A Brief Proof that
Pi is Irrational

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Abstract

It is shown that the limit of $\cos(j)$ and $\sin(j)$ as j goes to infinity does not exist. Using DeMoivre's theorem this implies the limits of $\sin(j!)$ and $\cos(j!)$ don't exist either. Assuming π is rational, its multiple can be expressed as a factorial. From this a contradiction is derived.

Proof

The following Lemma is taken from a youtube video: Dr. Barker video.

Lemma 1. *The limits*

$$\lim_{j\to\infty} \sin(j) \tag{1}$$

and

$$\lim_{j\to\infty} \cos(j) \tag{2}$$

don't exist.

Proof. Suppose (1) exists and equals L. Then

$$\lim_{j\to\infty} \sin(j+1) = \lim_{j\to\infty} \sin(j-1) = L.$$

Using trigonometric identities,

$$\sin(j+1) = \sin(j)\cos(1) + \cos(j)\sin(1) \tag{3}$$

and

$$\sin(j-1) = \sin(j)\cos(1) - \cos(j)\sin(1). \tag{4}$$

Using (3), we solve for $\cos(j)$:

$$\cos(j) = \frac{\sin(j+1) - \sin(j)\cos(1)}{\sin(1)}.$$

This gives

$$\lim_{j\to\infty} \cos(j) = \frac{L - L\cos(1)}{\sin(1)} = \frac{L(1-\cos(1))}{\sin(1)}, \tag{5}$$

implying the limit of $\cos(j)$ exists. Similarly, using (4), we solve for $\cos(j)$

$$\cos(j) = \frac{\sin(j)\cos(1) - \sin(j-1)}{\sin(1)}.$$

This gives

$$\lim_{j\to\infty} \cos(j) = \frac{L\cos(1) - L}{\sin(1)} = \frac{L(\cos(1)-1)}{\sin(1)}. \tag{6}$$

The only way (5) and (6) can be made consistent is if both limits are 0. Given that $1 - \cos(1) \neq 0$, this forces $L = 0$.

We now have

$$\lim_{j\to\infty} \sin(j) = \lim_{j\to\infty} \cos(j) = 0,$$

but this implies

$$\lim_{j\to\infty} \cos^2(j) + \sin^2(j) = 0,$$

contradicting the Pythagorean identity: giving $0 = 1$. Therefore the limits, (1) and (2) don't exist. $\square$

Lemma 2.

$$\lim_{j\to\infty} \sin(j!)$$

does not exist.

Proof. If it did exist, then using DeMoivre's theorem

$$\lim_{j\to\infty} (\cos(j!) + i\sin(j!))^{1/(j-1)!} = \lim_{j\to\infty} (\cos(j) + i\sin(j))$$

would exist, contradicting 1. $\square$

Remarks

Powers of $e^{in} = \cos n + i \sin n$ spin a point on the unit circle. If n was ever equal to π then the point on the unit circle would be -1 and powers of this would cycle and make convergence impossible. On the other hand, if $2n = \pi$, $e^{i2n} = 1$ and all whole number powers of this equal one allowing convergence. These observations show that for whole number increments in $\cos$ and $\sin$ arguments evaluations must become constants in order for convergence to take place. We use this idea to prove π is irrational next.

Theorem 1. *π is irrational.*

Proof. Assume $\pi = p/q$, then there exists a first $k!$ that has a factor of p in it and

$$n = k!\frac{q}{p}$$

for some integer n. This means $j! = m\pi$, and for all $j \geq k$ and some integer m. This implies $\sin(j!) = 0$ for all such j giving

$$\lim_{j\to\infty} \sin(j!) = 0,$$

contradicting Lemma 2.

□

References

[1] I. Niven, A simple proof that π is irrational, *Bull. Amer. Math. Soc.* **53** (1947) 509.

[2] ———, *Irrational Numbers*, Carus Mathematical Monographs, no. 11, Mathematical Association of America, Washington, DC, 1985.