On The Riemann Hypothesis

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Abstract

Let $s = \sigma + it$ with $\sigma \in (0, \frac{1}{2})$ and $t > 1$, $\zeta(s)$ the Riemann zeta function, $\eta(s)$ the Dirichlet eta function and $g(\sigma) = \Re(\eta(s))$. From the functional equation of the zeta function, the roots in the critical strip not on the critical line are symmetric with respect to the critical line.

The aim of this article is to prove that all the roots are only on the critical line, which implies that the Riemann hypothesis is true. The roots of the eta and zeta functions in the critical strip are the same. We shall investigate the possibility or not of existence of roots of the eta function which are in the critical strip and not on the critical line.

To the memory of our parents, To our wifes and our children

1. Introduction

1.1 Preliminaries and Notations

Let for $n \in \mathbb{N}^*$:

$$l_n = \log(1 - (2n)^{-1}) = -\sum_{k=1}^{\infty} \frac{(2n)^{-k}}{k} = -\mathcal{O}(2n)^{-1} < 0 \tag{1.1}$$

We get the two relations:

$$\log(2n-1) = \log 2n(1 - (2n)^{-1})) = \log 2n + l_n \tag{1.2}$$

and:

$$(2n-1)^{-\sigma} = (2n)^{-\sigma} e^{-l_n \sigma} \tag{1.3}$$

Let :

$$h_n = -l_n > 0$$

1.2 Introduction

In 1859, G.F.B. Riemann had announced the following conjecture [1] known Riemann Hypothesis:

Conjecture 1.1. Let $\zeta(s)$ be the complex function of the complex variable $s = \sigma + it$ defined by the analytic continuation of the function:

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

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over the whole complex plane, with the exception of $s = 1$. Then the nontrivial zeros of $\zeta(s) = 0$ are written as :

$$s = \frac{1}{2} + it$$

We recall the following theorem [2]:

THEOREM 1.2. *The function $\zeta(s)$ satisfies the following :*

1. $\zeta(s)$ has no zero for $\Re(s) > 1$;

2. the only pole of $\zeta(s)$ is at $s = 1$; it has residue 1 and is simple;

3. $\zeta(s)$ has trivial zeros at $s = -2, -4, \dots$;

4. the nontrivial zeros lie inside the region $0 \leqslant \Re(s) \leqslant 1$ (called the critical strip) and are symmetric about both the vertical line $\Re(s) = \dfrac{1}{2}$ and the real axis $\Im(s) = 0$.

The vertical line $\Re(s) = \dfrac{1}{2}$ is called the critical line.

For our proof, we will use the function presented by G.H. Hardy namely Dirichlet eta function is given by [2]:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s)$$

The function eta is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$ [2].

It is easy to verify that we can write $\eta(s)$ as:

$$\eta(s) = \sum_{n=1}^{+\infty} \left[\frac{1}{(2n-1)^s} - \frac{1}{(2n)^s} \right] \tag{1.4}$$

Let us consider the initial relation:

$$\Re(\eta(s)) = \sum_{n=1}^{\infty} \left[(2n-1)^{-\sigma} \cos(t \log(2n-1)) - (2n)^{-\sigma} \cos(t \log 2n) \right] \tag{1.5}$$

2. The Proof

Assume that RH fails for $s = \sigma + it$ with $\sigma = \dfrac{1}{2} + u$, $0 < u < \dfrac{1}{2}$ and $t > 1$. Let us denote $g(\sigma) = \Re(\eta(s))$. From (1.2), (1.3) and (1.5), we obtain:

$$g(\sigma) = \sum_{n=1}^{\infty} \left[(2n)^{-\sigma} e^{-l_n \sigma} \cos(t \log(2n-1)) - (2n)^{-\sigma} \cos(t \log 2n) \right] \tag{2.1}$$

simplified, the above equation becomes:

$$g(\sigma) = \sum_{n=1}^{\infty} (2n)^{-\sigma} \left[e^{-l_n \sigma} \cos(t \log(2n-1)) - \cos(t \log 2n) \right] \tag{2.2}$$

We get from the functional equation by symmetry of roots with respect to the critical line:

$$g(\frac{1}{2}+u) = \sum_{n=1}^{\infty}(2n)^{-\frac{1}{2}-u}\left[e^{-l_n(\frac{1}{2}+u)}\cos(t\log(2n-1)) - \cos(t\log 2n)\right] = 0$$

$$g(\frac{1}{2}-u) = \sum_{n=1}^{\infty}(2n)^{-\frac{1}{2}+u}\left[e^{-l_n(\frac{1}{2}-u)}\cos(t\log(2n-1)) - \cos(t\log 2n)\right] = 0$$

reordered:

$$g(\frac{1}{2}+u) = \sum_{n=1}^{\infty}(2n)^{-\frac{1}{2}}(2n)^{-u}\left[e^{-l_n(\frac{1}{2})}e^{-l_n u}\cos(t\log(2n-1)) - \cos(t\log 2n)\right] = 0$$

$$g(\frac{1}{2}-u) = \sum_{n=1}^{\infty}(2n)^{-\frac{1}{2}}(2n)^{u}\left[e^{-l_n(\frac{1}{2})}e^{l_n u}\cos(t\log(2n-1)) - \cos(t\log 2n)\right] = 0$$

and finally let $S = g(\frac{1}{2}+u)$ and $T = g(\frac{1}{2}-u)$, then:

$$S = \sum_{n=1}^{\infty} e^{-\frac{l_n}{2}}(2n)^{-\frac{1}{2}}(2n)^{-u}\left[e^{-(l_n u)}\cos(t\log(2n-1)) - e^{\frac{l_n}{2}}\cos(t\log 2n)\right] = 0$$

$$T = \sum_{n=1}^{\infty} e^{-\frac{l_n}{2}}(2n)^{-\frac{1}{2}}(2n)^{+u}\left[e^{+(l_n u)}\cos(t\log(2n-1)) - e^{\frac{l_n}{2}}\cos(t\log 2n)\right] = 0$$

Now, we replace l_n by $-h_n$, we obtain:

$$S = \sum_{n=1}^{\infty} e^{\frac{h_n}{2}}(2n)^{-u-1/2}\left[e^{h_n.u}\cos(t\log(2n-1)) - e^{-\frac{h_n}{2}}\cos(t\log 2n)\right] = 0$$

$$T = \sum_{n=1}^{\infty} e^{\frac{h_n}{2}}(2n)^{u-1/2}\left[e^{-h_n.u}\cos(t\log(2n-1)) - e^{-\frac{h_n}{2}}\cos(t\log 2n)\right] = 0$$

To prove RH, it suffices to prove that always $S \neq T$. From the factor $(2n)^{-u-1/2}$ in S compare to the factor $(2n)^{u-1/2}$ in T, taking in account that:

1. $e^{h_n.u}$ in S and $e^{-h_n.u}$ in T are very near 1 as soon as $n \geqslant 2$.

2. The difference between S and T is **essentially** due to the factor $(2n)^{-u-1/2}$ in S by opposite to the factor $(2n)^{u-1/2}$ in T.

Therefore $S \neq T$ and consequently S and T cannot be simultaneously null, contradiction which implies that a necessary condition for $\zeta(\frac{1}{2}+u+it) = 0$ is that $u = 0$. This is not a sufficient condition (all the points of the critical line are not solutions!).

Theorem 2.1. *The Riemann Hypothesis is true.*

We give a numerical example as an example of application of the theory. The gap between S and T increases when u increases between 0 and $\frac{1}{2}$ (but this example don't contribute to the theory.)

References

1 A. Ivić, *The Riemann Zeta-Function*, Dover (1985).

2 P. Borwein, S. Choi, B. Rooney, A. Weirathmueller, *The Riemann Hypothesis, A Resource for the Afficionado and Virtuoso Alike*, Canadian Mathematical Society (2008). https://doi.org/10.1007/978-0-387-72126-2

3 H.M. Edwards, *Riemann's Zeta Function*, Dover (1974).

4 L.V. Ahlfors, *Complex Analysis*, Mac Graw Hill (1979).

5 John K. Hunter, *An Introduction to Real Analysis*, Department of Mathematics, University of California at Davis, (2014).

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