# A simple equation to calculate the Gaussian curvature of space-time according to the Schwarzschild model

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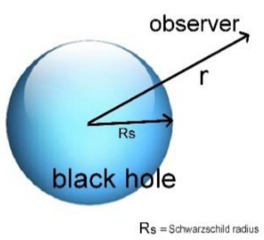
**Abstracs**

The Flamm paraboloid is a space-time solution to the problem posed by Schwarzschild concerning finding a solution other than the trivial one to Einstein's equations in an empty space. For this reason, it also represents the space that exists in the contour of a gravitational point mass. The solution of J. Droste that we study here, the Flamm paraboloid, leads to calculating values of the Gaussian curvature of its space-time that require a complicated process of obtaining. We have found a simple equation that allows calculating these values in a simple way. This equation obtained here by means of an algebraic expansion is exact in the space-time represented by the Flamm paraboloid.

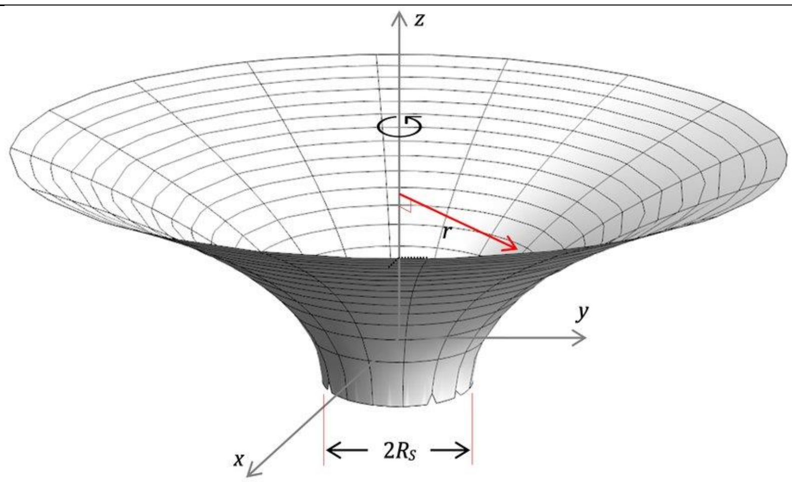
**Keywords:** Schwarzschild model. Gauss curvature of space-time. J. Droste solution

## 1.- Introduction

In Fig. 1, the physical problem that is posed here is represented, calculating the curvature of space-time at the points surrounding a supposed spherical gravitational mass that we have called a "black hole". In 1916, Schwarzschild carried out a study of Einstein's equations for this assumption. The solution studied here proposed by J. Droste is the Flamm paraboloid, represented in Fig. 1. It is a 2D surface of infinite measure and negative Gauss curvature. In addition, we will represent it by means of cylindrical coordinates, which is also a function of the Schwarzschild radius Rs of the gravitational mass that generates it. We will study it algebraically and we will find an equation that allows us to easily calculate the values of the Gauss curvature at each point. As we know, since it is a 2D surface, the value of the curvature scalar will be twice the value of the Gauss curvature.




The physical problem in Euclidean space




Space-time in the Schwarzschild metric Flamm paraboloid

**Fig. 1**

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## 2.-Resolution of the mathematical problem. Gaussian curvature and curvature scalar of the Schwarzschild spacetime in the J. Droste solution

The Flamm paraboloid, J. Droste's spacetime solution to the problem studied by Schwarzschild [1], is a 2D surface. Its geometry allows us to parameterize the paraboloid as a function of the observer's distance from the point mass "r" and the azimuth angle "φ". The problem admits a mathematical treatment of differential geometry of surfaces [2], and with it we are going to calculate the Gaussian Curvature. ($R_s$ = Schwarzschild radius). Since it is a 2D surface, the curvature scalar is obtained by multiplying the Gaussian curvature by two.

*The surface*

Surface parameters (r, φ)

$0 \leq r < \infty, \quad 0 \leq \varphi < 2\pi$

which has this parametric equation:

$x = r \cos\varphi$

$y = r \operatorname{sen}\varphi$

$z = 2(Rs\,(r- Rs))^{1/2}$

Vector Equation of the Flamm paraboloide

$f\,(x,y,z) = (r \cos\varphi,\ r \operatorname{sen}\varphi,\ 2(Rs(r- Rs))^{1/2})$

*Determination of velocity, acceleration, and normal vectors to the surface*

$\eth f/\eth\varphi = (-r \operatorname{sen}\varphi, r\cos\varphi, 0)$

$\eth f/\eth r = (\cos\varphi, \operatorname{sen}\varphi, (r/Rs -1)^{-1/2})$

$\eth^2 f/\eth\varphi^2 = (-r \cos\varphi, -r \operatorname{sen}\varphi, 0)$

$\eth^2 f/\eth r^2 = (0, 0, (-1/2R_s).\ (r/Rs -1)^{-3/2})$

$\eth f/\eth\varphi\eth r = (-\operatorname{sen}\varphi, \cos\varphi, 0)$

$$n = \frac{(\eth f/\eth\varphi \times \eth f/\eth r)}{[\frac{\eth f}{\eth\varphi} \times \frac{\eth f}{\eth r}]}$$

$(\eth f/\eth\varphi \times \eth f/\eth r) = (r\cos\varphi/\,(r/Rs\ -1)^{1/2},\ \ r\operatorname{sen}\varphi/(r/Rs\ -1)^{1/2},\ \ -r)$

$[\frac{\eth f}{\eth\varphi} \times \frac{\eth f}{\eth r}] = r\,((1/\,(r/Rs -1)) +1)^{1/2}$

*Curvature and curvature parameters*

**Gauss curvature**

$$\mathbf{K_{gauss} = LN\text{-}M^2/EG\text{-}F^2}$$

$L = ð^2f/ð\varphi^2.\ n = -r(r/Rs)^{-1/2}$

$N = ð^2f/ðr^2.\ n = (1/2Rs)\ (r/Rs)^{-1/2}\ (r/Rs - 1)^{-1}$

$M = (ðf/ð\varphi ðr).\ n = 0$

$E = ðf/ð\varphi.\ \ ðf/ð\varphi = r^2$

$G = ðf/ðr.\ ðf/ðr = 1 + (1/\ (r/Rs - 1))$

$F = ðf/ð\varphi.\ ðf/ðr = 0$

**An equation of Gauss curvature**

$$\mathbf{K_{gauss} = -R_S/2r^3}$$

**for Schwarzschild radius, $R_S = 2GM/c^2$**

**3.- Conclusions**

We have obtained a simple equation that, as a function of the Schwarzschild radius and the distance to the gravitational point mass, allows us to calculate the Gaussian curvature of the Schwarzschild space-time in the solution of J. Droste. To obtain this equation, we have made an extensive algebraic calculation of differential geometry of surfaces in which we have arrived at a simple equation that is expressed as a function of the variables r distance and $R_S$ Schwarzschild radius. This equation gives values of Gaussian curvature and therefore also in this case, since it is a 2D surface, values of the curvature scalar, the latter being equal to twice the Gaussian curvature.

**References**

[1] K. Schwarzschild. On the Gravitational Field of a Mass Point according to Einstein's Theory. 1916. **arXiv:physics/9905030v1 [physics.hist-ph] 1999-06-12**

[2] Differential surface geometry. Googlescholar [Crossref]. 2023. Available from [www.Wikipedia.org](www.Wikipedia.org)