit norm square $\|y\|_{\ell^2}^2 = \sum_\ell |\eta_\ell g(\ell)^{\frac{1}{4}+\frac{1}{2}it}|^2 = \sum_\ell \eta_\ell^2 \sqrt{g(\ell)} = \|\eta\|_{\sqrt{g}}^2 = 1$. Now

$$|F(it)| = \Big|\sum_{k,\ell} T_{k,\ell} x_k y_\ell\Big| = |\langle Tx, y\rangle_{\ell^2}| \le \|T\|_{\mathcal{L}(\ell^2,\ell^2)} \|x\|_{\ell^2} \|y\|_{\ell^2} \le M_0.$$

(iii) We have $F(1+it) := \sum_{k,\ell} T_{k,\ell} \xi_k f(k)^{-\frac{1}{4}-\frac{1}{2}it} \eta_\ell g(\ell)^{\frac{3}{4}+\frac{1}{2}it}$. The element

$$x := \sum_k \underbrace{\xi_k f(k)^{-\frac{1}{4}-\frac{1}{2}it}}_{=:x_k} e_k \in \ell^2_f \otimes \mathbb{C}$$

has ℓ^2_f-norm square $\|x\|_{\ell^2_f}^2 = \sum_k |\xi_k f(k)^{-\frac{1}{4}-\frac{1}{2}it}|^2 f(k) = \sum_k \xi_k^2 \sqrt{f(k)} = \|\xi\|_{\sqrt{f}}^2 = 1$. The element

$$y := \sum_\ell \underbrace{\eta_\ell g(\ell)^{-\frac{1}{4}+\frac{1}{2}it}}_{=:y_\ell} e_\ell \in \ell^2_g \otimes \mathbb{C}$$

has ℓ^2_g-norm square $\|y\|_{\ell^2_g}^2 = \sum_\ell |\eta_\ell g(\ell)^{-\frac{1}{4}+\frac{1}{2}it}|^2 g(\ell) = \sum_\ell \eta_\ell^2 \sqrt{g(\ell)} = \|\eta\|_{\sqrt{g}}^2 = 1$. Now

$$|F(1+it)| = \Big|\sum_{k,\ell} T_{k,\ell} x_k g(\ell) y_\ell\Big| = \Big|\langle Tx, y\rangle_{\ell^2_g}\Big| \le \|T\|_{\mathcal{L}(\ell^2_f,\ell^2_g)} \|x\|_{\ell^2_f} \|y\|_{\ell^2_g} \le M_1.$$

This proves the claim. $\square$

Lemma B.2 (The three line theorem)**.** *Let $F(z)$ be analytic on the open strip $0 < \mathrm{Re} z < 1$ and bounded and continuous on the closed strip $0 \le \mathrm{Re} z \le 1$. If*

$$|F(it)| \le M_0, \qquad |F(1+it)| \le M_1, \qquad -\infty < t < \infty,$$

then we have

$$|F(\tfrac{1}{2}+it)| \le \sqrt{M_0 M_1}, \qquad -\infty < t < \infty.$$

Proof. See e.g. [BL76, Le. 1.1.2]. $\square$

By parts (ii) and (iii) of the claim the three line theorem applies and gives

$$|F(\tfrac{1}{2})| \le \sqrt{M_0 M_1}.$$

By part (i) of the claim

$$|\langle T\xi, \eta\rangle_{\sqrt{g}}| \le \sqrt{M_0 M_1}.$$

Since ξ and η were arbitrary unit vectors in $\ell^2_{\sqrt{f}}$, respectively $\ell^2_{\sqrt{g}}$, this implies

$$\|T\|^2_{\mathcal{L}(\ell^2_{\sqrt{f}},\ell^2_{\sqrt{g}})} := \sup_{\|\xi\|_{\ell^2_{\sqrt{f}}}=1} \|T\xi\|_{\ell^2_{\sqrt{g}}} = \sup_{\substack{\|\xi\|_{\ell^2_{\sqrt{f}}}=1 \\ \|\eta\|_{\ell^2_{\sqrt{g}}}=1}} \langle T\xi, \eta\rangle_{\sqrt{g}} \le \sqrt{M_0 M_1}.$$

Here the equality holds since "$\ge$" is true by Cauchy-Schwarz, but "$=$" holds by choosing $\eta = \|T\xi\|^{-1}_{\ell^2_{\sqrt{g}}} T\xi$. This concludes the proof of Proposition B.1. $\square$

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