PICARD–FUCHS HYPERGEOMETRIC MANIFOLDS: A NEW LENS ON QUANTUM ENTANGLEMENT

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ABSTRACT

A unified mathematical framework, referred to as Picard–Fuchs Hypergeometric Manifolds (PFHM), is introduced to integrate modular symmetry, coupled dynamics and energy conservation. PFHM are constructed using a synthesis of Ramanujan's real period functions, Picard–Fuchs differential equations and Gaussian hypergeometric functions. We argue that PFHM provide an effective representation of two-dimensional coupled subsystems embedded in three-dimensional manifolds with dihedral symmetry. These coupled subsystems exhibit constrained energy reciprocity, making PFHM a robust tool for elucidating stable, closed and homoclinic orbits in Hamiltonian systems. An application of the proposed method is explored in the context of quantum entanglement. The intrinsic energetic reciprocity and symmetry of PFHM are shown to be analogous to the nonlocal correlations in entangled quantum systems. Modelling entangled pairs as constrained subsystems, the PFHM framework sheds new light on the energy dynamics and nonlocal correlations underpinning quantum entanglement.

KEYWORDS: dynamical systems theory; elliptic curves; homoclinic orbit; geometry.

INTRODUCTION

Hamiltonian systems capture the dynamics of physical phenomena through conservation laws. Yet, the behavior of coupled subsystems, particularly in higher-dimensional manifolds, presents significant analytical and computational challenges. This paper proposes a unified framework that combines modular symmetry, coupled dynamics, and energy conservation. This framework links three mathematical constructs that are known for their versatility in addressing complex systems:

1. Picard–Fuchs differential equations (henceforth PF). PF are simple, linear, ordinary differential equations whose solutions describe the periods of elliptic curves (Shen 2017; Kreshchuk and Gulden, 2019). Well-suited for defining periods of algebraic varieties, PF characterize the behavior of physical systems over parameter spaces. They have the unvaluable advantage of encompassing period-energy functions, allowing energy to be used as a parameter.

2. Gaussian hypergeometric functions. In particular, the Gaussian hypergeometric special function ${}_2F_1$ (a,b;c;z) is characterized by three regular singular points where the growth of solutions in the complex plane is bounded by an algebraic function (Becken and Schmelcher, 2000; Olde Daalhuis 2010). It allows the construction of manifolds with rich symmetry properties and smooth interrelations between subsystems (Ratner et al., 2001; Jong et al., 2015).
3. Ramanujan's real period functions. They provide a foundation for describing modular structures (Shen 2017).

Picard–Fuchs Hypergeometric Manifolds (PFHM) can be constructed as follows:

1. Hypergeometric manifolds. Begin with a two-dimensional base manifold M_2 defined using Gaussian hypergeometric functions. The base manifold is governed by the Picard–Fuchs equation:
$$\frac{d^2y}{dz^2} + \left(\frac{c}{z} + \frac{a+b-c+1}{z-1}\right)\frac{dy}{dz} + \frac{ab}{z(z-1)}y = 0,$$
where $a, b, c \in \mathbb{R}$.

2. Ramanujan's period functions. Define modular transformations on M_2, embedding it in a higher-dimensional space. The transformation $T(\tau) = \frac{a\tau+b}{c\tau+d}$ (with $ad - bc = 1$) generates symmetries corresponding to dihedral structures.

3. Three-dimensional coupled systems. Extend M_2 to a three-dimensional manifold M_3 by coupling two subsystems with Hamiltonian interactions:

$$H = H_1(x_1, p_1) + H_2(x_2, p_2) + V(x_1, x_2),$$

where $V(x_1, x_2)$ is the coupling potential.

The hypergeometric version of PF is satisfied by a set of integral period functions that define geometries characterized by simple, closed plane curves originating at regular points (Beukers and Heckman, 1989; Fürnsinn and Yurkevich, 2023). PFHM are characterized by a double-periodic elliptic function and a countable toric section of the Hamiltonian (Klee 2018a). Given the system's Hamiltonian $H(p, q)$, the associated real period function $T(\alpha)$ can be described by the integral-differential algorithm (Klee 2019):

$$T(\alpha) = {}_2F_1\left(\frac{1}{2}, \frac{s-1}{s}; 1; \alpha^2\right), \ s = 3, 4, \text{ or } 6$$

with the same signatures $s \in \{2, 3, 4, 6\}$ found in the Ramanujan theory of elliptic functions related to the integral period functions K_1, K_2 and K_3 (Ramanujan 1914; Shen 2017).
When the potential reaches its minimum, the phase curves form closed loops, which can be characterized using the period function $T(\alpha)$ (Klee 2019):

$$T(\alpha) = \oint dt = \oint dq \,/\, p(\alpha, q).$$

The invariant differential $dt = dq \,/\, p$ can be integrated on any loop around the Riemannian surface $\alpha = 2H(p, q), \ (q, p) \in \mathbb{C}^2$.
This means that the total Hamiltonian H enforces energetic reciprocity:

$$\Delta E_1 + \Delta E_2 = 0,$$

ensuring that energy shifts in one subsystem inversely mirror shifts in the other. This reciprocity leads to constrained dynamics that stabilize periodic orbits.
This framework is especially valuable for describing the behaviour of stable, periodic closed trajectories in multi-dimensional phase spaces, such as homoclinic orbits connecting saddle points and periodic orbits exhibiting dihedral symmetry of M_3. Additionally, the PFHM framework also facilitates chaotic dynamics under certain parameter conditions, where nonlinear interactions induced by the coupling potential $V(x_1, x_2)$ amplify sensitivity to initial conditions (Guo et al., 2021).

In sum, PFHM is a versatile tool that enables the generation of numerous algorithms and series of identities (Klee 2018a). Slight changes in the variable period $T(\alpha)$ and/or the Gaussian hypergeometric function lead to entirely different manifolds, which can be interpreted in the context of (physical or biological) dynamical systems exhibiting dihedral symmetry. Notably, the systems' energetic requirements and constraints can be calculated by plotting cross sections of the toric energy surface with plane curves. In the sequel, we will assess the phase space and the Hamiltonian of dynamical systems characterized by stable and closed orbits, with special emphasis on their energetic features.

FROM HYPERGEOMETRIC PHASE SPACES TO REAL SYSTEMS

We propose that PFHM could serve as a mathematical framework to elucidate the phase spaces and energetic dynamics of real systems. To provide a proof-of-concept example, consider the phase space trajectories of three-dimensional system composed of the orthogonal subsystems S1 and S2 characterized by homoclinic, stable, closed orbits, where period integrals along the contour curves enable the evaluation of time dynamics (Klee 2018b). See **Figure**. The S1's planar layer contains a family of Hamiltonian level curves indexed by energy $\alpha \in (0, 1)$ with the lowest energetic level located at the center and the higher on the borders. The higher the energy, the larger the closed orbit crossed by a hypothetical particle traveling in the S1's phase space. The HPFM procedure allows S1 and its family of Hamiltonian curves to be coupled with S2 Hamiltonian curves that reciprocally influence each other. The single complex dynamical system formed by the two subsystems S1 and S2 displays peculiar features:

1) The two-dimensional subsystems are located on perpendicular planes inside a three-dimensional phase space.
2) The system is finite and features dihedral symmetry.
3) The constrained trajectories of the subsystems follow homoclinic, stable and closed orbits.

4) The system could be treated either as continuous or quantized.
5) Due to the extreme value theorem, a real-valued function that is continuous on the closed and bounded interval must attain at least a maximum and a minimum on a compact manifold.
6) Plotting cross sections of the toric energy surfaces allows energy inversion symmetry. Differently from other Hamiltonian surfaces, the real period T(α) determines the complex period T(1-α) up to a rescaling, leading to energy inversion $\frac{\alpha}{1-\alpha}$ (Klee 2018a).
7) For every fixed value of the Hamiltonian, equipotential subspaces can be found in every subsystem. Therefore, changes in radius and energetic levels in S1 are inversely correlated with changes in radius and energetic levels in S2 (**Figure**). Contrary to S1, the S2 energetic values decrease from the center to the periphery.
8) The (physical or biological) subsystems S1 and S2 are coupled by an inverse linear function in such a way that the total energy of the system is conserved. Energetic increases in S1 must be balanced by inversely proportional energetic decreases in S2. This means that the dynamics result in linear, balanced, reciprocally induced energetic constraints between the S1 and S2 subsystems.

The closed orbits with coupled Hamiltonian dynamics could represent energetic levels of physical forces like gravitation or electromagnetism, energy exchanges between coupled plasma waves, coupled celestial orbits in three-body problems, oscillatory dynamics in coupled neurons, predator-prey systems with constrained energy exchanges, paths followed by social agents' networks, etc. In the sequel, we will focus on the special case of quantum entanglement.

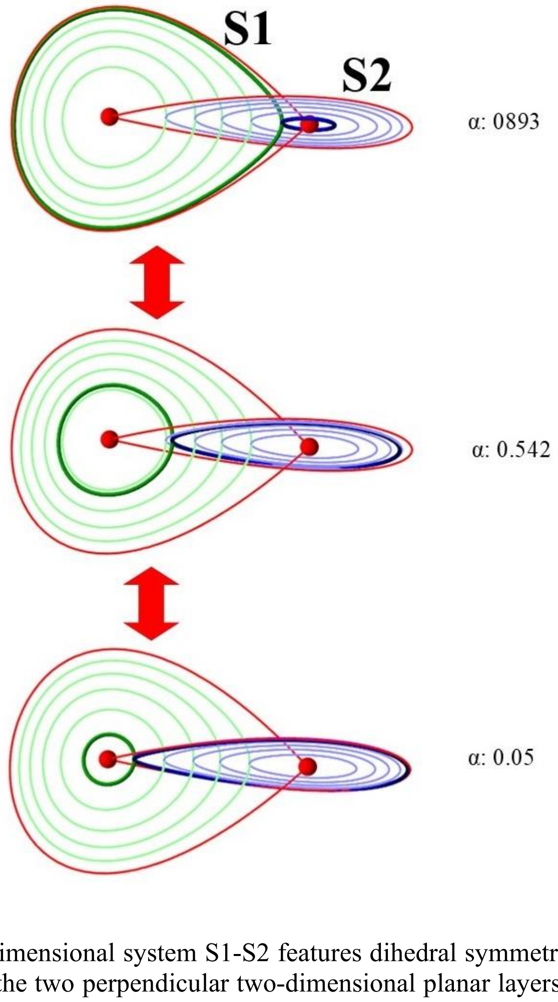

Figure 2. The dynamical three-dimensional system S1-S2 features dihedral symmetry with real period function T(α) = ${}_2F_1$ (1/6, 5/6; 1; α). Every one of the two perpendicular two-dimensional planar layers contains a family of Hamiltonian level curves indexed by energy. By adjusting the α value upward or downward, distinct and predictable time dynamics emerge within each subsystem (thicker circles). In contrast to subsystem S1, the energetic values of S2 decrease from the center to the periphery, such that changes in the energy of S1 are inversely proportional to those in S2. Modified from Klee (2018b); Klee (2019).

QUANTUM ENTANGLEMENT THROUGH THE LENS OF PICARD–FUCHS HYPERGEOMETRIC MANIFOLDS

Despite experimental validation, a comprehensive framework linking the nonlocal correlations of quantum entanglement to the underlying dynamics remains elusive (Nakata and Murao, 2020; Köhnke et al., 2021). Since PFHM are manifolds derived from the modular structures and symmetries inherent in hypergeometric functions, they naturally account for coupled dynamics and reciprocal interactions, making them a promising framework for modelling the behaviour of entangled particles. PFHM allows the description of the coordinate variables in terms of complex values matching each plane curve to a Riemann surface $S(\alpha, \epsilon)$ with nontrivial topology. This observation provides the mathematical and logical backbone to treat quantum entanglement in terms of paths traveling inside PFHM-like manifolds (Krutyanskiy et al., 2023).

PFHM for quantum systems is constructed using three core components:

1. Base manifold representation. The state space of two entangled particles is modeled using a two-dimensional hypergeometric manifold M_2 with state amplitudes defined as solutions to the Picard–Fuchs equation:

$$\frac{d^2\psi}{dz^2} + \left(\frac{c}{z} + \frac{a+b-c+1}{z-1}\right)\frac{d\psi}{dz} + \frac{ab}{z(z-1)}\psi = 0,$$

where $\psi(z)$ represents the quantum state amplitudes and a, b, c parameterize the system.

2. Extension to PFHM. By embedding M_2 into a three-dimensional manifold M_3, the coupled dynamics of entangled subsystems are described by a Hamiltonian:

$$H = H_1(\psi_1, p_1) + H_2(\psi_2, p_2) + V(\psi_1, \psi_2),$$

where $V(\psi_1, \psi_2)$ is the interaction potential that encapsulates entanglement.

3. Symmetry and reciprocity. Using modular transformations derived from Ramanujan's period functions, the energy dynamics of the coupled system satisfy reciprocity:

$$\Delta E_1 + \Delta E_2 = 0.$$

Once PFHM has been established for quantum systems, quantum entanglement can be modelled using the following procedure:

1. Representing entangled states. In the PFHM framework, an entangled quantum state $|\psi\rangle$ is represented as a point on M_3. For a two-particle system:

$$|\psi\rangle = \alpha|0\rangle_A|1\rangle_B + \beta|1\rangle_A|0\rangle_B,$$

where $|0\rangle$ and $|1\rangle$ are basis states. The amplitudes α and β evolve on the manifold according to the Picard–Fuchs equations, with the coupling potential $V(\psi_1, \psi_2)$ encoding the entanglement.

2. Energetic reciprocity in entangled systems. The constrained energetic reciprocity in PFHM reflects the conservation laws in quantum systems:

$$E_A + E_B = E_{\text{total}}.$$

In entangled states, changes in one subsystem's energy spectrum are instantaneously mirrored in the other, preserving the total system's symmetry. Within a PFHM-like phase space, the energetic configuration of particles in one subsystem can be predicted by examining the energetic configuration of particles in the coupled subsystem.

3. Quantum measurement as a geometric projection. Quantum measurement is modeled as a projection of the state vector onto a specific submanifold of M_3. The outcome of a measurement collapses the manifold's global symmetry into localized states, consistent with observed quantum correlations.

In sum, by embedding quantum systems in a higher-dimensional manifold, the dihedral symmetry of PFHM captures the nonlocal nature of entanglement. The simultaneous correlation of entangled particles becomes clear when viewed as embedded within hypergeometric coupled phase spaces, where their trajectories are governed by constraints imposed by conservation laws inherent in the Hamiltonian.

CONCLUSIONS

We introduce a new mathematical approach for modeling Hamiltonian dynamics in complex coupled systems while leaving room for empirical validation and interdisciplinary exploration. The concept of energetic reciprocity in coupled Hamiltonian subsystems with dihedral symmetry aligns with the conservation principles and the coupled behaviors observed in biological and physical systems, offering new insights into periodic and chaotic behaviors. For example, in cryptography and image processing, a message or visual image encoded within the invariant orbits of PHFM subsystems can be mapped onto the orbits of another subsystem, creating a hypergeometric projection of the original data. Similar to mapping a world chart, homoclinic or heteroclinic projections between two coupled hypergeometric manifolds can be performed, offering additional applications in digital imaging and memory storage.

This paper specifically explores PFHM as a structured framework for modelling quantum entanglement. Taking into account the constrained inverse reciprocity of trajectories within coupled subsystems, the modular structures, the energetic reciprocity and the higher-dimensional symmetries, PFHM offers a novel approach to understanding the dynamics and nonlocality of entangled systems. The framework is very flexible and can be adapted to multi-particle entanglement and extended to infinite-dimensional systems. The extension of PFHM to field-theoretic systems suggests applications in describing entangled fields, particularly in curved spacetime or scenarios involving the AdS/CFT correspondence

Our study has its limitations. While the concept offers several strengths, it ventures into complex theoretical terrain that demands rigorous validation and enhanced conceptual clarity. In particular, the mapping of quantum observables onto PFHM needs further refinement, especially for systems with high degrees of freedom. Developing experimental designs to test the predictions of the PFHM framework is an essential next step.

Future research should aim to extend PFHM to non-Hamiltonian systems and investigate computational techniques for solving PFHM-derived equations in high-dimensional spaces. Additionally, future work will focus on expanding this structured framework to incorporate quantum computing, specifically to model qubits in entangled states (Graham et al., 2022). Indeed, the geometric properties of PFHM can be used to optimize quantum gate operations by maintaining the reciprocity and symmetry crucial for preserving entanglement.

In summary, the connection between dynamical systems (e.g., homoclinic paths), Hamiltonian mechanics and quantum entanglement represents a significant leap from abstract mathematical structures to dynamic systems. This suggests a unifying framework that could provide a versatile approach to understanding complex interactions across disciplines.

DECLARATIONS

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Consent for publication: The Author transfers all copyright ownership, in the event the work is published. The undersigned author warrants that the article is original, does not infringe on any copyright or other proprietary right of any third part, is not under consideration by another journal, and has not been previously published.
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