MOND as a transformation between non-inertial reference frames via Sciama's interpretation of Mach's Principle

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Abstract

Milgrom's Modified Newtonian Dynamics (MOND) correction to Newtonian gravity or inertia is shown to be equivalent to a more fundamental transformation between a non-inertial local reference frame and the fixed background of the observable universe, complying with Mach's principle. Both Newton's gravitational constant and Milgrom's MOND acceleration parameter or scale constant are substituted for two varying, measurable, physical, and cosmological parameters under the justification of Mach's principle: causally connected mass and size of the universe. This Machian interpretation of MOND is mass, length, and time scale invariant at all regimes and free from fundamental constants and free parameters with the exception of the speed of light as the speed of causality and gravity. By extending the external field effect to include that of the rest of the universe, the correction is based on relative field intensities of the small and large scales. It respects the Machian effect by which, in absence of a background, rotational speed is undefined up to the speed of light. The Machian MOND interpretation is a necessary feature that a phenomenological non-linear theory of modified inertia or modified gravity which incorporates Mach's principle, in agreement with galaxy rotation curves, should effectively reduce to as an approximation.

Keywords: Machs principle; dark matter; galaxy rotation curves; MOND; modified Newtonian dynamics; modified gravity; modified inertia; gravitational constant; inertia

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1 Introduction

1.1 Modified Newtonian Dynamics MOND

The solution to the problem of the perihelion precession of Mercury first involved various failed attempts proposing the unobserved existence of a new small planet Vulcan, an asteroid belt, or a gas cloud orbiting the Sun. However, the resolution to the problem came as a modification to the laws of gravity by General Relativity (GR). Similarly, the observation of rotational velocities in disks of galaxies higher than predicted by Newtonian gravity, as depicted in galaxy rotation curves, has driven further research into modifications to the laws of gravity and mechanics. The discrepancies between observed luminance to gravitational mass ratio in galaxies and their missing Keplerian falloff in their velocity curves, together with faster radial velocities of galaxies in galaxy clusters than Newtonian predictions, among other evidence, are referred to as the 'dark matter' problem [1–3]. Modifying gravity or inertia instead of proposing the existence of physical cold dark matter is motivated by the Tully-Fisher law (the asymptotic velocities in rotation curves are proportional to the fourth root of their mass), the Renzo's rule (features in visible mass strongly correlate with features in velocity profiles), and the cuspy halo problem (Newtonian gravity correctly predicts velocity profiles at galactic cores). These recent modifications to the classical law of gravity or laws of mechanics are called Modified (or Mondified) Newtonian Dynamics (MOND) models (or MilgrOmiaN Dynamics), with the first and main MOND being Milgrom's MOND [4–6], which can be thought of as a modification to inertia (applied to Newton's second law of motion) or to the inverse square law of gravity. Almost all full-fledged versions of MOND are just modified gravity, in which the inverse-square law of gravity approximately changes to an inverse linear law, and not modified inertia. As modified inertia, MOND modifies the kinetic term in the particle Lagrangian $mv^2/2$ or the 'inertia term' mg, and measured kinematic accelerations are not simply proportional to forces per unit mass.

The reason for the anomalous motion of Mercury was that Newtonian gravity is not a good enough approximation for strong gravitational fields, such as near the Sun, where GR effects become significant. Milgrom's MOND exceptionally explains the observed rotation curves of galaxies by imposing a correction based on two observational relationships: the Tully-Fisher law, and that the mass (or velocity) discrepancies are always observed below a particular acceleration scale. The simplest correction that satisfies these observational conditions results in a constant asymptotic rotational velocity $v = \sqrt[4]{GMa_0}$ with the need to introduce a constant of acceleration a_0. The connection between the classical Newtonian regime where $v = \sqrt{GM/r}$ and Milgrom's deep MOND or low acceleration regime is determined by an 'interpolating function' $\mu\left(g/a_0\right) = 1/(1 + (a_0/g)^n)^{(1/n)}$ (with $n = 1$ for the simple and $n = 2$ for the standard interpolating functions) based on g as the 'true acceleration' and a constant of acceleration a_0 (also referred to as the Hubble acceleration or the acceleration scale constant), which is a free parameter adjusted by data fitting and is in principle considered a fundamental and universal constant in Milgrom's MOND. The interpolating function must satisfy the conditions $\mu(x) \to 1$ for $x \ll 1$ to recover Newtonian laws and $\mu(x) \to x$ for $x \gg 1$ to satisfy the Tully-Fisher law. A fixed-distance or fixed-mass (independent of the system under study) modification alone cannot account for the observed rotation velocities due to the Tully-Fisher law, i.e., it is not possible to simply set a constant of distance or a constant of mass to modify the laws of gravity to fit all rotation curves of galaxies (because some galaxies exhibit the effect of dark matter at small radius while others at large radius, and same happens with different galaxy masses). However, combining both mass and distance in acceleration or gravitational field intensity, Milgrom's MOND achieves this limit where the effect of dark matter in galaxies is generally observed. Still, Milgrom's MOND, being insufficient to explain all dark matter related phenomena such as galaxy cluster dynamics, [7, 8], being non-relativistic, and not explaining phenomenologically the origin of the interpolating function, is an effective theory or approximation to a more fundamental modification to gravitation or inertia.

The acceleration constant in Milgrom's MOND $a_0 \sim 10^{-10}\ m/s^2$, was already related to the cosmological scale through the Hubble constant H_0 (or the cosmological constant Λ) $a_0 \sim cH_0 \sim c^2\sqrt{\Lambda} \sim c^2/R_u$ originally by Milgrom [9], stating that this coincidence could point to a basic theory underlying MOND's phenomenology. In particular, he stated that "*an attractive possibility is that MOND results as a non-relativistic, small-scale expression of a fundamental theory by which inertia is a vestige of the interaction of a body with 'the rest of the Universe', in the spirit of Mach's principle.*", and "*We thus envisage inertia as resulting from the interaction of the accelerated body with some agent field, perhaps having to do with the vacuum fields, perhaps with an "inertia field" whose source is matter in the "rest of the universe"–in the spirit of Mach's principle*" [10], although he did not further develop his theory in relation to Machian ideas.

Progress in fundamental physics has been made by reducing the number of fundamental constants (such as the gravitational acceleration of the Earth and other planets for a universal gravitational constant, the unification of the speed of light with vacuum permittivity and permeability, and atomic constants for the Planck constant). Therefore, it is of considerable interest to explore modifications of the laws of gravity or mechanics without the need for more fundamental constants.

Various theories of modified gravity or inertia are based on Milgrom's MOND acceleration constant: John Moffat's MOG scalar–tensor–vector gravity approach to galaxy rotation curves is based on two parameters, which are set to fit galactic Milgrom's MOND's acceleration constant [11], Erik's Verlinde's entropic gravity attempts to propose an underlying framework for Milgrom's MOND's acceleration constant [12], and Deur's self-interacting gravitons model correction to Newtonian gravity contains a physical constant counterpart to Milgrom's MOND's acceleration constant in the form of $a = \Lambda/a_0 = \sqrt{\Lambda}/c^2$ [13]. Others, such as quantized inertia, rely on the acceleration parameter $a_0 = c^2/R_u$ where R_u is the radius of the cosmological comoving horizon [14].

1.2 Mach's Principle and modified inertia

Newtonian gravity and classical mechanics are founded on the principles of absolute space, absolute time, and absolute motion (including acceleration). These were originally opposed by Leibniz (and others, such as Berkeley in his 1721 *De Motu*, and Huygens) arguing that as observers, we can only epistemically access relative notions of space, time, and motion. Newton's justification of absolute space and motion is portraited in his rotating bucket of water experiment at the beginning of his 1687 masterpiece *Philosophiæ naturalis principia mathematica*, which can be summarized as follows: a bucket with water is set spinning around its axis and, at first, the walls of the bucket rotate relative to the stationary water while the surface of the water remains flat as prior to the spinning. After the water starts to rotate as well, it rises towards the walls and its shape is no longer flat when the spinning bucket and the water are at rest relative to each other. Interactions between water and walls (due to their relative velocities) cannot explain the shape of the water, and Newton assumed that this experiment could be used to measure rotation with respect to an absolute space, "*without relation to anything external, which remains always similar and immovable*".

The definition of absolute space, time, and motion suffer from circular reasoning. In Newtonian mechanics, absolute space is defined as the frame absent of inertial forces (also referred to as fictitious forces, which come from our choice of reference frame that is rotating in absolute space), but the absence of inertial forces is used to prove and identify the existence of absolute space in Newton's rotating bucket of water experiment. According to Newton's first law of motion, an object moves inertially if it is free from outside influences, but the fact that it is free from outside influences is inferred only by observing that it moves inertially. In Special Relativity (SR), light travels at the same speed in all inertial frames and absolute acceleration is defined relative to inertial frames, but inertial frames are defined as frames absent of acceleration (accelerometers always measure acceleration with respect to a reference frame of calibration, that must be assumed to be inertial). If a body is measured to have acceleration in one inertial frame, all other inertial frames will agree that it is accelerating.

Leibniz's relative space, time and motion ideas were further developed by Ernst Mach in his 1883 *The Science of Mechanics*, in which he criticized Newton's conclusion of his bucket of water experiment stating that "*No one is competent to say how the experiment would turn out if the sides of the vessel increased in thickness and mass till they were ultimately several leagues thick.*"[15]. According to Mach, the relative rotation of the water with respect to the bucket produces no noticeable centrifugal forces, and such forces are instead produced by its relative rotation with respect to Earth and the other celestial bodies (in this way, the relative motion between the several leagues thick bucket and the water could produce noticeable centrifugal forces in the water, and perhaps their non-relative motion could reduce these forces when both rotate with respect to the rest of the universe). In contrast to Newton, who attempted to explain the physical effects of inertia through a sort of resistance to motion within an unobservable absolute space with no physical properties, Mach conceived inertia as an interaction that required other external bodies to manifest. Newton did not suspect that the change in the water's shape could be due to rotation relative to the rest of the universe because his action at a distance gravitational force from his infinite and homogeneous universe acting on the water cancels out according to Newton's shell theorem and is independent of velocity or acceleration. But no local phenomena can ever be isolated from the rest of the universe, and its effect certainly reaches the water even if the sum of these forces is zero. According to Mach, and in opposition to Newton's conclusions, a spinning bucket of water in an empty universe would not change the water's shape (one could not detect relative motion in an empty universe, it would be undefinable), and if the rest of the universe was set spinning while the bucket was at rest, the water surface would curve, as both the spinning of the bucket and water or the spinning of the universe are indistinguishable systems without an absolute space. Thus, for Mach, the relativity symmetry of acceleration is broken by the existence of the background universe.

Moreover, there is an exact coincidence between the local measurement of the angular velocity of the Earth through Foucault's pendulum and the cosmological measurement through the apparent movement of distant stars and galaxies, which Newtonian gravity and mechanics cannot explain because it does not causally connect both measurements (and consider the determination of inertial frames by the fixed stars and Foucault's pendulum a coincidence), but constitutes the basic idea of Mach's principle: "*The universe, as represented by the average motion of distant galaxies, does not appear to rotate relative to local inertial frames.*"[16].

According to Mach, and known as Mach's principle, the inertia of a body is not an independent and intrinsic property of matter (unlike in Newtonian mechanics, where a particle in an empty universe has the usual inertial properties), but rather the result of the action of the universe as a whole. Mach suggested that the fixed background distribution of matter in the universe must exert the inertial forces on a local accelerating body. In this way, the reference system with respect to which the universe is at rest or in uniform and rectilinear motion is a true inertial reference system. In Ernst Mach words, "*I have remained to the present day the only one who insists upon referring the law of inertia to the Earth, and in the case of motions of great spatial and temporal extent to the fixed stars.*" [15]. Hence, inertial frames should be defined with respect to this rest frame, and local physical laws must be determined by the large-scale structure of the universe.

Einstein's pursuit of a relativistic gravitational theory was inspired by Mach, and he envisioned GR to fulfill his interpretation of Mach's principle: "*the metric tensor is completely caused and determined by the stress-energy tensor*" [17]. He noted that following Mach, it was expected that "*1. The inertia of a body must increase when ponderable masses are piled up in its neighbourhood, 2. A body must experience an accelerating force when neighbouring masses are accelerated, and, in fact, the force must be in the same direction as the acceleration, 3. A rotating hollow body must generate inside of itself a "Coriolis field", which deflects moving bodies in the sense of the rotation, and a radial centrifugal field as well*" [18]. He had hopes that by using tensors in GR, he would achieve the strong Machian version of relativity (in which the gravitational influence of the whole universe gave rise to inertia), but although points 2. and 3. were satisfied in GR (at least phenomenologically by frame-dragging), the first condition was not, because the immediate consequence of it is that a body in an otherwise empty universe should have no inertia, and the empty solution where the stress-energy tensor is zero everywhere in GR is the flat Minkowski spacetime of SR, in which test bodies have the usual inertia. Einstein then introduced the cosmological constant term (as the mass density o the universe) in his field equations in the hope that, with the cosmological term, they would have no solutions for a zero stress-energy tensor. However, de Sitter found in 1917 a solution for the field equations with the cosmological constant and with zero stress-energy tensor for an expanding universe, Einstein dismissed the cosmological constant as no longer justified, and he abandoned the ideas of Mach.

Einstein's equivalence principle (a frame linearly accelerated relative to an inertial frame in the Minkowskian spacetime of SR is locally identical to a frame at rest in a uniform gravitational field) axiomatically assumes that inertial and gravitational masses are equal. Gravity is the only infinite range force that cannot be screened due to the existence of only positive masses, unlike in electromagnetism, where positive and negative charges result in balance and neutrality, and astrophysical and cosmological structures are therefore not governed by electromagnetism. Moreover, there always exists a reference frame in which inertial forces vanish, just as for the case of gravitational forces in free fall. These equivalences indicate that both inertial and gravitational forces are of the same nature. Following Mach, in any two-body interaction, the influence of all other matter inside their causal spacetime should be taken into account. Inertia is thus a form of gravitational induction, appearing when a body is accelerated with respect to the rest of the universe, subjected to retarded action. But Newtonian gravity and GR were developed without considering today's acknowledgment of the mass content and size of the observable universe. Then, it seems that to describe, for example, the Earth's motion around the Sun, only local masses and distances are required, but the Machian perspective implies that the universe's action is already taken into account in the Newtonian laws. The only way this can be true is through the Newtonian gravitational constant, which, together with the choice of the inertial frame of absolute space, both constitute the two arbitrary choices of Newton.

In Newtonian gravity, the gravitational constant G is the only fundamental constant of the theory, and it is known with far less precision than any other fundamental dimensional constant in physics. The uncertainty in its measurement, together with its problematic in Quantum Field Theory in the Planck energy (which is used as a cutoff for the energy density of the vacuum and the theoretical quantum corrections estimation of the Higg's mass, leading to the vacuum catastrophe and hierarchy problems) has led to questioning its constant nature repeatedly over the last century; see [19] for constraints. GR carries this constant (a varying G would imply a violation of the strong equivalence principle, but only the weak principle is supported by the very precise Eötvös experiments) with the addition of the speed of light, which can be derived from the vacuum permittivity and permeability constants. Therefore, it is reasonable to question whether the gravitational constant is also a derived parameter.

The first historically suggestion of gravity arising necessarily as a consequence of the relativity of inertia came from Hans Reissner [20]. Schrödinger soon identified in

1925 [21] that GR did not fully implement Mach's principle, and proposed a relationship between the gravitational potential of distant masses and the speed of light: "*This remarkable relationship states that the (negative) potential of all masses at the point of observation, calculated with the gravitational constant valid at that observation point, must be equal to half the square of the speed of light.*". This conclusion was reached by imposing that the kinetic energy term had an origin in a potential-like interaction (such as all other forms of energy, which have an origin in an interaction) and was relational, so that $K = mv^2/2$ based on absolute mass and absolute velocity can be expressed through the gravitational Newtonian potential Gm/r so that $K = (Gm_i/r_{ij})(m_j v_{ij}^2/2c^2)$ with the need to introduce the square of the speed of light. The term $Gm_i/r_{ij}c^2 = GM_u/c^2R_u \sim 1$ is considered missing in the original Newtonian formulation, with M_u being the sum of all masses within R_u, the distance between the moving particle m_j and M_u. Schrödinger's formulation predicted a finite observable universe, since the potential of an infinite non-expanding eternal universe would also be infinite, even though the expansion of the universe was not known at the time. This relationship was later credited to Reissner's 1915 work by Schrödinger himself.

Schrödinger's formulation appeared repeatedly in several works after him [22–26], but it was popularized by Dennis Sciama, who developed in 1953 a modified cosmological vector potential theory of inertia in which the inertial law arises as a side effect of gravity [27]. Sciama realized that in the analogous case of Maxwell's equations applied to gravity, the rate of change of the vector potential leads to a term in the 'gravitoelectric' field that depends on the acceleration of an object relative to the rest of the masses of the universe. He showed that local inertial forces result from the gravitational induction of the universe, so that the dynamics of a rotating body would be affected by the large-scale distribution of the mass of the universe. Sciama postulated that "*In the rest-frame of any body the total gravitational field at the body arising from all the other matter in the universe is zero*". The induced inertia decreased with $1/r$ with r being the distance to the inertial body, so that the action of global matter dominates over the action of local matter, and it is not significantly modified by the acceleration with respect to local matter, leading to the illusion that inertia depends only on the body itself. The gravitational constant at any point was determined by the total potential of the distribution of matter of the universe, taking the form of $G \sim c^2/(\Phi_u + \phi)$ with $\Phi_u = M_u/R_u$ and local $\phi = M/r$ (assuming a linear superposition) which reduces to $G = c^2/(M_u/R_u)$ in most cases (when locally $\Phi_u >> \phi$, since only near neutron stars or black holes $\Phi_u \sim \phi$) with M_u and R_u the mass and radius of the observable universe, causally connected by the speed of light in the past light-cone of the point considered to evaluate G, for the equivalence of inertial and gravitational masses to hold. In this way, local phenomena were strongly coupled to global properties of the universe. Sciama knew about the Hubble expansion of the universe, but his model predicted an almost flat observable universe in which critical cosmic matter density is reached for $G\Phi_u/c^2 = 1$ to hold, even though the flatness of the universe was not observed at his time.

By pure dimensional analysis, the Reissner-Schrödinger- Sciama's relationship (henceforth Sciama's relationship) is the only possible derivation of the units of measurement of the gravitational constant through a mass, a distance, and the speed of light. If Sciama's relationship is derived from a more fundamental theory, it can be expected to also contain integers and mathematical constants, but these will be omitted for simplicity, since it does not significantly affect the resulting orders of magnitude of the comparison between M_u and R_u.

Robert H. Dicke and Carl H. Brans attempted to introduce Sciama's relationship in a relativistic scalar-tensor theory of gravitation in 1961 [28], inspired by Mach's principle, by interpreting the scalar field as an advanced wave integral over all matter. One interpretation of Brans-Dicke theory consists in the variation of the inertial mass with the scalar field if the gravitational constant is constant, so that Einstein's field equations are formally valid, but the most well-known interpretation allows the gravitational constant to be variable with position and time. The difference in predictions between the Brans-Dicke theory and GR has been locally tested in the solar system near the Sun through the Shapiro time delay effect measured by the Cassini experiment, showing that the Brans-Dicke dimensionless coupling factor of the scalar field to gravity, which differentiates between the theory and GR, must be so high that Brans-Dicke theory is almost indistinguishable from GR. All of these tests have been performed in the high-acceleration regime of the solar system.

Hans-Jürgen Treder developed a model similar to that of Schrödinger in a more complete inertia-free mechanics model [23] based on the Riemann potential as a velocity-dependent gravitational potential (instead of the Weber potential of Reissner and Schrödinger). This model implements the idea of inertia having a gravitational origin without Schrödinger's and Sciama's anisotropic inertial mass (which is ruled out by observations, such as the Hughes–Drever experiments) with Sciama's $G \sim c^2/(\Phi_u + \phi)$, in agreement with Mach's principle. It is worth noting that Treder was able to implement Mach's principle maintaining a scalar inertial mass instead of a tensorial one, exhibiting no preferred direction and thus avoiding inertial anisotropy. In this non-relativistic linear model, the weak equivalence principle is derived and not axiomatically postulated as in GR, the weakness of the gravitational constant is explained, and inertial mass is also derived from the model.

Sciama's relationship with today's measurements of the current cosmological model is satisfied only in terms of orders of magnitude, because the Hubble tension impedes a precise calculation. It yields the orders of magnitude of the observed gravitational constant when considering not the baryonic and dark matter content for M_u, but the total energy content (since all forms of energy must gravitate according to GR), which roughly corresponds to the critical density of the universe as measured to be almost flat. As Milgrom stated in one of his conferences, "*the only system that is strongly general relativistic and in the MOND regime is the Universe at large*". However, the critical density is calculated with the assumption of a *constant* gravitational constant itself and considering the dark matter content, which could be reduced through an effective correction to Newtonian laws, as will be proposed in the next section.

Few authors have attempted to develop a MOND for galaxy rotation curves based on Mach's principle [29], or relate Milgrom's MOND to Sciama's relationship and Mach's principle [30–36]. The most relevant ideas in the literature relating these topics are summarized onwards.

Alexander Unzicker explored galaxy dynamics in relation to Mach's Principle and the rotating bucket of water experiment, and proposed a basis for a MOND to solve flat rotation curves without considering Milgrom's MOND [29]. He proposes the following thought experiment. According to Mach, the system of two distant masses rotating around their center of mass in equilibrium of gravitational and centrifugal forces in an empty universe, since rotation between them is undefined due to absence of absolute space, must be equivalent to the system of those two distant masses not rotating and without gravity (only radial or relative velocities are measurable, and tangential velocities are impossible to measure instantaneously, as they are indistinguishable from a rotation of the coordinate system of the observer). The maximum possible angular velocity is limited by $w_{max} = c/r$ so that the maximum tangential speed is $v = c$ for one of the masses rotating around the other one, since the speed of light cannot be surpassed in any case. This luminal rotational speed in absence of a background universe from Unzicker's thought experiment is a feature of introducing Sciama's relationship in Newton's law of gravity, in which the Newtonian $v = \sqrt{GM/r}$ changes to $v = c\sqrt{(M/r)/(M_u/R_u)}$ after introducing $G = c^2/(M_u/R_u)$ and without a background universe $M_u \to M$ and $R_u \to r$, velocities are $v = c$. In other words, Sciama's relationship has the equivalent effect of decreasing inertia when decreasing M_u.

Jaume Gine explored a Machian interpretation of the modified second law of motion for the simple interpolating function of Milgrom's MOND based on the accelerated expansion of the universe [33], in which $a_0 = GM_u/{R_u}^2 = c^2/R_u$ through Sciama's relationship is the acceleration that an experimental body feels induced by the rest of the matter of the universe in its inertial frame of reference, and the value of acceleration for which inertial and gravitational masses of a body can differ. Thus, the equivalence principle between inertial and gravitational masses is broken for accelerations smaller than a_0. In a following paper [34], Gine attempted to derive a phenomenological version of Milgrom's MOND modification to Newton's second law. He considers a_0 to be the acceleration at which the edge of the universe at distance R_u goes away from a considered central inertial point due to the expansion of the universe. By relativizing acceleration considering the distant universe at rest, he derives a new interpolating function similar to Milgrom's MOND simple interpolating function.

A complete FundaMOND theory remains only a faint hope until the origin of Milgrom's MOND is correctly identified. Even though interesting ideas have been put forward relating MOND to Mach's principle, an exact equivalence based on the Machian consequence that the inertia of a body should decrease when masses are removed from its neighborhood (at the deep MOND regime) will be presented, which should arise as an approximation in any non-linear theory of modified inertia or gravity which attempts to explain galaxy rotation curves. The motivation behind this interpretation of Milgrom's MOND is that MOND can be reformulated with the same parameters of the mass and radius of the observable and causally connected universe that the Machian models of modified inertia contain.

2 Milgrom's MOND from Mach's Principle

Starting from the well-known form of Milgrom's MOND standard *or* simple interpolating function correction to Newtonian gravity *or* to inertia, the 'true' gravitational acceleration g *or* 'true' second law of inertia is

$$g = \frac{G}{\mu\left(\frac{g}{a_0}\right)}\frac{M}{r^2} = \frac{g_N}{\mu\left(\frac{g}{a_0}\right)} \quad or \quad F = \mu\left(\frac{g}{a_0}\right) m_i g \tag{1}$$

$$\mu\left(\frac{a}{a_0}\right) = \frac{1}{\sqrt{1+\left(\frac{a_0}{a}\right)^2}} \quad or \quad \mu\left(\frac{a}{a_0}\right) = \frac{1}{1+\frac{a_0}{a}} \tag{2}$$

with a_0 being Milgrom's MOND acceleration scale constant, $g_N = GM/r^2$ the Newtonian gravitational acceleration, M the active gravitational mass, m_i the inertial mass, and r the distance between the centers of mass of the active and passive gravitational masses. For the motion of a test particle in a gravitational field in the low acceleration or deep MOND regime, $\mu\left(a/a_0\right) \rightarrow a/a_0$ and $g = \sqrt{a_0 g_N}$, so that with $g = v^2/r$, $v = \sqrt[4]{GMa_0}$ for achieving flat galaxy rotation curves in agreement with the Tully-Fisher law in a remarkable simple way with a single new free parameter. For the high acceleration regime, $\mu\left(a/a_0\right) \rightarrow 1$ and Newtonian laws are recovered.

Milgrom's MOND acceleration scale constant a_0 is usually interpreted as a fundamental constant in MOND theory, but possibly related to the cosmological parameter of the cosmological constant $a_0 \sim c^2\sqrt{\Lambda}$ or the energy density of the vacuum ρ_{vac} with $\Lambda = 8\pi G\rho_{vac}/c^2$ according to GR, which coincides in orders of magnitude (a better match can be done with $a_0 = c^2\sqrt{\Lambda/3}$, although integers and Pi are onwards omitted for simplicity). This is due to Milgrom's wish to define absolute acceleration with respect to a smooth vacuum energy frame in MOND as modified inertia. But Mach's principle implies that acceleration should be defined with respect to the rest of the masses of the universe. Milgrom's reluctance to explore a Machian version could be attributed to the widespread false claim that Mach's principle necessarily implies anisotropy of inertia (due to non-uniform mass distributions), strongly constrained by the Hughes-Drever experiments. As shown by Treder, as long as inertial mass is a scalar, it exhibits no preferred direction, and no anisotropy of inertia. In order to express a_0 in terms of the mass and radius of the universe, ρ_{vac} is substituted for the matter density of the universe $\rho_{vac} \sim \rho_u \sim M_u/{R_u}^3$ with M_u the causally connected mass to the local system of study through the radius R_u of the observable universe (or particle horizon radius), a relationship which also coincides in orders of magnitude. Newton's gravitational constant G is substituted for Sciama's interpretation of Mach's principle $G \sim c^2/(M_u/R_u)$, so that $a_0 \sim GM_u/{R_u}^2 \sim c^2/R_u$, and by reverse engineering, $v \sim c\sqrt[4]{M/M_u}$ and $g \sim c^2\sqrt{M/M_u}/r$ in the equivalent deep MOND regime. The interpolating function is now scale-invariant in mass and length, resulting in a Machian MOND formulation

$$g = \frac{c^2}{\left(\frac{M_u}{R_u}+\frac{M}{r}\right)\,\mu\left(\sqrt{\frac{M/r^2}{M_u/{R_u}^2}}\right)}\frac{M}{r^2} \quad or \quad \frac{M}{r^2} = \frac{\left(\frac{M_u}{R_u}+\frac{M}{r}\right)}{c^2}\mu\left(\sqrt{\frac{M/r^2}{M_u/{R_u}^2}}\right) g \tag{3}$$

$$\mu\left(\sqrt{\frac{M/r^2}{M_u/{R_u}^2}}\right) = \frac{1}{\sqrt{1+\frac{M_u/{R_u}^2}{M/r^2}}} \quad or \quad \mu\left(\sqrt{\frac{M/r^2}{M_u/{R_u}^2}}\right) = \frac{1}{1+\sqrt{\frac{M_u/{R_u}^2}{M/r^2}}} \tag{4}$$

in which for most cases where local potentials are small, $\left(\frac{M_u}{R_u}+\frac{M}{r}\right) \sim \left(\frac{M_u}{R_u}\right)$.

It is trivial that, since only equivalent substitutions for G and a_0 according to Sciama's relationship based on Mach's principle which yield around the same values are done (omitting integers and mathematical constants), and M and r take the same meaning as in Milgrom's MOND, the resulting corrections (3) and (4) for galaxy rotation curves are equivalent to Milgrom's MOND. For $\frac{M_u/R_u{}^2}{M/r^2} << 1$, $\mu\left(\sqrt{\frac{M/r^2}{M_u/R_u{}^2}}\right) \to 1$ and Newtonian laws are restored with $v = c\sqrt{(M/r)/(M_u/R_u)}$. For $\frac{M_u/R_u{}^2}{M/r^2} >> 1$, $\mu\left(\sqrt{\frac{M/r^2}{M_u/R_u{}^2}}\right) \to \sqrt{\frac{M/r^2}{M_u/R_u{}^2}}$, $v = c\sqrt[4]{M/M_u}$, and the speed of rotating stars around galaxies far away from their galactic centers is determined by the relative active gravitational masses between the local and the global system.

It is also trivial that the denominator $M_u/R_u{}^2$ in (4) is the total field intensity (the gravitational field intensity without the gravitational constant G) of all masses of the universe (omitting dimensionless factors for simplicity), taking the same physical meaning as M/r^2 in the numerator, which is the field intensity of the masses of the local system at the point in which the correction is applied.

Thus, a transformation (4) can be identified between the non-inertial reference frame of a local system with a mass parameter M and distance r, and a global reference frame of observable mass M_u and size R_u of the universe. The variable changed by this transformation is either the gravitational force (equivalent to a variable gravitational constant) or the inertia of a body. Note that in (1), the interpolating function depends on true acceleration, while in Machian MOND it depends on field intensity. In particular, as modified inertia, it depends on the inertial mass, the local mass, and the mass of the universe, showing that inertia arises from the overall mass distribution à la Mach. It also makes use of the same parameters M_u and R_u as the modified inertia theories of Sciama and Treder.

By decreasing the mass of the universe within the observable radius, the transformation (4) decreases the gravitational force (or increases inertia) when $\frac{M_u/R_u{}^2}{M/r^2} >> 1$. But, considering also Sciama's relationship for the gravitational constant, the gravitational force increases (having the same observational consequence as considering Sciama's relationship alone, the decrease of inertia), and local velocities increase. The formal limit where $a_0 \to 0$ in which Milgrom's MOND approaches Newton's laws does not hold, since the mass and size of the universe (for which a_0 is effectively replaced) always exist. The mass of the universe can never be zero in the system under study, it can only be small as the very same mass of the system itself. Decreasing the mass of the universe until the absence of any background ($M_u \to M, R_u \to r$), the transformation (4) reduces to some simple number K which can be set to unity with the appropriate choice of integers in the formulation of the Machian transformation and in Sciama's relationship (a simple way to obtain $K = 1$ is considering Schrödinger's original definition of G and multiplying the relational field term of the transformation by 3), and velocities approach $v \to Kc$. This is already a Machian feature of Sciama's relationship by which in absence of a background universe, rotation is undefined, and tangential speed can take any value up to the limit of the speed of light. The proposed transformation respects this result and also respects the scale invariance in mass, length, and time that Sciama's relationship introduces in the Newtonian regime, while classical Newtonian gravity is scale-dependent. Milgrom's MOND is only length and time scale-invariant when $a_0 \to \infty$ fixing Ga_0 to achieve constant velocities independent of radius at the flat part of the galaxy rotation curves. Machian MOND is thus scale-invariant in mass, length, and time $(m, r, t) \to (\lambda m, \lambda r, \lambda t)$ at all regimes. Scaling the mass or size of both the universe and the local system always results in the same observational velocities.

3 Discussion

It is shown that Milgrom's MOND correction to Newtonian gravity or inertia can be reformulated based on Mach's principle without dimensional constants or free parameters except for the speed of light, ensuring mass, space and time scale invariance at all regimes.

Substituting both Newton's gravitational constant for Reissner-Schrödinger-Sciama's interpretation of Mach's principle $G \sim c^2/(M_u/R_u)$ with M_u and R_u (mass and radius of the observable universe) as global, measurable, variable and physical parameters relationship, and Milgrom's MOND acceleration constant for $a_0 \sim GM_u/{R_u}^2 \sim c^2/R_u$, yields equivalent results to Milgrom's MOND: in the deep-MOND regime, $v = \sqrt[4]{GMa_0} \sim c\sqrt[4]{M/M_u}$. The gravitational force or the inertial law depend on the relative field intensities of local and global mass distributions. In this way, the mass and size of the observable universe are always accounted for when considering non-inertial reference frames, in agreement with Mach's principle.

The relationship pointed out by Milgrom between his MOND's acceleration scale constant a_0 and the cosmological constant or the Hubble parameter is interpreted merely as accidental due to the cosmological coincidence by which, at the present epoch, the energy densities associated with dark energy (the cosmological constant) and visible matter are of the same orders of magnitude.

The Machian MOND's transformation is a function $\mu(M_u, R_u, M, r)$ (4) between different scales and non-inertial reference frames: the relationship between a local frame in spacetime and that of a global frame which is at rest (the background of the rest of the universe) with respect to the first one, with both being accelerated relative to each other. It expresses how gravity or inertia changes locally due to the background universe through relative masses and distances in both systems, in accordance with relational mechanics, and respects the weak equivalence principle since it does not depend on the inertial mass. The speed of light in Sciama's relationship takes the meaning of the speed of gravity and causality justifying the choice of R_u as the radius of the observable universe containing mass M_u, which is the part of the universe causally affecting local dynamics, for Mach's interpretation not to violate locality.

In Milgrom's MOND, the external field effect, which is behind the non-linearity of MOND, does not allow to ignore the external field (the field of the mother system) when considering internal dynamics. But somehow the field of the universe, in which all systems and embedded into, is ignored in Milgrom's MOND (if considered, all bodies in the universe would be above GM_u/R_u^2 and thus, in the high acceleration regime). By effectively replacing a_0 for M_u/R_u^2 in Machian MOND, the accelerated motion of the universe is considered due to the relativity of acceleration and inertia, when acceleration is defined with respect to the inertial frame of the universe à la Mach.

Although the origin of the Machian MOND transformation is not explained or derived here, it seems that it follows from Mach's principle's argument about which frame is acceleration defined with respect to. When the local field intensity is above that of the rest of the universe (at the cores of galaxies or within the solar system), the 'main frame' acceleration is defined with respect to is the local one, and one can describe the dynamics of the system without referring to the field intensity of the universe at large through the usual Newtonian laws. But, if the field intensity of the rest of the universe is greater than the one from the local system (in the deep MOND regime, such as the flat velocity part of the disk in galaxies), it seems that acceleration is defined with respect to the background frame of the universe, and the field intensity of the universe at large must be taken into account non-linearly together with that of the local frame, to explain the dynamics of the local system (in the case of Machian MOND, effectively through the Machian MOND transformation (4)). In between both regimes, the identification of the 'main frame' is shared between the local and global frames, described approximately by the slope of the Machian MOND transformation. This suggests that wherever local field intensities are higher than those of the universe at large, the Machian effects vanish (be that the one from Machian MOND, or from a varying gravitational constant in Brans-Dicke theory, in which tests showing the theory to be almost identical to GR are performed only at the high field intensity regime). And according to Mach, wherever local field intensities are below those of the universe at large, inertia is smaller and velocities are higher. This points towards the long-discussed problem of boundary conditions of the universe at large in GR, which has been linked to Mach's principle in the past.

Machian MOND satisfies several definitions of Mach's principle: Newton's gravitational constant G is a dynamical field (and the inertial mass of a body increases with the agglomeration of masses in its neighborhood), local inertial frames are affected by the cosmic distribution of matter, and rotation is undefined up to the speed of light in the absence of the rest of the universe (or equivalently, a body in an empty universe has no inertia). In agreement with Mach's principle, if inertial mass is defined by the potential of the mass distribution (following Sciama's argument), it is reasonable to think that it could also be affected by the field intensity of the mass distribution. Thus, Machian MOND introduces an even stronger (and non-linear, due to the external field effect of MOND) dependence of local dynamics on the distribution of matter than linear theories that just consider Sciama's relationship, and it can be considered more Machian.

Milgrom himself favors modifying inertia (which is the true aim of Mach's principle), and not just gravity. An example of modified inertia is that of SR $F = m_i d(\gamma v)/dt$ with γ being the Lorentz factor [37]. The original Milgrom's MOND as modified inertia with the inertial term of the equation of motion $\mu\,(g/a_0)\, m_i g$ is translation, rotation, and Galilei invariant, but leads to non-conservation of momentum in a many-body systems because the equation of motion is not derivable from an action. As a modification of the kinetic term in the Lagrangian, it conserves momentum, energy, and angular momentum, but is not Galilei invariant. A non-relativistic MONDian kinetic action for a particle with conservation laws of momentum, energy, and angular momentum, and the boost symmetry being Galilean, must be time non-local [10]. But this only occurs if it is a function of the trajectory $r(t)$ (since the function requires knowledge of the entire trajectory) and it is required to have a MOND limit when $a_0 \to 0$, which is neither the case in Machian MOND. We have omitted the original negative sign of Sciama's relationship $c^2 \sim -GM_u/R_u$ for simplicity. But by considering it and changing the sign in the Machian MOND transformation (4) from positive to negative for consistency, the resulting transformation

$$\mu\left(\sqrt{\frac{M/r^2}{M_u/{R_u}^2}}\right) = \frac{1}{\sqrt{1-\left(\frac{-M_u/{R_u}^2}{M/r^2}\right)}} \tag{5}$$

resembles the structure of a Lorentz-like transformation $\gamma = 1/\sqrt{1-\beta}$. If a function depending on velocities, such as the Lorentz factor, arises from Lorentz invariance in SR, where accelerations are absolute for consistency of the theory, one would expect a function depending on accelerations or field intensities, such as Machian MOND, to arise when relativizing accelerations. Lorentz invariance implies that the speed of light is constant in all inertial frames, and it seems that Machian MOND implies that the total field intensity at any point in the universe is always $> M_u/R_u^2$. It may be that Machian MOND arises from a new boost symmetry generalizing Lorentz invariance, for instance, through the relativity of acceleration, which is broken by the presence of the universe at large. Without a background universe, this symmetry would reduce to the Galilean or Lorentzian one. If inertia has an origin in gravitation, then the modification on the kinetic term can be based only on field intensities, without modifying the Poisson equation (and leaving gravity intact), which is shown in Machian MOND as modified inertia. The resulting Machian MOND formulation can be thought of as an in-between interpretation of modified inertia and modified gravity.

Machian MOND, or a similar and equivalent approximate formulation, is considered to be a necessary non-linear feature that a phenomenological theory of modified inertia or modified gravity which incorporates Mach's principle reduces to as an approximation, in order to agree with galaxy rotation dynamics. This might arise by imposing Mach's principle (for instance, through the relativity of accelerations and inertia of local objects being determined by the sources of the gravitational field at the global scale) to a non-Machian non-linear Lagrangian or to a theory of gravity such as GR (in which solutions to local dynamics do not depend on the mass or size of the observable universe). In fact, analogous interpolating functions in SR and GR have, as independent variables, dimensionless ratios of the form v/c and MG/Rc^2, similar to Machian MOND. The transformation function should arise naturally as an approximation, without the need of introducing it by hand (as in AQUAL or QUMOND), but the global M_u and R_u parameters governing local dynamics should appear in the complete FundaMOND theory, which is not the case in GR. So far, only linear and non-Lorentz invariant Machian theories of modified inertia with Sciama's relationship have been constructed, such as those of Treder and Schrödinger.

It is well known that, according to Bekenstein, a higher value for a_0 could resolve the tension between Milgrom's MOND and galaxy cluster's core dynamics (and that a_0 seems to grow with scale), where Newtonian accelerations take around that same value (provided that a_0 stays the same in galaxies for agreement with rotation curves). Machian MOND, although being equivalent to Milgrom's MOND, is effectively a model of a varying a_0 with time scale and could in principle help resolving this issue of Milgrom's MOND. It could also in principle solve the cosmological issues of Milgrom's MOND [38], such as those arising from a fixed scale-dependence on a_0. As modified gravity, the equivalent variation of the Newtonian G is both temporal due to the expanding universe and dependence on the mass and radius of the observable universe, and spatial since local potentials and local field intensities should also be included in Sciama's relationship and the Machian MOND transformation, respectively.

It would be interesting to compare Machian MOND with observational constraints of a varying effective gravitational constant at the early universe considering a cosmological model without most or all physical dark matter (assuming that Machian MOND solves at least partly the need for dark matter), due to its dependence on the mass and radius of the observable universe, which have both varied over time. A similar study has been done for quantized inertia by showing the prediction of a specific increase in the galaxy rotation anomaly at higher redshifts [39], but quantized inertia only makes the a_0 of Milgrom's MOND dependent on R_u through $a_o \sim c^2/R_u$. The formulation of quantized inertia is similar to the one proposed (without Sciama's relationship), but its interpretation is not based on Mach's principle. In fact, no Planck constant appears in the formulation, which suggests that its origin is not related to quantum phenomena, while the need to account for the universe at large directly points towards Mach's principle. More recently, it has been observed that there is no significant evolution in the baryonic Tully-Fisher law for redshifts up to $z \sim 2$ [40], implying an almost constant a_o in MOND. In contrast, Machian MOND shows a difference dependence compared to quantized inertia: the Machian MOND function depends not only on R_u, but also on M_u. If the term M_u/R_u^2 is more constant than the simple dependence on R_u, this observation would not rule out Machian MOND (the observable radius was smaller in the past, but so was the mass contained in the volume defined by the observable radius). But as explained before, Machian MOND depends on many uncertainties, such as the Hubble tension, whether some physical dark matter is present in M_u, missing integers and Pi in the formulation, and relativistic effects of the universe as a whole.

The form of the transformation (4) is not unique and depends on the chosen Milgrom's MOND interpolating function for which the Machian substitutions are made. Although (4) might be in conflict with solar system constraints, as it approaches $\mu(x) \to 1$ too slowly at the high acceleration regime, this is a problem of the slope of the transformation, which can be chosen to be different by another form for it.

The exact form of the transformation should not be fundamental but derived as an approximation from a complete FundaMOND theory of modified inertia á la Mach considering the relativity of acceleration and inertia. Furthermore, which integers and mathematical constants play a role within the Machian transformation and within Sciama's relationship is left to be discussed for a better agreement with observations.

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