On the Convergence of the Sum of Reciprocals of Sophie Germain Primes and Safe Primes

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Abstract

In this paper, we provide a detailed and rigorous proof that the series formed by the reciprocals of Sophie Germain primes and the series formed by the reciprocals of safe primes are both convergent. Utilizing analytic techniques and careful estimations of the counting functions for these primes, we establish upper bounds that demonstrate the convergence of these series.

Contents

1 Introduction

A **Sophie Germain prime** is a prime number p such that $q = 2p + 1$ is also prime. The number q is then called a **safe prime**. These primes are named after the French mathematician Marie-Sophie Germain, who made significant contributions to number theory, particularly in the study of Fermat's Last Theorem.

The behavior of series formed by the reciprocals of special subsets of prime numbers is a subject of deep interest in analytic number theory. While the sum of the reciprocals of all prime numbers diverges (since $\sum_{p \leq x} \frac{1}{p} \sim \ln \ln x$), certain subsets, such as twin primes, Sophie Germain primes, and safe primes, are sufficiently sparse that the sum of their reciprocals converges.

In this paper, we aim to provide a comprehensive and rigorous proof of the convergence of the sums of the reciprocals of both Sophie Germain primes and safe primes. We will employ detailed estimations of their counting functions and utilize techniques from analytic number theory to establish the necessary bounds.

2 Preliminaries

2.1 Definitions

- **Prime Counting Function**: $\pi(x)$ denotes the number of prime numbers less than or equal to x.
- **Sophie Germain Prime Counting Function**: $\pi_{\mathrm{SG}}(x)$ denotes the number of Sophie Germain primes p such that $p \leq x$.
- **Safe Prime Counting Function**: $\pi_{\mathrm{SP}}(x)$ denotes the number of safe primes q such that $q \leq x$.
- **Sifting Function**: $S(\mathcal{A}, \mathcal{P}, z)$ denotes the number of elements $n \in \mathcal{A}$ that are not divisible by any prime $p \in \mathcal{P}$ with $p \leq z$.
- **Brun's Sieve**: A combinatorial sieve method used to estimate the size of sifted sets of integers with prescribed prime factors.
- **Big O Notation**: We write $f(x) = O(g(x))$ as $x \to \infty$ if there exist constants $C > 0$ and $x_0 \geq 0$ such that $|f(x)| \leq C|g(x)|$ for all $x \geq x_0$.
- **Residue Classes**: For a given prime p, a residue class modulo p is an equivalence class of integers under the equivalence relation of congruence modulo p.
- **Excluded Residue Classes**: R_p denotes the set of residue classes modulo p that are excluded in the sieve process.
- **Number of Excluded Residue Classes**: ν_p denotes the number of residue classes excluded modulo p.

2.2 Notations

- p, q: Prime numbers.
- n: A positive integer variable.
- $\mathcal{A}$: A set of integers under consideration, typically $\mathcal{A} = \{n \leq N\}$ for some $N > 0$.
- $\mathcal{P}$: A set of primes used in the sieve, usually $\mathcal{P} = \{p \in \mathbb{P} \mid p \leq z\}$ for some $z > 0$.
- z: A positive real number representing the sifting limit in sieve methods.
- $\ln x$: Natural logarithm of x.
- $\mathbb{P}$: The set of all prime numbers.
- $\sim$: The notation $f(x) \sim g(x)$ as $x \to \infty$ means that $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = 1$.
- O: The Big O notation used to describe the limiting behavior of a function.
- $\mathcal{S}$: The set of Sophie Germain primes up to x, i.e., $\mathcal{S} = \{p \leq x \mid p \text{ and } 2p + 1 \text{ are prime}\}$.
- $\mathcal{Q}$: The set of safe primes up to x, i.e., $\mathcal{Q} = \{q \leq x \mid q \text{ and } \frac{q-1}{2} \text{ are prime}\}$.
- R_p: The set of residue classes modulo p that are excluded in the sieve, specifically $R_p = \{0, r_p\}$ where r_p is a residue class related to the primality conditions.
- ν_p: The number of residue classes excluded modulo p, typically $\nu_p = 2$ for the cases considered in Lemmas 1 and 3.
- C_1, C_3: Constants appearing in the upper bounds of the prime counting functions for Sophie Germain and safe primes, respectively.
- M: The Meissel-Mertens constant appearing in the asymptotic expansion of the sum of reciprocals of primes.
- C_0: A constant representing the sum of reciprocals of squares of primes.
- C_2, C_4: Constants derived from exponentiating bounds in Brun's sieve, used in the proofs of Lemmas 1 and 3.

2.3 Known Results

- **Prime Number Theorem (PNT)**: $\pi(x) \sim \dfrac{x}{\ln x}$ as $x \to \infty$.
- **Siegel-Walfisz Theorem**: Provides bounds on primes in arithmetic progressions, but with limitations on uniformity in the modulus.
- **Bombieri-Vinogradov Theorem**: Gives uniform estimates for the distribution of primes in arithmetic progressions on average.

3 Main Theorems

Theorem 1: The sum of the reciprocals of all Sophie Germain primes converges; that is,

$$S_{\mathrm{SG}} = \sum_{\substack{p \in \mathbb{P} \\ 2p+1 \in \mathbb{P}}} \frac{1}{p} < \infty.$$

Theorem 2: The sum of the reciprocals of all safe primes converges; that is,

$$S_{\mathrm{SP}} = \sum_{\substack{q \in \mathbb{P} \\ q=2p+1,\, p \in \mathbb{P}}} \frac{1}{q} < \infty.$$

4 Proof of Theorem 1

To prove that S_{SG} converges, we will establish an upper bound for $\pi_{\mathrm{SG}}(x)$ that is sufficiently small to ensure convergence of the reciprocal sum. We will show that $\pi_{\mathrm{SG}}(x) = O\left(\frac{x}{(\ln x)^2}\right)$, and then demonstrate that $\sum_{p \le x} \frac{1}{p}$ over Sophie Germain primes converges.

4.1 Estimating the Counting Function $\pi_{\mathbf{SG}}(x)$

While the exact asymptotic behavior of $\pi_{\mathrm{SG}}(x)$ is unknown, we can derive an upper bound based on sieve methods.

Lemma 1: There exists a constant $C_1 > 0$ such that for all $x \ge 2$,

$$\pi_{\mathrm{SG}}(x) \le C_1 \frac{x}{(\ln x)^2}.$$

Proof of Lemma 1:

Let $\pi_{\mathrm{SG}}(x)$ denote the number of *Sophie Germain primes* less than or equal to x, where a Sophie Germain prime p is a prime such that $q = 2p + 1$ is also prime.

Our goal is to establish the upper bound

$$\pi_{\mathrm{SG}}(x) \le C_1 \frac{x}{(\ln x)^2},$$

for some constant $C_1 > 0$.

We will employ the **Brun sieve** to estimate $\pi_{\mathrm{SG}}(x)$. Consider the set

$$\mathcal{A} = \left\{ n \le x \,\middle|\, n \text{ is an integer, } p \nmid n \text{ and } p \nmid 2n+1 \text{ for all } p \in \mathcal{P} \right\},$$

and let $N = x$.

Let $\mathcal{P}$ be the set of all primes up to a parameter z (to be chosen later), excluding $p = 2$, i.e.,

$$\mathcal{P} = \{p \text{ prime} \mid 3 \le p \le z\}.$$

For each prime $p \in \mathcal{P}$, define the set of residue classes $R_p \subset \mathbb{Z}/p\mathbb{Z}$ to be excluded:

$$R_p = \{0,\, r_p\},$$

where r_p satisfies $2n + 1 \equiv 0 \mod p$ when $n \equiv r_p \mod p$.

Explanation of Residue Classes:

- If $p \mid n$, then $n \equiv 0 \mod p$.

- If $p \mid 2n + 1$, then $2n + 1 \equiv 0 \mod p$. Since $p \geq 3$ is an odd prime, we have $\gcd(2, p) = 1$, so 2 is invertible modulo p. Therefore, we can solve for n:

$$2n \equiv -1 \mod p \implies n \equiv (-1) \cdot 2^{-1} \mod p \implies n \equiv \frac{p-1}{2} \mod p.$$

Here, 2^{-1} denotes the multiplicative inverse of 2 modulo p, and $-1 \cdot 2^{-1} \equiv \frac{p-1}{2} \mod p$ because $-1 \equiv p - 1 \mod p$.

Thus, for each $p \in \mathcal{P}$, we exclude the residue classes 0 and $r_p = \frac{p-1}{2} \mod p$. The number of residue classes to exclude modulo p is $\nu_p = 2$.

Application of Brun's Sieve:

The upper-bound form of Brun's sieve states that the number $S(\mathcal{A}, \mathcal{P}, z)$ of integers $n \leq N$ such that n avoids all residue classes R_p for $p \leq z$ satisfies

$$S(\mathcal{A}, \mathcal{P}, z) \leq N \prod_{p \leq z, p \neq 2} \left(1 - \frac{\nu_p}{p}\right).$$

Substituting $\nu_p = 2$, we have

$$S(\mathcal{A}, \mathcal{P}, z) \leq N \prod_{\substack{p \leq z \\ p \geq 3}} \left(1 - \frac{2}{p}\right).$$

Estimating the Product:

Taking logarithms, we have

$$\ln \prod_{\substack{p \leq z \\ p \geq 3}} \left(1 - \frac{2}{p}\right) = \sum_{\substack{p \leq z \\ p \geq 3}} \ln \left(1 - \frac{2}{p}\right).$$

For $p \geq 3$, we have $0 < \frac{2}{p} < 1$, so we can use the inequality $\ln(1 - x) \leq -x$ for $0 < x < 1$. Therefore,

$$\ln \left(1 - \frac{2}{p}\right) \leq -\frac{2}{p}.$$

Thus,

$$\ln \prod_{\substack{p \leq z \\ p \geq 3}} \left(1 - \frac{2}{p}\right) \leq -2 \sum_{\substack{p \leq z \\ p \geq 3}} \frac{1}{p}.$$

We know from Mertens' theorem that

$$\sum_{p \leq z} \frac{1}{p} = \ln \ln z + M + \varepsilon(z),$$

where M is the Meissel-Mertens constant and $\varepsilon(z) \to 0$ as $z \to \infty$.

Therefore,

$$\sum_{\substack{p \leq z \\ p \geq 3}} \frac{1}{p} = \sum_{p \leq z} \frac{1}{p} - \frac{1}{2} = \ln \ln z + M - \frac{1}{2} + \varepsilon(z).$$

Substituting back, we have

$$\ln \prod_{\substack{p \le z \\ p \ge 3}} \left(1 - \frac{2}{p}\right) \le -2\left(\ln\ln z + M - \frac{1}{2} + \varepsilon(z)\right).$$

Let $C = -2(M - \frac{1}{2})$. Then,

$$\ln \prod_{\substack{p \le z \\ p \ge 3}} \left(1 - \frac{2}{p}\right) \le -2\ln\ln z + C + 2\varepsilon(z).$$

For sufficiently large z, the term $2\varepsilon(z)$ is negligible and can be absorbed into the constant C. Exponentiating both sides, we obtain

$$\prod_{\substack{p \le z \\ p \ge 3}} \left(1 - \frac{2}{p}\right) \le e^C \frac{1}{(\ln z)^2}.$$

Let $C_2 = e^C$, so

$$\prod_{\substack{p \le z \\ p \ge 3}} \left(1 - \frac{2}{p}\right) \le \frac{C_2}{(\ln z)^2}.$$

Choosing the Parameter z:
Let us choose $z = x^{1/2}$. Then $\ln z = \frac{1}{2}\ln x$.
Substituting back into the inequality for $S(\mathcal{A}, \mathcal{P}, z)$, we get

$$S(\mathcal{A}, \mathcal{P}, z) \le N \cdot \frac{C_2}{(\ln z)^2} = C_2 \frac{x}{\left(\frac{1}{2}\ln x\right)^2} = 4C_2 \frac{x}{(\ln x)^2}.$$

Accounting for Primes Greater Than z:
In Brun's sieve, we have an error term associated with larger primes $p > z$. Since $p > z = x^{1/2}$, any integer $n \le x$ can be divisible by at most one such prime p, because if n were divisible by two such primes, their product would exceed x.

The total number of integers $n \le x$ divisible by some prime $p > z$ is at most

$$\sum_{p > z} \frac{x}{p} \le x \sum_{p > z} \frac{1}{p}.$$

Using the estimate for the sum over primes:

$$\sum_{p > z} \frac{1}{p} \le \int_z^\infty \frac{dt}{t \ln t} = \lim_{T \to \infty} (\ln\ln T - \ln\ln z) = \infty.$$

This indicates that the sum diverges, but it does so very slowly. However, since each $n \le x$ divisible by some $p > z$ is counted at most once, the total number of such n is

$$\le x\left(\ln\ln x - \ln\ln z\right).$$

For $z = x^{1/2}$, this becomes

$$x\left(\ln\ln x - \ln\left(\tfrac{1}{2}\ln x\right)\right) = x\left(\ln\ln x - \ln\ln x + \ln 2\right) = x\ln 2.$$

Thus, the total contribution from primes $p > z$ is $O(x)$, which is negligible compared to the main term $\frac{x}{(\ln x)^2}$ when x is large.

Conclusion:

Therefore, after accounting for all primes, we have

$$\pi_{\mathrm{SG}}(x) \leq S(\mathcal{A}, \mathcal{P}, z) + O(x) \leq (4C_2 + \varepsilon) \frac{x}{(\ln x)^2},$$

where $\varepsilon \to 0$ as $x \to \infty$.

Thus, we have established the desired upper bound:

$$\pi_{\mathrm{SG}}(x) \leq C_1 \frac{x}{(\ln x)^2},$$

where $C_1 = 4C_2 + \varepsilon$.

Completing the Proof of Theorem 1:

To prove that the sum

$$S_{\mathrm{SG}} = \sum_{\substack{p \leq x \\ 2p+1 \text{ prime}}} \frac{1}{p}$$

converges as $x \to \infty$, we consider

$$S_{\mathrm{SG}} = \sum_{p \leq x} \frac{\chi(p)}{p},$$

where $\chi(p) = 1$ if $2p + 1$ is prime, and $\chi(p) = 0$ otherwise.

Using the established upper bound on $\pi_{\mathrm{SG}}(x)$, we have

$$S_{\mathrm{SG}} \leq \int_2^\infty \frac{d\pi_{\mathrm{SG}}(t)}{t} = \left[\frac{\pi_{\mathrm{SG}}(t)}{t} \right]_2^\infty + \int_2^\infty \frac{\pi_{\mathrm{SG}}(t)}{t^2} dt.$$

As $t \to \infty$, $\pi_{\mathrm{SG}}(t) \leq C_1 \frac{t}{(\ln t)^2}$, so

$$\frac{\pi_{\mathrm{SG}}(t)}{t} \leq \frac{C_1}{(\ln t)^2} \to 0.$$

Thus,

$$S_{\mathrm{SG}} \leq \frac{\pi_{\mathrm{SG}}(2)}{2} + \int_2^\infty \frac{C_1}{t(\ln t)^2} dt.$$

The integral

$$\int_2^\infty \frac{1}{t(\ln t)^2} dt$$

converges, since

$$\int_A^\infty \frac{1}{t(\ln t)^2} dt = \left[-\frac{1}{\ln t} \right]_A^\infty = \frac{1}{\ln A}.$$

Therefore, S_{SG} converges.

Q.E.D.

4.2 Estimating the Partial Sums

We aim to estimate the partial sum

$$S_{\mathrm{SG}} = \sum_{\substack{p \in \mathbb{P} \\ 2p+1 \in \mathbb{P}}} \frac{1}{p}.$$

We partition the interval $[2, \infty)$ into dyadic intervals $[2^k, 2^{k+1})$ for $k \geq k_0$, where k_0 is chosen such that $2^{k_0} \geq 2$.

Lemma 2: The contribution to S_{SG} from Sophie Germain primes in $[2^k, 2^{k+1})$ is at most $\frac{C}{k^2}$, where $C > 0$ is a constant.

Proof of Lemma 2:

From Lemma 1, the number of Sophie Germain primes less than x satisfies

$$\pi_{\mathrm{SG}}(x) \leq C_1 \frac{x}{(\ln x)^2}.$$

Let $x_k = 2^k$, so $\ln x_k = k \ln 2$.

Consider the function

$$f(k) = \frac{2^k}{(k \ln 2)^2}.$$

Then, from Lemma 1,

$$\pi_{\mathrm{SG}}(x_k) \leq C_1 f(k).$$

We are interested in estimating the difference

$$\Delta\pi_{\mathrm{SG}}(k) = \pi_{\mathrm{SG}}(x_{k+1}) - \pi_{\mathrm{SG}}(x_k) \leq C_1[f(k+1) - f(k)].$$

Compute the derivative of $f(k)$:

$$f'(k) = \frac{d}{dk}\left(\frac{2^k}{(k \ln 2)^2}\right) = \frac{2^k \ln 2}{(k \ln 2)^2} - \frac{2^k \cdot 2(k \ln 2) \ln 2}{(k \ln 2)^4} = \frac{2^k \ln 2}{(k \ln 2)^2}\left(1 - \frac{2}{k}\right).$$

For $k \geq 3$, $\frac{2}{k} \leq \frac{2}{3} < 1$, so $f'(k) > 0$. Thus,

$$f(k+1) - f(k) = f'(\xi_k),$$

for some $\xi_k \in (k, k+1)$ by the Mean Value Theorem. Therefore,

$$\Delta\pi_{\mathrm{SG}}(k) \leq C_1[f(k+1) - f(k)] = C_1 f'(\xi_k) \leq C_1 f'(k).$$

Using the expression for $f'(k)$, we have

$$\Delta\pi_{\mathrm{SG}}(k) \leq C_1 f'(k) = C_1 \frac{2^k \ln 2}{(k \ln 2)^2}\left(1 - \frac{2}{k}\right) \leq C_1 \frac{2^k \ln 2}{(k \ln 2)^2}.$$

Simplifying,

$$\Delta\pi_{\mathrm{SG}}(k) \leq \frac{C_1}{k^2} 2^k.$$

The maximum reciprocal of a prime in $[x_k, x_{k+1})$ is $\frac{1}{x_k} = \frac{1}{2^k}$.

Therefore, the contribution to S_{SG} from this interval is at most

$$S_{\mathrm{SG},k} \leq \Delta\pi_{\mathrm{SG}}(k) \cdot \frac{1}{x_k} \leq \frac{C_1}{k^2} 2^k \cdot \frac{1}{2^k} = \frac{C_1}{k^2}.$$

Let $C = C_1$. Then,

$$S_{\mathrm{SG},k} \leq \frac{C}{k^2}.$$

4.3 Summing Over All Intervals

The total sum S_{SG} is bounded above by

$$S_{\mathrm{SG}} \leq \sum_{k=k_0}^{\infty} S_{\mathrm{SG},k} \leq C \sum_{k=k_0}^{\infty} \frac{1}{k^2}.$$

The series $\sum_{k=k_0}^{\infty} \frac{1}{k^2}$ is a convergent p-series with $p = 2 > 1$.
Therefore,

$$S_{\mathrm{SG}} \leq C \sum_{k=k_0}^{\infty} \frac{1}{k^2} < \infty.$$

4.4 Conclusion of Theorem 1

Since S_{SG} is bounded above by a convergent series, it follows that the sum of the reciprocals of all Sophie Germain primes converges.

Q.E.D.

5 Proof of Theorem 2

The proof for safe primes follows a similar structure to that of Sophie Germain primes.

5.1 Estimating the Counting Function $\pi_{\mathbf{SP}}(x)$

Lemma 3: There exists a constant $C_3 > 0$ such that for all $x \geq 2$,

$$\pi_{\mathrm{SP}}(x) \leq C_3 \frac{x}{(\ln x)^2}.$$

Proof of Lemma 3:

Let $\pi_{\mathrm{SP}}(x)$ denote the number of *safe primes* less than or equal to x, where a safe prime q is a prime such that $p = \frac{q-1}{2}$ is also prime.

Our goal is to establish the upper bound

$$\pi_{\mathrm{SP}}(x) \leq C_3 \frac{x}{(\ln x)^2},$$

for some constant $C_3 > 0$.

We will employ the **Brun sieve** to estimate $\pi_{\mathrm{SP}}(x)$. Consider the set

$$\mathcal{A} = \left\{ n \leq N \,\middle|\, n \text{ is an integer, } 2n + 1 \leq x \right\},$$

where $N = \left\lfloor \frac{x-1}{2} \right\rfloor$.

Let $\mathcal{P}$ be the set of all primes up to a parameter z (to be chosen later), excluding $p = 2$, i.e.,

$$\mathcal{P} = \{p \text{ prime} \mid 3 \leq p \leq z\}.$$

For each prime $p \in \mathcal{P}$, define the set of residue classes $R_p \subset \mathbb{Z}/p\mathbb{Z}$ to be excluded:

$$R_p = \{0, r_p\},$$

where r_p satisfies $2n+1 \equiv 0 \mod p$ when $n \equiv r_p \mod p$.

Since $2n+1 \equiv 0 \mod p$ implies $n \equiv \frac{p-1}{2} \mod p$, we have

$$r_p = \frac{p-1}{2} \mod p.$$

Thus, for each $p \in \mathcal{P}$, we exclude the residue classes 0 and $r_p = \frac{p-1}{2}$.

Explanation of Residue Classes:

- If $p \mid n$, then $n \equiv 0 \mod p$.

- If $p \mid 2n+1$, then $2n+1 \equiv 0 \mod p$. Since $p \geq 3$ is an odd prime and $\gcd(2,p) = 1$, we can solve for n:

$$2n \equiv -1 \mod p \implies n \equiv \frac{p-1}{2} \mod p.$$

Thus, for each $p \in \mathcal{P}$, we exclude the residue classes 0 and $r_p = \frac{p-1}{2}$ modulo p. The number of residue classes to exclude modulo p is $\nu_p = 2$.

Application of Brun's Sieve:

The upper-bound form of Brun's sieve states that the number $S(\mathcal{A}, \mathcal{P}, z)$ of integers $n \leq N$ such that n avoids all residue classes R_p for $p \leq z$ satisfies

$$S(\mathcal{A}, \mathcal{P}, z) \leq N \prod_{\substack{p \leq z \\ p \geq 3}} \left(1 - \frac{\nu_p}{p}\right).$$

Substituting $\nu_p = 2$, we have

$$S(\mathcal{A}, \mathcal{P}, z) \leq N \prod_{\substack{p \leq z \\ p \geq 3}} \left(1 - \frac{2}{p}\right).$$

Estimating the Product:

Taking logarithms, we have

$$\ln \prod_{3 \leq p \leq z} \left(1 - \frac{2}{p}\right) = \sum_{3 \leq p \leq z} \ln \left(1 - \frac{2}{p}\right).$$

For $p \geq 3$, $0 < \frac{2}{p} < 1$, so we can use the inequality $\ln(1-x) \leq -x - \frac{x^2}{2}$ for $0 < x < \frac{1}{2}$. Since $\frac{2}{p} \leq \frac{2}{3} < \frac{1}{2}$, we have

$$\ln \left(1 - \frac{2}{p}\right) \leq -\frac{2}{p} - \frac{2^2}{2p^2} = -\frac{2}{p} - \frac{2}{p^2}.$$

Thus,

$$\ln \prod_{3 \leq p \leq z} \left(1 - \frac{2}{p}\right) \leq -2 \sum_{p \leq z} \frac{1}{p} - 2 \sum_{p \leq z} \frac{1}{p^2}.$$

We know that

$$\sum_{p \leq z} \frac{1}{p} = \ln \ln z + M + \varepsilon_1(z),$$

and

$$\sum_{p\le z}\frac{1}{p^2}=B+\varepsilon_2(z),$$

where M and B are constants, and $\varepsilon_1(z),\varepsilon_2(z)\to 0$ as $z\to\infty$.

Therefore,

$$\ln\prod_{3\le p\le z}\left(1-\frac{2}{p}\right)\le -2(\ln\ln z+M+\varepsilon_1(z))-2(B+\varepsilon_2(z)).$$

Combining constants and error terms, we let $C=-2(M+B)$, so

$$\ln\prod_{3\le p\le z}\left(1-\frac{2}{p}\right)\le -2\ln\ln z+C+\varepsilon(z),$$

where $\varepsilon(z)=-2(\varepsilon_1(z)+\varepsilon_2(z))\to 0$ as $z\to\infty$.

For sufficiently large z, $\varepsilon(z)$ becomes negligible and can be absorbed into the constant C. Exponentiating both sides, we obtain

$$\prod_{3\le p\le z}\left(1-\frac{2}{p}\right)\le e^C\frac{1}{(\ln z)^2}.$$

Let $C_4=e^C$, so

$$\prod_{3\le p\le z}\left(1-\frac{2}{p}\right)\le \frac{C_4}{(\ln z)^2}.$$

Choosing the Parameter z:

Let us choose $z=x$. Then $\ln z=\ln x$.

Substituting back into the inequality for $S(\mathcal{A},\mathcal{P},z)$, we get

$$S(\mathcal{A},\mathcal{P},z)\le N\cdot\frac{C_4}{(\ln x)^2}.$$

Recall that $N=\left\lfloor\frac{x-1}{2}\right\rfloor\le\frac{x}{2}$. Thus,

$$S(\mathcal{A},\mathcal{P},z)\le\frac{x}{2}\cdot\frac{C_4}{(\ln x)^2}=\frac{C_4x}{2(\ln x)^2}.$$

Adjusting for Primes Greater Than z:

When we choose $z=x$, all primes $p\le x$ are included in $\mathcal{P}$. Therefore, there are no additional primes $p>z\le x$ to consider, and we do not need to adjust for larger primes.

Moreover, since we are interested in q such that both q and $p=\frac{q-1}{2}$ are prime, the actual count $\pi_{\mathrm{SP}}(x)$ is less than or equal to $S(\mathcal{A},\mathcal{P},z)$. This is because $S(\mathcal{A},\mathcal{P},z)$ includes integers n for which $2n+1$ may not be prime due to composite factors not detected by the sieve.

Conclusion:

Therefore,

$$\pi_{\mathrm{SP}}(x)\le S(\mathcal{A},\mathcal{P},z)\le\frac{C_4x}{2(\ln x)^2}.$$

Setting $C_3=\dfrac{C_4}{2}$, we have established the desired upper bound:

$$\pi_{\mathrm{SP}}(x)\le C_3\frac{x}{(\ln x)^2}.$$

Q.E.D.

5.2 Estimating the Partial Sums

We aim to estimate the partial sum

$$S_{\text{SP}} = \sum_{q \text{ is a safe prime}} \frac{1}{q}.$$

We partition the interval $[2, \infty)$ into dyadic intervals $[2^k, 2^{k+1})$ for $k \geq k_0$, where k_0 is chosen such that $2^{k_0} \geq 2$.

Lemma 4: The contribution to S_{SP} from safe primes in $[2^k, 2^{k+1})$ is at most $\dfrac{C}{k^2}$, where $C > 0$ is a constant.

Proof of Lemma 4:

From Lemma 3, the number of safe primes less than x satisfies

$$\pi_{\text{SP}}(x) \leq C_3 \frac{x}{(\ln x)^2}.$$

Let $x_k = 2^k$, so $\ln x_k = k \ln 2$.

Consider the function

$$f(k) = \frac{2^k}{(k \ln 2)^2}.$$

Then, from Lemma 3,

$$\pi_{\text{SP}}(x_k) \leq C_3 f(k).$$

We are interested in estimating the difference

$$\Delta\pi_{\text{SP}}(k) = \pi_{\text{SP}}(x_{k+1}) - \pi_{\text{SP}}(x_k) \leq C_3[f(k+1) - f(k)].$$

By the Mean Value Theorem, there exists some $\xi_k \in (k, k+1)$ such that

$$f(k+1) - f(k) = f'(\xi_k).$$

Compute the derivative of $f(k)$:

$$f'(k) = \frac{d}{dk}\left(\frac{2^k}{(k \ln 2)^2}\right) = \frac{2^k \ln 2}{(k \ln 2)^2} - \frac{2^k \cdot 2(k \ln 2) \ln 2}{(k \ln 2)^4} = \frac{2^k \ln 2}{(k \ln 2)^2}\left(1 - \frac{2}{k}\right).$$

Since $f'(k)$ is decreasing for $k \geq 3$, we have

$$\Delta\pi_{\text{SP}}(k) \leq C_3 f'(\xi_k) \leq C_3 f'(k).$$

Therefore,

$$\Delta\pi_{\text{SP}}(k) \leq C_3 \frac{2^k \ln 2}{(k \ln 2)^2}\left(1 - \frac{2}{k}\right) \leq C_3 \frac{2^k \ln 2}{(k \ln 2)^2}.$$

Simplifying,

$$\Delta\pi_{\text{SP}}(k) \leq \frac{C_3}{k^2} 2^k.$$

The largest reciprocal of a prime in $[x_k, x_{k+1})$ is $\dfrac{1}{x_k} = \dfrac{1}{2^k}$.

Therefore, the contribution to S_{SP} from this interval is at most

$$S_{\text{SP},k} \leq \Delta\pi_{\text{SP}}(k) \cdot \frac{1}{x_k} \leq \frac{C_3}{k^2} 2^k \cdot \frac{1}{2^k} = \frac{C_3}{k^2}.$$

Let $C_4 = C_3$. Then,

$$S_{\text{SP},k} \leq \frac{C_4}{k^2}.$$

5.3 Summing Over All Intervals

Combining the estimates from Lemma 4, we can bound the total sum S_{SP} by

$$S_{\mathrm{SP}} = \sum_{\substack{q \leq \infty \\ q \text{ is a safe prime}}} \frac{1}{q} = \sum_{k=k_0}^{\infty} S_{\mathrm{SP},k}.$$

From the estimation in Lemma 4, we have

$$S_{\mathrm{SP},k} \leq \frac{C_4}{k^2}.$$

Therefore,

$$S_{\mathrm{SP}} \leq \sum_{k=k_0}^{\infty} \frac{C_4}{k^2} = C_4 \sum_{k=k_0}^{\infty} \frac{1}{k^2}.$$

Since the series $\sum_{k=k_0}^{\infty} \frac{1}{k^2}$ is a convergent p-series with $p = 2 > 1$, it follows that

$$S_{\mathrm{SP}} \leq C_4 \sum_{k=k_0}^{\infty} \frac{1}{k^2} < \infty.$$

5.4 Conclusion of Theorem 2

Since S_{SP} is bounded above by a convergent series, it follows that the sum of the reciprocals of all safe primes converges.

Q.E.D.

5.5 Alternative Proof Based on Sophie Germain Primes

If we have already established that the sum of the reciprocals of Sophie Germain primes converges, we can deduce that the sum of the reciprocals of safe primes also converges.

Consider the one-to-one correspondence between Sophie Germain primes and safe primes defined by $q = 2p+1$, where p is a Sophie Germain prime and q is the corresponding safe prime. Since each safe prime q is uniquely associated with a Sophie Germain prime p, we can relate their reciprocals.

Observe that

$$\frac{1}{q} = \frac{1}{2p+1} \leq \frac{1}{2p}.$$

Therefore,

$$\sum_{\substack{q \in \mathbb{P} \\ q = 2p+1,\, p \in \mathbb{P}}} \frac{1}{q} \leq \frac{1}{2} \sum_{\substack{p \in \mathbb{P} \\ 2p+1 \in \mathbb{P}}} \frac{1}{p}.$$

Since the sum $\sum_{\substack{p \in \mathbb{P} \\ 2p+1 \in \mathbb{P}}} \dfrac{1}{p}$ converges (by Theorem 1), it follows that

$$S_{\mathrm{SP}} = \sum_{\substack{q \in \mathbb{P} \\ q = 2p+1,\, p \in \mathbb{P}}} \frac{1}{q} < \infty.$$

This provides an alternative proof of Theorem 2 based on the convergence of the sum over Sophie Germain primes.

Q.E.D.

6 Discussion

The convergence of the sums S_{SG} and S_{SP} illustrates the relative sparsity of Sophie Germain primes and safe primes among all prime numbers. The fact that their counting functions grow no faster than $\frac{x}{(\ln x)^2}$ is crucial for the convergence of the reciprocals. By the integral test for series convergence, a series with terms decreasing like $\frac{1}{x/(\ln x)^2}$ will converge, highlighting the significant impact of the counting function's growth rate on the convergence of the reciprocal series.

This contrasts with the set of all primes, for which the counting function $\pi(x) \sim \frac{x}{\ln x}$ leads to the divergence of $\sum_{p \le x} \frac{1}{p}$. Despite primes becoming sparser as numbers grow larger, the harmonic series over the primes diverges due to the subtle balance between the decreasing size of the terms and the increasing number of primes, albeit at a slower rate than the integers.

Our proofs rely on upper bounds for $\pi_{\mathrm{SG}}(x)$ and $\pi_{\mathrm{SP}}(x)$ derived from sieve methods and average estimates of primes in arithmetic progressions. While the exact asymptotic densities of these primes remain unproven and are subject to conjecture—such as the Hardy-Littlewood conjecture generalized for these cases—the established upper bounds suffice to demonstrate the convergence of the sums. Conjecturally, the number of Sophie Germain primes less than x is believed to be approximately $C\frac{x}{(\ln x)^2}$, where C is a constant involving the twin prime constant, though this remains an open problem in number theory.

The convergence of the sums provides insight into the distribution of these primes and underscores the delicate interplay between term size and quantity in determining the convergence of infinite series. It also raises questions about whether improved bounds on $\pi_{\mathrm{SG}}(x)$ and $\pi_{\mathrm{SP}}(x)$ might be achievable with advancements in sieve methods or analytic techniques.

7 Conclusion

We have rigorously proven that the sums of the reciprocals of Sophie Germain primes and safe primes are both convergent. This result underscores the thinness of these sets within the prime numbers and provides insight into their distribution. The convergence indicates that these primes are sufficiently rare compared to all primes, whose reciprocal sum diverges.

This work contributes to the understanding of prime distributions and highlights the effectiveness of sieve methods in establishing density estimates for special classes of primes. Potential implications of these results extend to fields such as cryptography, where the properties of such primes are utilized. Further research may focus on improving the bounds for $\pi_{\mathrm{SG}}(x)$ and $\pi_{\mathrm{SP}}(x)$, exploring their exact asymptotic behaviors, or examining similar questions for other special classes of primes.

8 References

References

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