Resolving The Cosmological Constant: A Conjecture for Homogeneous Infinitesimals

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Abstract: The discovery in 1998 that the universe is paradoxically accelerating its expansion has led some cosmologists to question the correctness of the non-Euclidean geometric theory of gravity, General Relativity. Physically assigning the term Dark Energy to the Cosmological Constant, sometimes viewed as a constant of integration, as the source of this acceleration has only produced even more questions. In the 17th century, there was also a great paradox between two views for the geometric constituents of a line, heterogeneous (made of points) versus homogeneous (made of infinitesimal segments). Evangelista Torricelli elucidated his logical reasoning on why lines must be made of infinitesimal segments instead of points and created one particular fundamental example among many. In this paper, I produce unknown corollaries to Torricelli's argument allowing me to falsify the relationship between his infinitesimals and the Archimedean axiom, resolve L'Hôpital's paradox, as well as redefine the Fundamental Theorem of Calculus, scale factor/metrics, n-spheres and Gaussian curvature. I conjecture that the intractability of Dark Energy is due to the points of coordinate systems within General Relativity actually being a logically flawed heterogeneous interpretation. I propose that Euclidean and non-Euclidean geometry, and the physics equations based upon them, should be rewritten from the perspective of homogeneous infinitesimals. I introduce the geometrical logic in this paper in order to pave the way for the physical logic.

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Keywords:

1 Introduction

The Dark Energy Task Force[1], a committee of scientists tasked with advising the DOE, NASA and NSF on Dark Energy, has stated, "The acceleration of the Universe is, along with dark matter, the observed phenomenon which most directly demonstrates that our fundamental theories of particles and gravity are either incorrect or incomplete." The theoretical value for the Cosmological Constant (CC) is well known by now as the worst prediction ever made in physics for good reason:

> An alternative explanation of the accelerating expansion of the Universe is that general relativity or the standard cosmological model is incorrect. We are driven to consider this prospect by potentially deep problems with the other options. A cosmological constant leaves unresolved one of the great mysteries of quantum gravity and particle physics: If the cosmological constant is not zero, it would be expected to be 10^{120} times larger than is observed.

If these problems are fundamental enough for the Task Force to advise that General Relativity (GR) itself could be incomplete or incorrect then it also begs the question: *How* could it be either? I propose an answer: GR could be incorrect if our concept of infinitesimals has always been incomplete.

2 Background

The meaning behind dx (although the notation was invented by Leibniz[1] for the Calculus it has also become ubiquitous in GR)[2] can be traced back to concepts from over 2500 years ago and more rigidly to Bonaventura Cavalieri in 1635[3]. One of the great arguments during his time was whether lines were made of non-dimensional points (heterogeneous) or made of infinitesimal segments of lines (homogeneous)[4]. In the same vein, it was also debated whether area would be composed of infinitesimally thin slices of area versus stacked lines and whether volume was made of infinitesimally thin sheets of volume versus stacked planes. Evangelista Torricelli, a brilliant scientist and inventor in his own right and well known to Galileo, is also known in these debates for his talent at taking a difficult concept and explaining it in many different ways. This has been said to have enabled the transfer of fundamental concepts more so than the voluminous writings of Cavalieri. Torricelli's analysis of the heterogeneous/homogeneous debate [6] landed him firmly on the infinitesimal segment side as recent authors have pointed out[5].

> All indivisibles seem equal to one another, that is, points are equal to points, lines are equal in thickness to lines, and surfaces are equal in

[1] See Katz[11] for a discussion on who invented the term "infinitesimal".

[2] It would seem to me it is taken for granted. Often the Einstein field equation in compact form doesn't even bother to include the infinitesimal notation $dx_\mu dx_\nu$ with the metric notation $g_{\mu\nu}dx_\mu dx_\nu$ such as $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = kT_{\mu\nu}$.

[3] See [2] p303 for timeline.

[4] See [10] p. 4 for discussion.

[5] See [2] and [10] page 125.

> depth to surfaces is an opinion that in my judgment is not only difficult to prove, but false.

By this he meant that it would seem we should be using infinitesimal segments[6] instead of non-dimensional points. Whereas points can't be distinguished from each other, the segments can have infinitesimal length and that length isn't necessarily the same from one line to another thus distinguishing them (as would be his similar argument for area and volume).

One example in particular that he used to demonstrate his reasoning, prior to his early death at the age of 39, has been called by Francois De Gandt the “condensed" “fundamental example" for Torricelli's view on the heterogeneous/homogeneous paradox[7]. While I have come to very much agree with the sentiment that this is a “condensed paradox", my examination of Torricelli's example has also revealed startling unknown similarities with the chain of logic that was used to create non-Euclidean geometry and ultimately General Relativity. This chain seems to have a fundamental difference with the infinitesimals of non-standard analysis (NSA). NSA seems primarily concerned with incorporating infinitesimals into the preexisting concept of the real number $\mathbb{R}$ whereas the homogeneous real line is simply the sum of primitive notion infinitesimals. I am not aware that the homogeneous/heterogeneous distinction has ever been examined within the context of NSA. Regardless, I introduce this research to the NSA community since it has the most recent research and historical background of the infinitesimal concept itself.

3 Flatness, Curvature and HIs

Imagine that you could have a single line and it is itself composed solely of “infinitesimal” line segments *SEG* and that the sum of their magnitudes $|SEG|$ is defined as the “length" of that line (i.e. as opposed to Bernhard Riemann's definition [17] that the length of every line is “measurable by every other line” which includes no mention of the infinitesimals of which it is composed),

$$\sum |SEG| \equiv \textbf{line length}. \tag{1}$$

[6]There was a philosophical distinction between indivisibles and infinitesimals. I do not expand upon the indivisible concept as I view this a geometrical and philosophical red herring. See page 24[14].

[7]See [6] page 164.

Suppose that the magnitude $|SEG^n|$ of a segment n could either be of equal relative magnitude (as in Eqn.2) to another segment $n-1$ or could have a different value (as in Eqn.3) *even within the line itself.* A simple to state introductory hypothesis is whether Euclidean geometry can be derived from

$$|SEG^n| - |SEG^{n-1}| = 0 : \text{intrinsically flat} \tag{2}$$

which I will call *intrinsically flat*. Allow me to define a “point" as simply an infinitesimal *SEG* that is of null magnitude in the direction along the line so that we can also understand that Euclid’s definition of a straight line (Euclid’s Elements, Book I, Definition 4) is one that “lies evenly with the points upon itself” and in this case both terms are equal or even (with a point between the two segments and at their respective ends). I could then propose that non-Euclidean geometry could be derived from

$$|SEG^n| - |SEG^{n-1}| \neq 0 : \text{intrinsically curved} \tag{3}$$

so that the points would no longer be equally spaced and which I will call *intrinsically curved*.

Now also suppose that I can examine the number of segments in one line versus another and I called this number the Relative Cardinality (RC) so that I can write (in the simplifying case that the line is intrinsically flat) the equation

$$\sum |SEG| = RC * |SEG| \equiv \textbf{line length}. \tag{4}$$

Consider the conversion of the ratio $\frac{\Delta Y}{\Delta X} \to \frac{dy}{dx}$. One of the most immediately striking things from this research is that Leibniz’s notation is flawed in that I can show that this is actually a ratio of relative cardinalities of intrinsically flat homogeneous infinitesimals:

$$|SEG|_Y = |SEG|_X \tag{5}$$

so that

$$\frac{RC_Y|SEG|_Y}{RC_X|SEG|_X} = \frac{\Delta Y}{\Delta X} \to \frac{\Delta RC_Y}{\Delta RC_X} = \frac{dy}{dx}. \tag{6}$$

Also consider the equation

$$ds^2 = \sum_i^j g_{ij} dq_i dq_j \tag{7}$$

where the metric g is equivalent to the square of a “scale factor" h[8],

$$h_i h_j = g_{ij}. \tag{8}$$

[8]See relative 6.2.1 and absolute strain 6.4.1 for evidence that scale factors as presented in modern geometry textbooks are logically incomplete based on this research

The right side of the scale factor equation

(9) $$ds_i = h_i dq_i$$

is said to have a "dimension of length"[9]. Relative Cardinality with infinitesimal magnitude represents line length and thus the transition [10] of

(10) $$\Delta S \to ds$$

stands at odds with

(11) $$\Delta S = RC_S ds \to RC_{S^*} ds = 1ds$$

where

(12) $$RC_S$$

is an arbitrary cardinal number greater than 1 and

(13) $$RC_{S^*} = 1.$$

This demonstrates that ΔS is the sum of infinitesimals whereas ds is a singular infinitesimal and this implies the scale factor h is a logically flawed mixture of cardinality and scaling properties. While this may seem to be an extraordinary conjecture, I point out that the major part of the energy within our known universe is theorized[1] to have something to do with a scalar multiple of the metric, Λg. I propose it is then logical to leave no stone unturned as I compare the properties of

(14) $$h_i dq_i$$

with

(15) $$RC_i * |SEG|_i.$$

Qualitatively, in Equation 7, ds^2 seems to be able to have properties of both intrinsically curved and flat HIs, the metric g seems to have the properties of both relative cardinality and scaling, and dq seems to have the properties of flat HIs.

In more general descriptive terms, pairing the properties of Equations 4, 6 and 7 with that of Torricelli's homogeneous infinitesimal (HI) concept creates an interesting perspective. If the logically true portions of non-Euclidean straight and curved lines were actually based on properties of HIs then curved space-time could provide certain accurate predictions within a neighborhood yet still suffer from paradoxes as a flawed interpretation of the underlying geometry. I have found that Torricelli's HIs have a

[9] See equation 2.9 and following paragraph within [3].
[10] i.e. the transition between the two equations within Box 13.1 of [16]

striking resemblance to both coordinate systems and basis vectors. For coordinate systems, I can derive the real number line with HIs, similar to non-standard analysis, but also show how it can contract and extend, similar to Bernhard Riemann's analogies of stretched surfaces[17] using a new concept called voluminal lines. If comparing with basis vectors, HIs also possess direction and relative magnitude, but with HIs their absolute magnitudes $|SEG|$ are arbitrary[11] whereas with basis vectors their flat boundary condition magnitude seems to be defined as "1" ("orthonormal") within GR[12].

More to the point, the Einstein Field Equation is commonly written as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = kT_{\mu\nu} \tag{16}$$

with

$$g_{\mu\nu}dx^{\mu}dx^{\nu} \tag{17}$$

as the metric g with the space-time terms dx. Some of the characteristics of the left side would be more easily understood using my conjectures. As examples, this research allows me to predict that the left side:

(1) First term $R_{\mu\nu}$ would be rewritten as a measure of the change in magnitude of voluminal homogeneous infinitesimals.

(2) Second term of $\frac{1}{2}Rg_{\mu\nu}$ would be proven to have the same properties as what I call a measure of the change of "Relative Cardinality" of the HIs.

(3) Third term $\Lambda g_{\mu\nu}$ is represented as an ad-hoc add on instead of the actual boundary condition from which the intrinsic curvature should be measured (instead of $g_{\mu\nu}$). This work would show it is equivalent to a constant of integration and this means approximating Newtonian gravity by using an equation similar to $\Lambda - \Lambda 2\phi$ instead of $1 - 2\phi$.

(4) Demonstrates that the properties of space-time are actually the fundamental properties of HIs. For example the Schwarzschild Radius of the solution $1-2\phi = 0$ is just the solution when the magnitude of a radial voluminal homogeneous infinitesimal goes from an arbitrarily assigned boundary value of "1" to "0" (null). In other words, the points are no longer evenly spaced but instead get close enough to overlap.

(5) Having "solutions" that must be found for the Einstein Field Equation is fundamentally due to a postulate that homogeneous infinitesimals must follow

[11] Similar to Riemann's "two magnitudes can only be compared when one is a part of the other; in which case also we can only determine the more or less and not the how much"[17].

[12] See Figure 2.3 [16].

Eudoxus' Theory of Proportions (which the Transfer Principle of NSA would be an extension of).

(6) Utilizes conservation of voluminal HI magnitudes and their cardinality as a geometric language to represent conservation of physical properties (ie energy-momentum).

As a historical analogy, HIs would be to curved and straight lines underlying space-time as ellipses were to the perfect circles upon which epicycles and deferents relied[13]. Since infinitesimals have been almost entirely replaced by the concept of the mathematical limit[14][15][16], then the CC problem would effectively stem from premature abandonment of HI research by the time the Calculus was developed and thus never made it in to the consideration of non-Euclidean geometry nor relativity. This seems to be the same argument for the infinitesimals of non-standard analysis.

I do not introduce any hypotheses about the right side $kT_{\mu\nu}$ yet as that introduces a discussion concerning the historical use of perfect fluid analogies present within GR[17]. The introduction for this hypothesis lies solely in the philosophical understanding of infinitesimals/infinity, their ability to have differing values and the concept of relative cardinality. Therefore this is a foundational issue for geometry and set theory and not yet the physical philosophy which rests upon it. I will introduce that later should this work be found geometrically compelling.

4 Structure

The logical order for this work is not the same as the historical order of the development of Euclidean/Calculus/non-Euclidean geometry so some portions may seem out of sync initially. I outline the structure here:

(1) Introduce the name and acronym for this research (5),

(2) Explain Torricelli's Parallelogram and his logic for HIs (6.1).

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[13] This was inspired by Yale University ASTR 160 Lect. 24 class video of C. Bailyn @15:30 https://oyc.yale.edu/astronomy/astr-160/lecture-24.

[14] See [10] pages 359-364 for discussion.

[15] As Keisler states[12] "banished".

[16] See [5] page xii

[17] Although the reader is free to brush up on the paradoxes such as those presented within the Report of the DETF[1].

(3) Produce two new corollaries to his example and derive the concept of line length (6.2), intrinsic flatness and curvature (6.2), relative strain (6.2.1), Relative Cardinality, absolute strain (6.4.1).

(4) Demonstrate that flat HIs with RC are compatible with the Archimedean axiom (6.6.2) using L'Hôpital's paradox and non-standard analysis (6.6.3) as evidence that I can derive the real number line.

(5) Set the HI as a primitive notion (7) and give postulates (7.1).

(6) Analyze lineal (8.1), areal (8.2), and voluminal lines (8.3).

(7) Introduce Concept of Background and Foreground geometry (8.3.1).

(8) Using a flat background, relax the point postulate and hypothesize how to derive the Fundamental Theorem of Calculus (Appendix A.1): Use Relative Cardinality to define Euclidean geometry, Leibniz's $\frac{dy}{dx}$ and the process of integration.

(9) Using foreground geometry, describe the similarities between voluminal lines and principal curvature K of Gaussian curvature (8.3.2).

One obvious section that should be included in this paper is an analysis of prior research by Gauss, Riemann, Bolyai and Lobachevsky etc. during the development of non-Euclidean geometry when they analyzed Torricelli's HIs. Unfortunately, I only have access to commonly published works and not perhaps unpublished notes. I currently find no published evidence that any of them analyzed Torricelli's work. However, absence of evidence is not evidence of absence and thus will have to rely upon the peer review process to enlist mathematical historians. My efforts at assistance prior to submission of this paper has not been fruitful.

In light of the breadth of this research versus the readability of an introductory paper, I have decided to follow Torricelli's example and try to err on the side of simplicity. His known simplifications have proven to be more effective for the initial spread of ideas than a dense but unread "Geometria Indivisibilibus".

5 CPNAHI: The Calculus, Philosophy and Notation of Axiomatic Homogeneous Infinitesimals

I have chosen the term CPNAHI because of my view that while *Notation* provides economy of thought, my actual equations are of geometric concepts such as summation, differences, ratios, etc. of HIs (*Calculus*) equated to *Philosophical* concepts such as money, population, space, time, force, velocity, change in wavelength, change in clock rate, strain, pressure, momentum, density and change in density, energy, *ad infinitum*. In simpler words, HIs are a *language*.[18] It is also my view that if the notation giving us economy of thought does not properly represent the underlying geometry, then the philosophical interpretations will be of poor and misleading value.

6 Geometry: The Calculus of Homogeneous Infinitesimals

6.1 Historical Analysis of Torricelli's Parallelogram

In order to give background to an uninitiated reader, I could recreate the historical explanation for Torricelli's parallelogram involving area and area of lines[19] but it isn't necessary to get to the crux of his philosophy. Condensing this example[20] and adding in a bit more notation:

Instead of a line having points on it, assume that a line is made of points and that the number of points in a line determine the length. Two lines that are of the same length have the same number of points. A shorter line has less points and a longer line has more points.

Now take a parallelogram with the four corner points labeled A,B,C, and D. Draw a line BD down the diagonal of it as shown in Figure 1. Let us make a point E on the diagonal line BD. Now draw perpendicular lines from E to a point F on AD, and a second line to a point G on CD. Move these two lines point by point simultaneously so that E moves toward D until they meet, keeping the lines EF and EG always parallel to AB and BC respectively. When we move the lines EF and EG, we are moving their ends simultaneously from point to point on AD, CD and BD.

Since line AD is shorter than the line CD, the number of points that the line AD contains is less than the number of points that line CD contains. However, this creates a paradox. Since we moved the lines point by point and with both points F and G ending up together at point D then this shows that lines AD and CD must also have the same number of points as shown in the equations in Figure 2.

[18] It may be informative if you consider the hypothetical argument between a "hard-nosed physicist" and a "hard-nosed mathematician" [16] p. 230 if neither views geometry as a language.

[19] Some of the arguments Torricelli made are how lines logically seemed to have non-zero width but hopefully my argument makes the resolution of all those trivial.

[20] It may be possible that I read a notationless explanation similar to the one presented here. If so, I am unable to find it again and my apologies to that author.

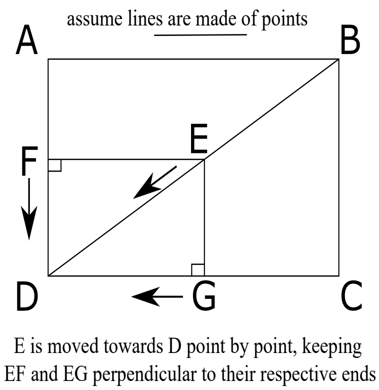

Figure 1: Torricelli's parallelogram paradox

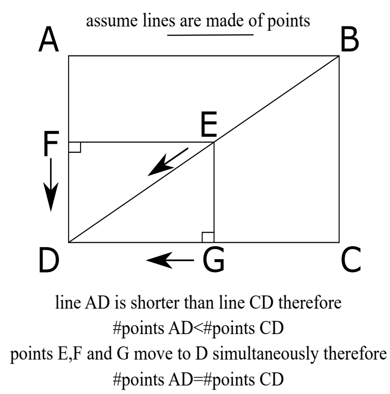

Figure 2: Torricelli's parallelogram paradox

Torricelli's meaning was that the lines AD and CD must be made up of infinitesimal segments (and not dimensionless points) and that these segments must consist of the same number in each line even if they are not of the same magnitude.

The current most advanced explanation for this paradox is said to be the difference between cardinality and magnitude[21]. While I very much agree with the presence of these fundamental properties, let us introduce some notation to gain further insight.

Specifically avoiding Leibniz's notation I need something basic to designate that I am referring to the cardinal number of segments $\#SEG$ so that I can write

$$\#SEG_{AD} = \#SEG_{CD}. \tag{18}$$

[21] See [10] page 125

Expressing that the magnitudes of these segments thus cannot be equal I write

$$|SEG_{AD}| \neq |SEG_{CD}|. \tag{19}$$

While this can and has been said to have lead to the development of the Calculus, let us dig further.

6.2 Homogeneous Infinitesimal Magnitudes: Their Sums, Differences, Strains and Relative Cardinality

Assume that the relative length of a line is defined by the sum of the magnitudes of the segments within it

$$\sum |SEG_{AD}| \equiv \mathbf{length}_{AD}. \tag{20}$$

Now let us define that the magnitude of any segment *even within the same line* can be compared to the magnitude of any other segment such that I can write the equation or inequality of one segment n vs an adjacent (or non-adjacent) segment $n-1$

$$|SEG^{n}_{AD}| - |SEG^{n-1}_{AD}| = 0 : \text{intrinsically flat} \tag{21}$$

which I will call *intrinsically flat* and

$$|SEG^{n}_{AD}| - |SEG^{n-1}_{AD}| \neq 0 : \text{intrinsically curved} \tag{22}$$

which I will call *intrinsically curved. I am unable to find an equivalent definition for curvature within any theory since the advent of the infinitesimal or indivisible concept*. While the most immediate subject to discuss concerning these equations would be the Archimedean axiom[22] (and perhaps Bernhard Riemann's definition of curvature and flatness [17]), let us hold off for a bit.

Just to enhance clarification, let us bring in line BD into our consideration also. Figures 3,4 and 5 are a visual aid for understanding the previous two equations. Let us just assume that the transfer principle is applicable so that I can use these diagrams of non-infinitesimal line segments to represent the magnitudes of infinitesimals and their cardinality. If by the property of congruence we can lay the lines BD, CD and AD next to each other and they are of unequal length, let us then imagine that we can

[22] See [10] page 50 for discussion of Euclid, Elements, book V, definition IV

use the vertical dividing lines to help denote the infinitesimal segments within each line. Torricelli's example is represented by Figure 3 so that we could understand that the magnitudes of the segments within BD are all the same (intrinsically flat). The same for CD and AD. However, the magnitudes of the segments within BD must not be the same as CD, nor AD. Again, this is what Torricelli meant when he said that points are indistinguishable whereas segments can differ by their magnitude. They are intrinsically curved relative to each other. We can then also understand that the cardinality (this can be thought of as the "number" for now) within AD must be the same as BD as well as for CD.

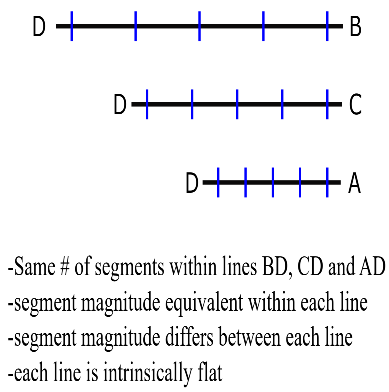

Figure 3: Intrinsically Flat Lines With Equal Cardinality

6.2.1 Relative Strain

Although strain is normally considered a physical concept, we will consider it first as a geometric one. Figure 3 is an example of relative strain ϵ_{rel} if we consider the concept that line AD is being stretched out to the length of line BD but our measurement system is another line AD that is NOT being stretched out. Thus the cardinality between our stretched line and the measurement system stays the same but the line is longer due to greater magnitude segments.

In this case a scale factor f multiplied with the length of line AD is increasing only the magnitude of the infinitesimals and NOT the cardinality,

$$f * AD = \#SEG_{AD} * (f * |SEG_{AD}|). \tag{23}$$

6.3 Torricelli's Parallelogram Theorem

Let us refer to his parallelogram setup as Torricelli's Parallelogram Theorem (as it will become within a new axiomatic framework) so that I can assign these equations as a description of it. I use the term “parallelogram" instead of “rectangle" since his Italian use of "parallelogrammo" seems to translate to the former.

(24) $$\#\text{segments}_{BD} = \#\text{segments}_{CD} = \#\text{segments}_{AD}$$

(25) $$|SEG^{n}_{BD}| - |SEG^{n-1}_{BD}| = 0$$

(26) $$|SEG^{n}_{CD}| - |SEG^{n-1}_{CD}| = 0$$

(27) $$|SEG^{n}_{AD}| - |SEG^{n-1}_{AD}| = 0$$

(28) $$|SEG_{BD}| > |SEG_{CD}| > |SEG_{AD}|$$

6.4 First Corollary to Torricelli's Theorem

However, this also means that another way to compare lines (which are not representative of Torricelli's example) would be to set the magnitudes of all the segments within the lines equivalent not only within the lines (intrinsically flat) but between the lines also. The longer the line is, the more segments it has (as opposed to dimensionless points) as in Figure 4.

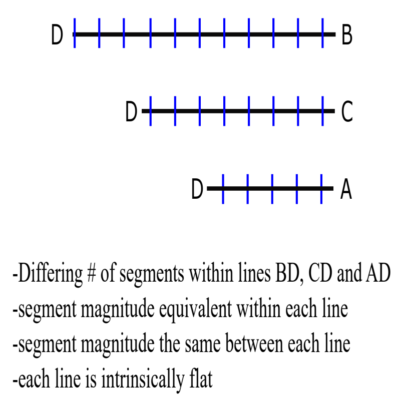

Figure 4: Intrinsically Flat Lines With Differing Cardinality

I assign these equations as a description of the First Corollary:

$$\#\text{segments}_{BD} > \#\text{segments}_{CD} > \#\text{segments}_{AD} \tag{29}$$

$$|SEG^{n}_{BD}| - |SEG^{n-1}_{BD}| = 0 \tag{30}$$

$$|SEG^{n}_{CD}| - |SEG^{n-1}_{CD}| = 0 \tag{31}$$

$$|SEG^{n}_{AD}| - |SEG^{n-1}_{AD}| = 0 \tag{32}$$

$$|SEG_{BD}| = |SEG_{CD}| = |SEG_{AD}| \tag{33}$$

6.4.1 Absolute Strain

Figure 4 is an example of absolute strain ϵ_{abs} if we consider the concept that line AD is being stretched out to the length of line BD but our measurement system is another AD that is *not* being stretched out. The magnitudes of the segments stay the same but the cardinality increases relative to our measurement system line. Thus *relative* strain is from *changing magnitude* and *absolute* strain is from *changing cardinality*.

In this case a scale factor j multiplied with line AD is increasing only the cardinality of the infinitesimals and NOT the magnitudes of the infinitesimals,

$$j * AD = (j * \#SEG_{AD}) * |SEG_{AD}|. \tag{34}$$

A scale factor as understood in regular Euclidean geometry does not distinguish between whether it is scaling the relative cardinality or the magnitude of the infinitesimals.

6.5 Second Corollary to Torricelli's Theorem

Likewise, we could also set the magnitudes of the segments to differ *within each line* (intrinsically curved). It may be helpful to imagine that the line is either compressed or stretched out in different places and to understand my hypothesis is that Riemann's example of areas that are stretched out [17] can be derived from intrinsic curvature. I do not bother with cardinality in this example as it may obfuscate the simple meaning I am attempting to demonstrate (although it is no less important). See Figure 5.

$$\#\text{segments}_{BD} <=> \#\text{segments}_{CD} <=> \#\text{segments}_{AD} \tag{35}$$

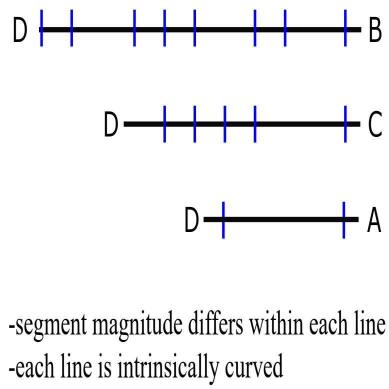

Figure 5: Intrinsically Curved Lines

(36) $$|SEG^n_{BD}| - |SEG^{n-1}_{BD}| <=> 0$$

(37) $$|SEG^n_{CD}| - |SEG^{n-1}_{CD}| <=> 0$$

(38) $$|SEG^n_{AD}| - |SEG^{n-1}_{AD}| <=> 0$$

(39) $$|SEG^{BD}| <> |SEG^{CD}| <> |SEG^{AD}|$$

Note that for the limited example here, it may be possible for the line to be flat between some HIs and curved between others, unlike the Theorem and First Corollary where they are always flat within the line.

6.6 Line Length: Sum of Segments and Relative Cardinality of Flat Lines

If I choose to build a line out of equal magnitude segments (flat), then I can represent Eqn. 20 as the number of segments or *Relative Cardinality* (RC),

(40) $$\#SEG \equiv RC,$$

times the magnitude of a representative segment $|SEG|$ so that I can write

(41) $$\sum |SEG| = RC * |SEG| \equiv \textbf{line length}.$$

Thus, in our normal sense of line length, if we wanted to compare the length of the three lines from Torricelli's Theorem we would have to define that each line is composed of segments of equal magnitude and that the line with the greatest Relative Cardinality is the longer line within Figure 4.

6.6.1 Segment Notation

See Appendix Section A.2 for my argument that Leibniz's notation $\frac{dy}{dx}$ masks the concept of measuring a change in relative cardinality[23] and can be derived from

$$\frac{\Delta RC_y}{\Delta RC_x} \equiv \frac{dy}{dx} \tag{42}$$

and thus his notation is not always logically representative of HIs.

In order to distinguish HIs, I borrow his notation and modify it to

$$|SEG| \equiv \overleftrightarrow{dx} \tag{43}$$

where the double ended arrow above *dx* indicates that it is the *magnitude* of a HI.

In order to have notation to indicate that we are examining the *difference of magnitude* between two HIs, I introduce the notation in the equation

$$\overleftrightarrow{dx}_a - \overleftrightarrow{dx}_b \equiv \overset{\Delta}{\overleftrightarrow{dx}} \tag{44}$$

where the Δ combined with the double ended arrow indicates the difference of magnitude between two HIs. This allows us to rewrite Eqn. 21 as

$$\overset{\Delta}{\overleftrightarrow{dx}} = 0 \tag{45}$$

for intrinsically flat and Eqn. 22 as

$$\overset{\Delta}{\overleftrightarrow{dx}} \neq 0 \tag{46}$$

for intrinsically curved.

6.6.2 Archimedean Axiom

It has been said that infinitesimals do not follow the Archimedean axiom[24]. As a counterargument, I am going to falsify the following statement (bold mine)[25]:

> it shows that for any o infinitesimal relative to a, the ratio $a + o : a$ determines the same upper set as $a : a$ but differs from the latter in having an empty "middle set";**since the quantities** *no* **are obviously all infinitesimal in relation to** *a*

[23] This work also allows me to predict that Christoffel symbols can be derived from this.
[24] See [13] for discussion.
[25] See page 171 [18].

Let us define the real line by Equation 41 and rewrite (**bold**) the definition given in Ref.[4]:

> Magnitudes (**infinitesimal magnitudes**) $[a, b]$, are said to have a ratio with respect to one another (**and Relative Cardinalities,** $[n, m]$**, are said to have a ratio with respect to one another**) which, being multiplied $[na]$ are capable of exceeding one another $[na > b]$ (**[na>bm]**).

Examing the following quote with respect to the rewrite,

> In the contrary case, there is an element $q > 0$ called an infinitesimal such that no finite sum $q + q + ... + q$ will ever reach 1

and stating that any number n times an infinitesimal q cannot be greater than or equal to c, $nq > c$, is simply a demonstration of the confusion of trying to compare RC n, magnitudes q and line length c with poor definitions. These are three different types of proportion: cardinality, magnitudes and their product. I propose that Eudoxus' Theory of Proportions must apply: cardinality to cardinality, homogeneous infinitesimals to homogeneous infinitesimals AND homogeneous sums to homogeneous sums such as line length to line lengths, area to area, volume to volume etc.[26].

Perhaps the easiest way to grasp this is to review absolute 6.4.1 and relative 6.2.1 strain. The Archimedean/infinitesimal confusion lies in trying to scale only infinitesimal magnitudes and comparing that with line length. Both cardinality and magnitude are required in order to represent line length (it takes at least two infinitesimals to define intrinsic flatness or curvature). Scaling only magnitude and trying to compare that to line length is a violation of Eudoxus' Theory of Proportions.

The modern mathematical notation[4] of

$$(\forall \epsilon > 0)(\exists n \in \mathbb{N})[n\epsilon > 1] \tag{47}$$

or

$$(\exists \epsilon > 0)(\forall n \in \mathbb{N})[\epsilon \leq \frac{1}{n}] \tag{48}$$

appears to not have been helpful in understanding the distinction between a segment magnitude ϵ, a cardinal number n and a line length of 1. It also does not appear to have led to the concept that infinite sets can have differing cardinality similar to finite sets nor that infinitesimals ϵ can have different magnitudes and multiple directions.

[26] I may refer to this as the Theorem of Homogeneous Infinitesimal Relativity in future papers.

6.6.3 L'Hôpital's Paradox and non-standard analysis

As further evidence, let us use the definition for line length to resolve L'Hôpital's paradox[27] and non-standard analysis. L'Hôpital's paradox is given by the equation

$$X + dx = X \tag{49}$$

where X is a non-infinitesimal line segment and dx is an infinitesimal segment. We can instead write

$$X = X \tag{50}$$

becomes

$$RC_1 \overset{\rightarrowtail\!\leftarrowtail}{dx}_1 = RC_2 \overset{\rightarrowtail\!\leftarrowtail}{dx}_2 \tag{51}$$

with

$$RC_1 = RC_2 \tag{52}$$

and

$$\overset{\rightarrowtail\!\leftarrowtail}{dx}_1 = \overset{\rightarrowtail\!\leftarrowtail}{dx}_2. \tag{53}$$

Assume for every $\overset{\rightarrowtail\!\leftarrowtail}{dx}_1$ that is added to the left side of Eqn.51, increasing the Relative Cardinality in comparison to the right side, the magnitude of the left hand side segments are also correspondingly reduced (cardinality increases in the same proportion that the infinitesimal magnitude decreases). I can then write

$$RC_1 \overset{\rightarrowtail\!\leftarrowtail}{dx}_1 + 1 \overset{\rightarrowtail\!\leftarrowtail}{dx}_1 = RC_1' \overset{\rightarrowtail\!\leftarrowtail}{dx}_1 \tag{54}$$

meaning that

$$RC_1 + 1 = RC_1'. \tag{55}$$

This gives

$$\frac{RC_1'}{RC_2} = \frac{RC_1 + 1}{RC_2} = \frac{\overset{\rightarrowtail\!\leftarrowtail}{dx}_2}{\overset{\rightarrowtail\!\leftarrowtail}{dx}_1}. \tag{56}$$

This proves that ratios of relative cardinalities can be algebraically compared to ratios of homogeneous infinitesimals in accordance with the theory of proportions.

[27] See [10] page 13 for discussion.

For non-standard analysis, "Two hyperreal numbers are infinitely close if their difference is an infinitesimal"[13] can be proven if we do *not* resize $\overset{\bowtie}{dx}_1$. Equation 51 still becomes Equation 54 but Equation 56 is instead

$$RC_1'\overset{\bowtie}{dx}_1 - RC_2\overset{\bowtie}{dx}_2 = 1\overset{\bowtie}{dx} \tag{57}$$

with the HIs still equal,

$$\overset{\bowtie}{dx}_1 = \overset{\bowtie}{dx}_2 = \overset{\bowtie}{dx}, \tag{58}$$

$$RC_1' - RC_2 = \frac{\overset{\bowtie}{dx}_2}{\overset{\bowtie}{dx}_1} = 1. \tag{59}$$

6.7 Pandora's Box of HIs

With all of this we can understand that Torricelli's actual argument was that he was choosing to define and examine the lines as having equivalent RC with differing segment magnitude[28]. With notation, mentally examine the two cases of holding constant and varying the terms within

$$\frac{RC_Y * |SEG|_Y}{RC_X * |SEG|_X}. \tag{60}$$

In the normal concept of scaling lines, RC_Y and RC_X are varied while $|SEG|_Y$ and $|SEG|_X$ are held constant. Torricelli's paradox results from the opposite case of holding RC_Y and RC_X constant and varying $|SEG|_Y$ and $|SEG|_X$.

While this hopefully seems to be a fairly simple analysis, I also view it as opening Pandora's box as it casts suspicion on the derivation of everything[29] that is based upon infinitesimals. Although it may not be obvious yet, I am not only saying that the $\frac{dy}{dx}$ of Leibniz's notation can be defined as $\frac{\Delta RC_y}{\Delta RC_x} \equiv \frac{dy}{dx}$ but also that infinity can have different magnitudes (I hypothesize that Cantor's transfinite numbers [9] can be derived from this), that Hilbert's "betweenness" postulate, basis vectors[30] and even tensors are all flawed representations of the magnitude of HIs and/or their cardinality. This path could obviously meander forever so instead let us restart by enlisting the aid of primitive notions within an axiomatic framework.

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[28] As we will see, later in this paper, there is another way to view the actual area of the rectangles based only on RC by using flat areal HIs.

[29] See [13] for examples just viewed through the lense of Calculus alone.

[30] See [16] pages 52 and 229 for examples.

7 CPNAHI: Postulates

As one author states[7], axioms should be given without justification. However I see no way to simply launch into a proof using CPNAHI as even Leibnizian notation would seem to be flawed. I know of nothing within the body of knowledge of geometry nor mathematics[31] that already contains analysis of both Torricell's work and non-Euclidean geometry from which to launch my hypotheses. Therefore, I instead will justify my primitive notions and postulates by resolving certain historical paradoxes that presently seem unsolved and/or present a different method of describing them. It is my hope that these similarities and rudimentary proofs will motivate individuals to my view that CPNAHI has compelling features. There are many things that I have not yet attempted (such as even simply deriving contravariance and covariance from CPNAHI) but I fear letting the perfect be the enemy of the good. Some of these postulates will inevitably be found to be derivable from each other but I think the practice of deducing that will be more informative than an absolutely correct set here.

7.1 Primitive Notion and Postulates

CPNAHI Primitive Notion Let a homogeneous infinitesimal (HI) be a primitive notion.

CPNAHI Postulates

(1) Postulate of Homogeneity: HIs can have the property of length, area, volume etc. Only HIs of length can sum to create lines. Only HIs of area can sum to create area. Only HIs of volume can sum to create volume. etc..[32].

(2) HIs have no shape.[33]

(3) Postulate of HI proportionality: RC, HI magnitude and the sum each follow Eudoxus' theory of proportion.

[31]Nor alternate axiom systems. See Chapter 15 [15].

[32]This is also in accordance with Eudoxus' theory of proportions which I view as equivalent to not being possible to sum heterogeneous infinitesimals. In simpler words, "stacked" two dimensional planes cannot integrate into a volume. Geometers of Torricelli's era seemed to have been tempted by summing segments to creates lines but led astray by trying to sum lines to create area and thick planes to create volume instead of just fundamental primitive notions of length, area, volume etc.

[33]No "protruding parts" to paraphrase Descartes. See [10] page 169 for discussion.

(4) HIs can be adjacent or non-adjacent to other HIs.

(5) A set of HIs can be a closed set.

(6) A lineal line is defined as a closed set of adjacent HIs (path) with the property of length. These HIs have one direction.

(7) An areal line is defined as a closed set of adjacent HIs (path) with the property of area. These HIs possess two orthogonal directions.

(8) A voluminal line is defined as a closed set of adjacent HIs (path) with the property of volume. These HIs possess three orthogonal directions.

(9) Higher directional lines possess higher orthogonal directions.

(10) The cardinality of these sets is infinite.

(11) The cardinality of these sets can be relatively less than, equal to or greater than the cardinality of another set and is called Relative Cardinality(RC).

(12) The magnitudes of a HI can be relatively less than, equal to or the same as another HI.

(13) The magnitude of a HI can be null.

(14) If the HI within a line is of the same magnitude as the corresponding adjacent HI, then that HI is intrinsically flat relative to the corresponding HI.

(15) If the HI within a line is of a magnitude other than equal to or null as the corresponding adjacent HI, then that HI is intrinsically curved relative to the corresponding HI.

(16) A HI that is of null magnitude in the same direction as a path is defined as a point.

(17) Adjacent points within adjacent areal lines are said to create an arc. These points have the property of length in one direction and are null in the second.

(18) Adjacent points within adjacent voluminal lines are said to create a surface. These points have the property of area in two directions and are null in the third.

8 CPNAHI Lines

8.1 Lineal Lines

A lineal line is defined as a path consisting of lineal HIs (i.e the segments). A point in a lineal line is defined as a HI null in the direction of the line. There are no other directions (undefined). If each adjacent non-null HI is of equivalent magnitude then the line is intrinsically flat. If an adjacent non-null HI is of differing magnitude then the line is intrinsically curved.

Since I know of no properties to distinguish between a one-dimensional $\mathbb{R}^1$ and a flat lineal line, I view this as why Torricelli's example works when examining the line segments comparatively. Although his parallelogram possessed area, geometrically he was simply examining one of the issues that arise from not clearly defining relative cardinality and magnitudes. Lineal line points are similar to points within $\mathbb{R}$ since both have null infinitesimal magnitude. See Lineal Line Pseudo-Point Geometry, Appendix Section A.1 for a discussion of $\mathbb{R}^n$ for $n \geq 2$.

Let us switch notation from x to y. Define a "line" as a set or "path" of adjacent HIs that possess length and an "intrinsically straight line" (Figure 6) as a path of "adjacent" HIs with the property of

$$\overleftrightarrow{dy}_n - \overleftrightarrow{dy}_{n+1} = \overset{\triangle}{dy} = 0. \tag{61}$$

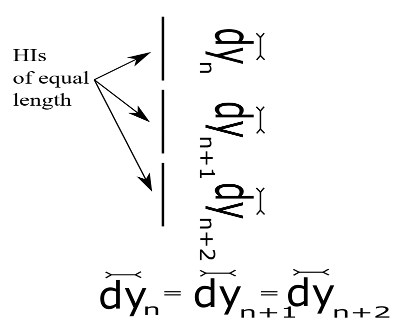

Figure 6: Intrinsically straight lineal line made of equal HIs

An intrinsically curved line made of HIs of length (Figure 7) could be represented by HI $\overleftrightarrow{dy}_n$ having greater infinitesimal length than HI $\overleftrightarrow{dy}_{n+1}$. This allows us to write the inequality

$$\overleftrightarrow{dy}_n > \overleftrightarrow{dy}_{n+1} \tag{62}$$

which gives us

$$\overleftrightarrow{dy}_n - \overleftrightarrow{dy}_{n+1} = \overset{\triangle}{dy} \neq 0. \tag{63}$$

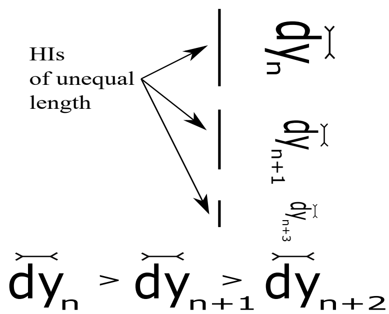

Figure 7: Intrinsically curved lineal line made of unequal HIs

8.2 Areal Lines

An areal line is defined as a path consisting of areal HIs. A point in an areal line is null in the direction of the line but is non-null in the orthogonal direction. If an adjacent non-null HI is of equivalent magnitude in the direction of the line, it is intrinsically flat. If an adjacent HI is of differing non-null magnitude then the line is intrinsically curved.

8.2.1 Euclidean lines vs CPNAHI areal lines

Figure 8 demonstrates the conception of columns of intrinsically flat areal HIs that makeup areal lines that can be summed to create area. This is in opposition to the heterogeneous view of stacked lines of zero width being summed to create area.

Figure 9 is a representation of a 2 dimensional Cartesian Coordinate system. If I assume the background is composed of flat areal HIs, then we can represent the line lengths for coordinates using the CPNAHI equation for a flat line. Thus the coordinates are actually just the relative cardinality.

8.2.2 Euclidean lines vs CPNAHI Area Lines for a Circle

Figure 10 demonstrates that an areal line composed of HIs of area can define the radius of a circle[34]. Areal points, which are HIs that are null in the line direction (radial here), can be defined as forming the circumference of circles as in Figure 10. Every circle has the same number of points. The reason that the circumference grows as the radius

[34] also see [10] p. 28 Al-Ghazalis wheel

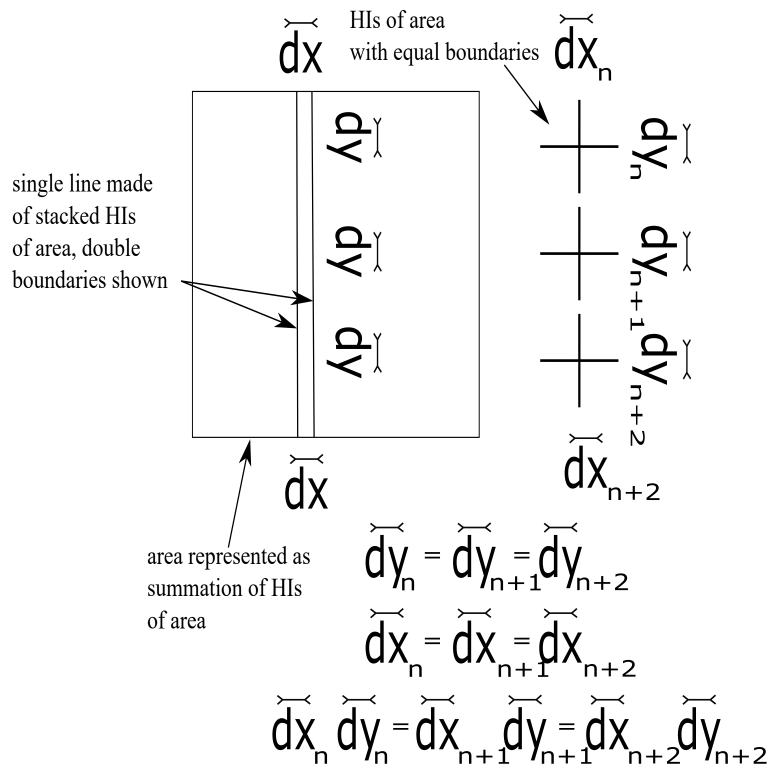

Figure 8: Columns of Flat Areal HIs Summed To Create Areal Lines That Are Summed To Create Area Of A Parallelogram

increases is due to the increase in the relative magnitude of the HIs in the orthogonal direction[35]. Note that the circumference of every circle is intrinsically flat. A note in the margin of Torricelli's Opere essentially describes the properties of these areal lines ("tapering") as they make up a circle[36]. I predict that a 1-sphere can be proven to have the same properties as Figure 10.

Figure 11 is a conceptual comparison between multiple lines penetrating the circumferences of concentric circles versus single lines that penetrate and form the circumference of radial circles within CPNAHI. Understand that the dual "lines" in the figure on the right indicating the increasing magnitudes of the HIs can be graphically misleading. Only a single line is represented in the figure on the right and it has the property of area. The lines radiating out from the figure on the left possess no width but only

[35]Kepler viewed these as triangles. See [10] p. 62.
[36]see [10] p. 126 for discussion

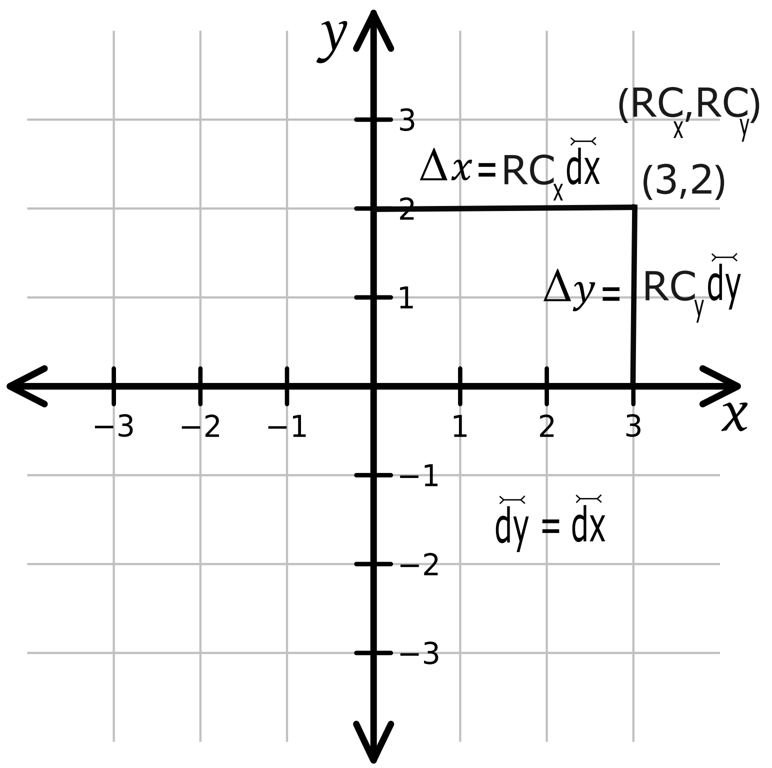

Figure 9: Intrinsically Flat Areal HIs Creating Cartesian Coordinate System

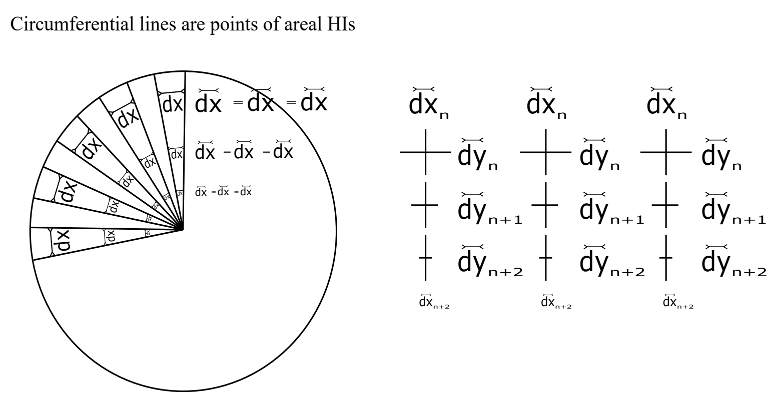

Figure 10: Intrinsically Curved Areal Lines Summed To Create Area Of A Circle

length. The figure on the left has no explanation for why there would be an increasing number of non-dimensional points in the circumference as the radius increases whereas the circumference of the circles on the right increases because the magnitude of the infinitesimals that make it up increases even while the number of segments stays the same.

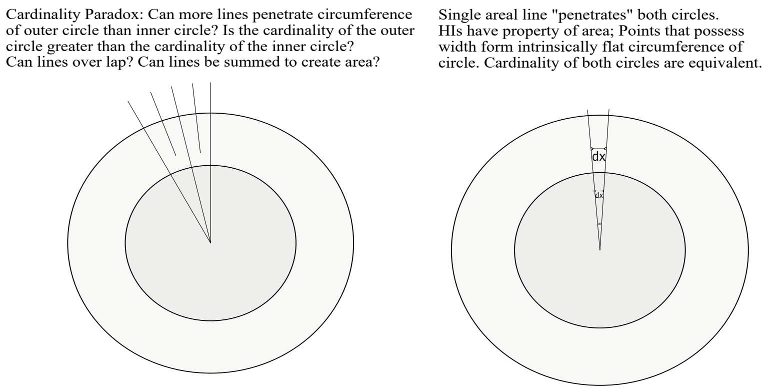

Figure 11: Euclidean Circle Paradox Versus CPNAHI Areal Lines

8.3 Voluminal Lines

A voluminal line is defined as a path consisting of voluminal HIs. Figure 12 is a graphical aid in understanding a voluminal line. A point in a voluminal line is null in the direction of the line (z) but is non-null in the two orthogonal directions (x and y). If an adjacent HI is of equivalent magnitude in the direction of the line, it is intrinsically flat. If an adjacent HI is of differing non-null magnitude then the line is intrinsically curved.

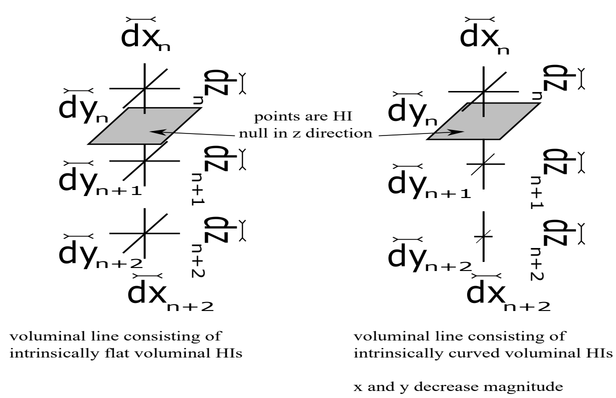

Figure 12: Intrinsically Straight and Curved Voluminal Lines

In Figure 12, the image on the left is intrinsically flat so that we can write

$$\overset{\succ\!\prec}{dx}_{n} - \overset{\succ\!\prec}{dx}_{n+1} = \overset{\succ\!\triangle\!\prec}{dx} = 0, \tag{64}$$

$$\overset{\succ\!\prec}{dy}_{n} - \overset{\succ\!\prec}{dy}_{n+1} = \overset{\succ\!\triangle\!\prec}{dy} = 0, \tag{65}$$

and

$$\overset{\succ\!\prec}{dz}_{1} - \overset{\succ\!\prec}{dz}_{n+1} = \overset{\succ\!\triangle\!\prec}{dz} = 0. \tag{66}$$

The one on the right is intrinsically curved in the orthogonal directions. As we examine down in the z direction we write

$$\overset{\succ\!\triangle\!\prec}{dx} <> 0, \tag{67}$$

$$\overset{\succ\!\triangle\!\prec}{dy} <> 0, \tag{68}$$

and

$$\overset{\succ\!\triangle\!\prec}{dz} = 0. \tag{69}$$

In both we have a voluminal point represented by the shaded parallelogram. It is null in the z direction. I could also show a voluminal line that is intrinsically curved in the z direction but that isn't necessary for the similarity we are examining.

I find it has the same properties as the constituent of a sphere as areal lines do when they constitute a circle (see Figure 11). If the points of adjacent voluminal lines make up the "surface" of a sphere then each concentric surface would have the same cardinality as the radius increases because the points now have the property of area and that area increases as the radius increases.[37]

8.3.1 Foreground and Background Geometry

Let us assume that we can have a sum of voluminal HIs to create volume. From this we can philosophically create the concept of a coordinate system. "Events" can take place on this coordinate system which do not effect the background itself. Events are

[37] The affine connection of rolling a "Euclidean plane" along the surface would seem to be derivable from this definition.

distinguished via changes in relative cardinality only if the HIs are flat. I refer to this as (Flat) Background (Voluminal) Geometry.

Let us also assume that we can have a different sum of voluminal HIs to create volume. No events take place “on" this geometry. Instead, all events are simply changes in relative cardinality and magnitudes of the HIs themselves. I refer to this as Foreground (Voluminal) Geometry.

With this Foreground/Background understanding and that voluminal lines can be intrinsically curved not only in the direction of the line but also with respect to the orthogonal directions of their HIs, let us examine the properties of a set of these as compared to Gaussian curvature.

8.3.2 Similarity Between Principal Curvature of Points on a Surface and Intrinsic Curvature Across Points Within Voluminal Lines

There are four basic types of points on surfaces within Gaussian curvature: elliptic, hyperbolic, parabolic and planar. If we examine the principal curvature K properties for each type of point, we can see a pattern in Table 1 that matches the orthogonal intrinsic curvature of voluminal lines within CPNAHI:

Table 1: Gaussian Principal Curvature Vs Voluminal Line Orthogonal Direction Intrinsic Curvature

	principal curvature K_1 and K_2	intrinsic curvature of $\overset{\triangle}{dx}$ and $\overset{\triangle}{dy}$
elliptic	same sign	same sign
hyperbolic	opposite sign	opposite sign
parabolic	one is 0 and other is pos or neg	one is flat and other is pos or neg
planar	both are 0	both are flat

8.3.3 Manifolds and 2-spheres vs CPNAHI voluminal point surfaces

Compare a pictorial concept of rays emanating from a source within a differentiable manifold as shown[38] in the 2-sphere from Figure 9.3 p. 241 in *Gravitation*[16] . Note the similarities of “rays" and voluminal lines if S^2 is a “manifold" and the *“rays" are the points of* S^2. I hypothesize that a “2 sphere" is equivalent to saying the surface is

[38] Not included due to inability to obtain copyright permission.

made of the area of the voluminal points that are contained within the voluminal lines radiating as "rays" from the origin. Note that I do not show the z direction in these drawings varying in magnitude along with x and y for the sake of simplicity.

If one considers that the voluminal points that make up the area of the sphere (see Figure 12) and the points are within the voluminal lines, then there is a similarity between the "rays" of manifolds and CPNAHI voluminal lines. I predict that Gaussian curvature can be proven to be a flawed view of voluminal lines.[39]

I am assuming for now that the need for the two infinitesimal terms $dx_\mu dx_\nu$ in GR is that these can be derived from $\overset{\rightarrowtail\leftarrowtail}{dx}$ and $\overset{\rightarrowtail\leftarrowtail}{dy}$ and that $\overset{\rightarrowtail\leftarrowtail}{dz}$ is considered to be the same magnitude[40].

9 Summary

This paper has introduced the homogeneous infinitesimal as an axiomatic primitive notion, viewing it through the concept of background and foreground geometry and comparing all of that with the concept of Euclidea/non-Euclidean geometry. The motivation for this is the Cosmological Constant problem. Using the logic of CPNAHI I have attempted to provide compelling evidence that Euclidean and non-Euclidean geometry, the Calculus, as well as our laws of physics are all based upon the undocumented properties of homogeneous infinitesimals within this axiomatic framework. Euclidean geometry and the Calculus would be based upon a flat HI background where lines are composed of LLPP geometry. Functions are defined based on the Relative Cardinality of these lines. Non-Euclidean geometry would be derived from flat and intrinsically curved foreground HIs where lines are lineal, areal, voluminal etc. The intractability of the Cosmological Constant problem would ultimately stem from the enormous amount of research that resulted in the local accuracy of the relationship between energy/momentum and "curved space-time" but also contains the inherent systemic flawed understanding of homogeneous infinitesimals and the required redefining of energy/momentum.

[39] It may also be helpful to view the similarity with Figure 9.2 on page 232 [16].

[40] In a future paper I will explore if this is what is meant by a "Riemannian" metric.

Appendix A Issues With The Distinction Between Areal HIs And Euclidean Geometry

A.1 LLPP on Flat Background HIs

There is an issue between drawings such as Figure 9 and points as defined in CPNAHI. Conceptually, a point in Euclidean geometry has no dimension no matter how many directions of real space it exists in. In other words two lines in two dimensional Euclidean geometry can intersect at a point. With CPNAHI lineal lines, nothing conceptually exists off the line so having two lineal lines “intersect" is an undefined concept. For area in CPNAHI this is still not possible since two dimensional area requires the summation of areal HIs and a point is defined as null magnitude HI in the path of an area line. In other words, these points DO have dimension (infinitesimal) and so two areal lines would seem to be able to intersect, but this non-point intersection would seem to have the property of area[41].

Therefore let us consider a background geometry consisting of areal HIs. For this introduction we will consider only areal lines and that they are all flat. If we relax the CPNAHI point postulate for lines so that the points are always null in all directions, then this seems to me to be equivalent to points in Euclidean geometry and I call this a pseudopoint. We allow a lineal line to be drawn on this background so that any and all drawn lineal lines are flat and can now intersect at the pseudopoints. See Figure 11 for the issue this causes with cardinality. Let us call these special conditions Lineal Line PseudoPoint geometry (LLPP).

Philosophically, an “absolute object" can exist “on" this background and line lengths as defined by Equation 41 can be used to define a coordinate system. In other words, an objects position can be defined via the coordinate system created from the background areal HIs. Note that for the Second Corollary to Torricelli’s Theorem this does not apply because I intentionally analyzed the lines from their components comparatively without background areal HIs limiting our ability to change the magnitude of the segments (although this can and should be done at some time in the future just for clarification).

[41] I have no issue with standard Euclidean geometry and points as a geometric method to convey information. Because of the apparent developing consistency of CPNAHI, I now have an issue when Euclidean geometry is combined with the concepts of the infinitesimal and infinity.

A.2 Areal Homogeneous Infinitesimals Versus Leibnizian/Newtonian Differentiation

In Figure 13 I have Keisler's example[12] of a tangent line drawn to a line $y = f(x)$. I purposefully use his example as I hypothesize that non-standard analysis can be derived from CPNAHI. Extrinsic curvature in LLPP[42] is defined as a change in length of an areal line segment with respect to two lines. This must be a change in RC since the background is flat. Any lines drawn upon this flat background are in and of themselves flat from segment to segment within the path of the line.

Hypothesis: This image can be described as using LLPP to describe a change in the RC of areal HIs.

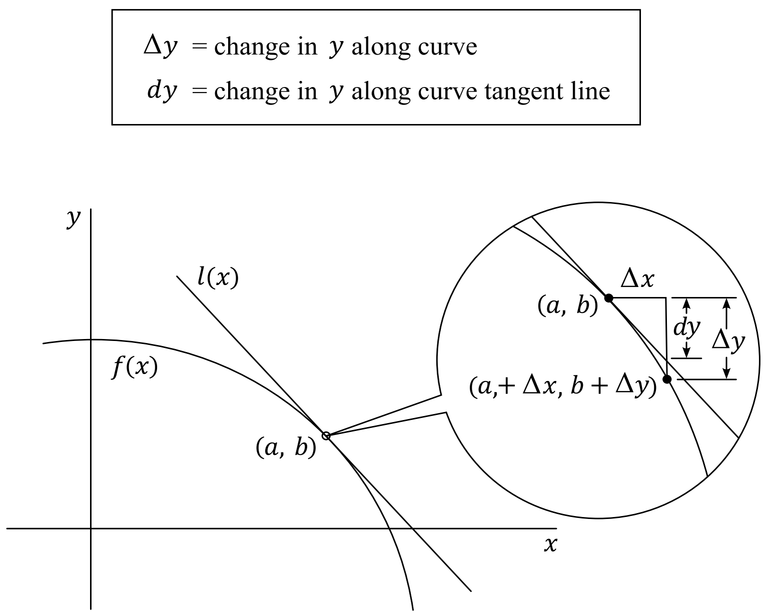

Figure 13: Keisler differential Figure 2.2.3

In Figure 14 I have placed what look like + signs in Keisler's image to indicate that the background consists solely of flat areal HIs. These signs represent the $\overset{\asymp}{dy}$ and $\overset{\asymp}{dx}$ of the background areal HIs. The drawn lines all use LLPP to exist. From Equation 41, we can see that within this diagram

$$\Delta y = RC_y \overset{\asymp}{dy} \tag{70}$$

[42] which does not appear to be the same as MTW [16] 21.5 after a perfunctory examination

and

(71) $$\Delta x = RC_x \overset{\succ\!\!-\!\!\prec}{dx}\ .$$

Since the background areal HIs, $\overset{\succ\!\!-\!\!\prec}{dy}$ and $\overset{\succ\!\!-\!\!\prec}{dx}$ are defined as flat then we know that

(72) $$\overset{\succ\!\!-\!\!\prec}{dy} = \overset{\succ\!\!-\!\!\prec}{dx}$$

and thus we get the equality of ratios

Δy = **change in** y **along curve**
dy = **change in** y **along tangent line**

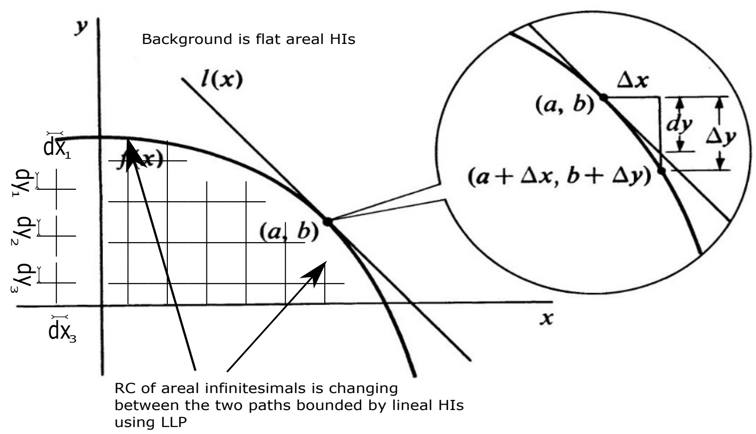

Figure 14: Keisler Differential Explained Via LLPP

(73) $$\frac{\Delta y}{\Delta x} = \frac{RC_y \overset{\succ\!\!-\!\!\prec}{dy}}{RC_x \overset{\succ\!\!-\!\!\prec}{dx}}.$$

Note that Δx and Δy are line segment representing a change in a and b. So although I could write

(74) $$\frac{\Delta RC_y}{\Delta RC_x} \equiv \frac{dy}{dx}$$

it would be correct to also set $\Delta RC_x = 1$ and to view differentiation as the column of areal HIs that is ΔRC_y high.

We can then understand that the following non-CPNAHI notation is a special case of flat HIs where every segment is divided into equal segments:

$$\lim_{1\to\infty} \Delta x_n = dx. \tag{75}$$

By definition, on a flat background, any line with which you would compare the length with would have identical segment magnitude and thus the line length depends solely on RC.

This also means that if the RC is constant, then there is no change of cardinality and the ratio of Equation 74 is equal to 0, identical to the property of differentiating a constant of integration. Bringing in the concept of a "scalar multiple of a metric Λ", the relationship between this, Equation 8 and constant relativity cardinality can be understood. For simplicity sake, I do not expand upon that in this paper.

A.3 Leibniz Notation, Integration, Fundamental Theorem of Calculus, Euclid's Parallel Postulate, Straight Voluminal Lines in a Curved Foreground

It may be obvious by now that basic integration can be derived as the summation of columns of areal HIs $RC_y \overset{\asymp}{dy}$ high by $RC_x \overset{\asymp}{dx}$ wide,

$$\sum RC_y \overset{\asymp}{dy}\, RC_x \overset{\asymp}{dx}\,. \tag{76}$$

Since $RC_x = 1$ for a column and with the RC function $y = f(x)$[43], we can see the notational difficulties that necessitated new notation different from Leibniz since

$$\sum RC_y \overset{\asymp}{dy}\, \overset{\asymp}{dx} = \int f(x)dx. \tag{77}$$

I do not put in here a discussion of the length of $y = RC_y \overset{\asymp}{dy}$ from the origin as a function of the length of $x = RC_x \overset{\asymp}{dx}$ from the origin as it should be obvious from the previous equations.

A.3.1 Fundamental Theorem of Calculus

Thus, within CPNAHI, we can understand the Fundamental Theorem of Calculus is defined by using LLPP on a flat background: As a simple explanation, for

[43] see [13] for a discussion of when Euler helped move the Calculus from being about variables of geometry to functions

differentiation it is the change in cardinality of columns of areal HIs bounded between two lines. For integration it is the summation of columns of areal HIs bounded between two lines. Leibniz's notation seems to work because we are always comparing the Relative Cardinality of y with the Relative Cardinality of x but RC_x is defined as 1 in differential notation of $\frac{dy}{dx}$. This can easily be seen by changing a graph from x and y and using a Cartesian Coordinate plot instead.

A.3.2 Euclid's Parallel Postulate

Similarly, Euclid's parallel postulate can be defined through Geometric Conservation, in that two LLPP lines are parallel provided the Relative Cardinality of the column of flat background areal HIs that they bound between them does not change.

A.3.3 Straight Voluminal Lines in a Curved Foreground

In foreground geometry straight voluminal lines can be defined, using Geometric Conservation, by the magnitude of the three orthogonal components as shown in Figure 12. Even if $\overset{\bowtie}{dy}_n$ and $\overset{\bowtie}{dx}_n$ are not equal in magnitude relative to each other but do not change relative to themselves, $\overset{\bowtie}{dy}_n = \overset{\bowtie}{dy}_{n+1}$ and $\overset{\bowtie}{dx}_n = \overset{\bowtie}{dx}_{n+1}$ along the voluminal line path, then this is how a line can be straight in an intrinsically curved foreground. I will expand upon this and the similarities/differences between GR world lines in a future paper.

Appendix B HI functions versus RC functions

Suppose that we are using LLPP, and that the length of one line is dependent on the length of another line orthogonal to it. I call this an RC function and write

$$\Delta RC_y = f(\Delta RC_x). \tag{78}$$

I detect no difference between this and $y = f(x)$. Essentially the length of y is a dependent on the length of x.

Suppose that we have a lineal HI $\overset{\bowtie}{dx}$ and that the magnitude of second HI $\overset{\bowtie}{dy}$ is dependent upon the magnitude of the first. I write

$$\overset{\Delta}{dy} = f(\overset{\Delta}{dx}) \tag{79}$$

and call this a HI function.

B.1 Proposed Overview of HI functions versus RC functions

Imagine two sets of flat voluminal HIs that sum up to create two volumes, one background and one foreground. I hypothesize that upon the background flat voluminal HIs, LLPP is used to create $\mathbb{R}$, Euclidean geometry and the Calculus with RC functions upon which Newtonian physics relies. With the foreground we are free to change magnitudes of the HIs themselves, creating intrinsic curvature and geometric waves (maxima and minima) including using HI functions.

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