Understanding when the correlations imply the predictability for the multiple Gaussian

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Abstract

In this paper, we will expose for the Gaussian multiple a theorem relating the predictability to correlations. This theorem is based on another equality which will be also proven. For the correlations to be predictability, the proof will show that the variance-covariance matrix must be located onto the boundary of the positive semi-definite matrix cone with only one zero eigenvalue.

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1 Introduction

The aim of this paper is to propose a theorem relating the conditions in which the correlations imply the predictability. Correlations will imply predictability when the variance-covariance matrix will be located onto the boundary of the cone of positive semi-definite symmetric matrices with only one zero eigenvalue.

2 Conditional variance and marginal variance

In what follows, we will make the link between the conditional variance (Schur's complement) $\Sigma_{X^2} - \Sigma_{X\Omega}.\Sigma_{\Omega^2}^{-1}.\Sigma_{\Omega X}$ and the marginal variance $Var(.)$ of the difference between the current variable X and the conditional mean $E[X|\Omega]$:

$$\boxed{\Sigma_{X^2} - \Sigma_{X\Omega}.\Sigma_{\Omega^2}^{-1}.\Sigma_{\Omega X} = Var(X - E[X|\Omega])}$$

<u>Proof</u>:

$\Sigma_{X^2} - \Sigma_{X\Omega}.\Sigma_{\Omega^2}^{-1}.\Sigma_{\Omega X} = Var(X - E[X|\Omega])$

$\Sigma_{X^2} - \Sigma_{X\Omega}.\Sigma_{\Omega^2}^{-1}.\Sigma_{\Omega X} = \Sigma_{X^2} - 2.Cov(X, E[X|\Omega]) + Var(E[X|\Omega])$

$2.Cov(X, E[X|\Omega]) - \Sigma_{X\Omega}.\Sigma_{\Omega^2}^{-1}.\Sigma_{\Omega X} = Var(E[X|\Omega]$

$2.Cov(X, \Sigma_{X\Omega}.\Sigma_{\Omega^2}^{-1}.\vec{\omega} + \mu_X - \Sigma_{X\Omega}.\Sigma_{\Omega^2}^{-1}.\mu_\Omega) - \Sigma_{X\Omega}.\Sigma_{\Omega^2}^{-1}.\Sigma_{\Omega X} = Var(E[X|\Omega]$

$2.Cov(X, \Sigma_{X\Omega}.\Sigma_{\Omega^2}^{-1}.\vec{\omega}) - \Sigma_{X\Omega}.\Sigma_{\Omega^2}^{-1}.\Sigma_{\Omega X} = Var(E[X|\Omega]$

Using the bilinearity of the covariance we obtain:

$2.\Sigma_{X\Omega}.\Sigma_{\Omega^2}^{-1}.\Sigma_{\Omega X} - \Sigma_{X\Omega}.\Sigma_{\Omega^2}^{-1}.\Sigma_{\Omega X} = Var(E[X|\Omega]$

$\Sigma_{X\Omega}.\Sigma_{\Omega^2}^{-1}.\Sigma_{\Omega X} = Var(E[X|\Omega]$

We will now develop $Var(E[X|\Omega])$:

$Var(E[X|\Omega])$

$= Var(\Sigma_{X\Omega}\Sigma_{\Omega^2}^{-1}.\vec{\omega} + \mu_X - \Sigma_{X\Omega}.\Sigma_{\Omega^2}^{-1}.\mu_\Omega)$

$= Var(\Sigma_{X\Omega}\Sigma_{\Omega^2}^{-1}.\vec{\omega})$

As we have the following relationship:

$Var(A.X) = Var(A.X.X^t.A^t) - A\mu.\mu^t.A^t = A.(E[X.X^t] - \mu.\mu^t).A^t = A.var(X).A^t$

we obtain:

$Var(E[X|\Omega])$

$= Var(\Sigma_{X\Omega}\Sigma_{\Omega^2}^{-1}.\vec{\omega})$

$= \Sigma_{X\Omega}\Sigma_{\Omega^2}^{-1}.\Sigma_{\Omega^2}.\Sigma_{\Omega^2}^{-1}.\Sigma_{\Omega X}$

$= \Sigma_{X\Omega}.I.\Sigma_{\Omega^2}^{-1}.\Sigma_{\Omega X}$

$= \Sigma_{X\Omega}.\Sigma_{\Omega^2}^{-1}.\Sigma_{\Omega X}$

We have proven the relationship: $\Sigma_{X^2} - \Sigma_{X\Omega}.\Sigma_{\Omega^2}^{-1}.\Sigma_{\Omega X} = Var(X - E[X|\Omega])$

3 Context when the Correlations are predictability for the multiple Gaussian

Theorem:

If $\Omega \equiv \vec{\omega}$ is a set of Gaussian variables with $\#\Omega \geq 2$ and X a variable then there exists a predictable relationship between the variables Ω and the variable X:

$$X = E[X|\Omega] = \Sigma_{X\Omega}.\Sigma_{\Omega^2}^{-1}.\vec{\omega} + \mu_X - \Sigma_{X\Omega}.\Sigma_{\Omega^2}^{-1}\mu_\Omega = \sum_{i=1}^{\#\Omega} \beta_{X\omega_i}.\omega_i + \beta_X$$

If:

$$K_{X\Omega}.K_\Omega^{-1}.K_{\Omega X} = 1$$

Where $\Sigma_{(\Omega,X)^2}$ is the covariance-variance matrix between the variables (Ω, X) and $K_{(\Omega,X)^2} = (diag^{-1}(\Sigma_{(\Omega,X)^2}))^{\frac{1}{2}}.\Sigma_{(\Omega,X)^2}.(diag^{-1}(\Sigma_{(\Omega,X)^2})^{\frac{1}{2}}$ is the correlation matrix between the variables (Ω, X).

The predictable relationship is obtained when the variance covariance matrix $\Sigma_{(\Omega,X)^2}$ lies on the boundary of the cone of positive semi-definite matrices with only one zero eigenvalue.

Proof:

We will consider a symmetric matrix $\Sigma_{(\Omega X)^2}$ strictly positive definite or a symmetric matrix $\Sigma_{(\Omega X)^2}$ having a single negative eigenvalue:

$$\Sigma_{(\Omega X)^2} = \begin{pmatrix} \Sigma_{\Omega^2} & \Sigma_{\Omega X} \\ \Sigma_{X\Omega} & \Sigma_{X^2} \end{pmatrix}$$

We will project this matrix onto the boundary of the cone of symmetric positive semi-definite matrices (see paper [4] page 9).The projection is done by spectral decomposition and by putting the last eigenvalue equal to 0:

$$\Sigma_{(\Omega X)^2}^+ = P_S(\Sigma_{(\Omega X)^2}) = \begin{pmatrix} \Sigma_{\Omega^2}^+ & \Sigma_{\Omega X}^+ \\ \Sigma_{X\Omega}^+ & \Sigma_{X^2}^+ \end{pmatrix}$$

The symmetric matrix $\Sigma_{(\Omega X)^2}^+$ is **singular** because it is onto the boundary of the cone of the positive semi-definite matrix, we obtain therefore:

$$det(\Sigma_{(\Omega X)^2}^+) = det(\Sigma_{\Omega^2}^+).(\Sigma_{X^2}^+ - \Sigma_{X\Omega}^+.(\Sigma_{\Omega^2}^+)^{-1}.\Sigma_{\Omega X}^+) = 0$$

We suppose that $\Sigma_{\Omega^2}^+$ is strictly positive definite, we can therefore deduce:

$$det(\Sigma_{\Omega^2}^+) > 0 \Longrightarrow \Sigma_{X^2}^+ - \Sigma_{X\Omega}^+.(\Sigma_{\Omega^2}^+)^{-1}.\Sigma_{\Omega X}^+ = 0$$

From the theorem proof in the previous section, we obtain therefore:

$$\Sigma^+_{X^2} - \Sigma^+_{X\Omega}.(\Sigma^+_{\Omega^2})^{-1}.\Sigma^+_{\Omega X} = Var(X - E[X|\Omega]) = 0$$

The equality implies that we have the predictable relationship:

$$X = E[X|\Omega] = \Sigma^+_{X\Omega}.(\Sigma^{-1}_{\Omega^2})^+.\vec{\omega} + \mu_X - \Sigma^+_{X\Omega}.(\Sigma^{-1}_{\Omega^2})^+\mu_\Omega = \sum_{i=1}^{\#\Omega} \beta_{X\omega_i}.\omega_i + \beta_X$$

We will factorize the variance $\Sigma^+_{X^2}$ into quadratic form $\Sigma^+_{X^2} - \Sigma^+_{X,\Omega}.(\Sigma^+_{\Omega^2})^{-1}.\Sigma^+_{X,\Omega}$ to show the correlations $K_{X,\Omega}.K^{-1}_{\Omega^2}.K_{\Omega,X}$:

$\Sigma^+_{X^2} - \Sigma^+_{X,\Omega}.(\Sigma^+_{\Omega^2})^{-1}.\Sigma^+_{X,\Omega}$

$= \Sigma^+_{X^2} - \Sigma^+_{X,\Omega}.(diag^{-1}(\Sigma^+_{\Omega^2}))^{\frac{1}{2}}.K^{-1}_{\Omega^2}.(diag^{-1}(\Sigma^+_{\Omega^2}))^{\frac{1}{2}}.\Sigma^+_{\Omega,X}$

$= \Sigma^+_{X^2} - (\Sigma^+_{X^2})^{\frac{1}{2}}.K_{X,\Omega}.K^{-1}_{\Omega^2}.(\Sigma^+_{X^2})^{\frac{1}{2}}.K_{\Omega,X}$

$= \Sigma^+_{X^2}.(1 - K_{X,\Omega}.K^{-1}_{\Omega^2}.K_{\Omega,X})$

The expression $X = E[X|\Omega]$ is valid when we have:

$$\Sigma^+_{X^2} - \Sigma^+_{X\Omega}.(\Sigma^+_{\Omega^2})^{-1}.\Sigma^+_{\Omega X} = \Sigma^+_{X^2}.(1 - K_{X\Omega}.K^{-1}_{\Omega^2}.K_{\Omega X}) = 0$$

This implies that we have the predictable relationship when the quadratic form $K_{X\Omega}.K^{-1}_{\Omega^2}.K_{\Omega X}$ verifies the following equality:

$$K_{X\Omega}.K^{-1}_{\Omega^2}.K_{\Omega X} = 1$$

We have proven the theorem.

4 Conclusion

In this paper and a Gaussian context, we have proved a theorem giving the conditions under which correlations imply the predictability. This implication is true when the covariance matrix lies on the boundary of the cone of positive semi-definite matrices with only one zero eigenvalue. The paper therefore aimed to relate the notion of correlation to that of predictability.

[1]Elements of information theory. Author: Thomas M.Cover and Joy A.Thomas. Copyright 1991 John Wiley and sons.

[2]Optimal stastical decisions. Author: Morris H.DeGroot. Copyright 1970-2004 John Wiley and sons.

[3]Matrix Analysis. Author: Roger A.Horn and Charles R.Johnson. Copyright 2012, Cambridge university press.

[4]Computing the nearest correlation Matrix-A problem from finance. Author: Nicholas Higham.copyright 2002, The university of Manchester