Integration of rational functions

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November 22, 2024

Abstract

This article has been based on a lecture handout for high school students. Most of the calculus books mention the method of partial fractions in an algorithmic way. I have described the reason behind the method. It has been mentioned as a theorem without proof in Problems in analysis by Prof Maron published by Mir Publications of the Soviet Era. cited in the text of the article. It is possible that the proof is also included in some book maybe in Russian language but I have not come across the reasoning in any English language book and thus is novel to the best of my knowledge. The proof uses arguments accessible to high school students who have seen polynomials and complex numbers before, such as the class I was lecturing in India.

1 Introduction

Antiderivatives of a large class of functions called the rational functions will be discussed in this article. Rational functions of a single variable, $R(x)$ are of the form $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials over the reals.

2 The theorem

We will discuss the main result[1] which will enable us to compute the antiderivative of any rational function. The idea is to express any rational function of a single variable as a sum of a polynomial and fractions of the forms $\frac{A}{x-a}^k$ and $\frac{Mx+N}{x^2+cx+d}^r$ all of whose antiderivatives are known and easier to compute. The polynomial $Q(x)$ in its most general form can be written as product of powers of linear and quadratic factors. A zero of any polynomial over $\mathbb{R}$ either is real or complex whose imaginary part in non-zero, say a. If a is real, a power of the linear polynomial $x-a$ is a factor and if a is complex then a power of the quadratic polynomial over $\mathbb{R}$, $(x-a)(x-\overline{a}) = x^2 - (a+\overline{a}x + a\overline{a}) = x^2 - 2\mathrm{Re}(a) + |a|^2$

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is a factor since complex roots of a polynomial with coefficients in $\mathbb{R}$ occur in conjugate pairs. If a is a root of $f(z) = 0$ then $\overline{f(z)} = 0$ has $\overline{a}$ as a root and $\overline{f(z)} = f(z)$ since the coefficients are real. So all the factors are either linear or quadratic which cannot be factorized further over $\mathbb{R}$.

So, $Q(x)$ can be written as $(x-a)^k(x-b)^l \ldots (x^2+c_1x+d_1)^{r_1}(x^2+c_2x+d_2)^{r_2}\ldots$ If $\frac{P(x)}{Q(x)}$ is a proper fraction, that is degree of $P(x)$ is less than degree of $Q(x)$ then $\frac{P(x)}{Q(x)}$ can be written as sum of fractions called partial fractions as follows:

$$\begin{aligned}\frac{P(x)}{Q(x)} =& \frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \ldots + \frac{A_k}{(x-a)^k} + \\ & \frac{B_1}{(x-b)} + \frac{B_2}{(x-b)^2} + \ldots \frac{B_l}{(x-b)^l} + \ldots + \\ & \frac{M_1x+N_1}{(x^2+cx+d)} + \frac{M_2x+N_2}{(x^2+cx+d)^2} + \ldots \frac{M_{r_1}x+N_{r_1}}{(x^2+cx+d)^{r_1}} + \\ & \frac{R_1x+L_1}{(x^2+px+q)} + \frac{R_2x+L_2}{(x^2+px+q)^2} + \ldots \frac{R_{r_2}x+L_{r_2}}{(x^2+px+q)^{r_2}} + \ldots\end{aligned} \tag{1}$$

where $A_1, A_2 \ldots A_k, B_1 B_2 \ldots B_k \ldots M_1, N_1 \ldots M_{r_1}, N_{r_1}, \ldots, R_1, L_1, \ldots R_{r_2}, L_{r_2} \ldots$ are real numbers to be determined.

A proper rational expression such as $\frac{P(x)}{(x-a)^2}$ can be written as $\frac{bx+c}{(x-a)^2}$ since the denominator is square of a linear factor and thus is a quadratic and so the numerator must be a linear polynomial or a constant. This can also be written as $\frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2}$ since adding up and equating coefficients of the numerators lead to a unique solution $A_1 = b$ and $A_2 = ab + c$. Similarly partial fractions can be written for for other powers of linear factors in the denominators and all the powers quadratic factors in the denominators. $\frac{P(x)}{Q(x)}$ can be written as sum of proper rational functions with the factors of $Q(x)$ as their denominators.

A rational function which is not proper can be written as a sum of a polynomial and a proper rational function. This is because the numerator can be written as the sum of product of the denominator with the quotient polynomial and remainder polynomial whose degree is less than that of the denominator.

3 Examples

One in which the integrand is not a proper fraction:

Problem 1.

$$I = \int \frac{x^4 - 3x^2 - 3x - 2}{x^3 - x^2 - 2x}$$

Solution. The integrand is not a proper fraction. $x^4 - 3x^2 - 3x - 2 = x(x^3 - x^2 - 2x) + (x^3 - x^2 - 2x) + (-x - 2) = (x+1)(x^3 - x^2 - 2x) - (x+2)$, so the integrand can be written as

$$\frac{x^4-3x^2-3x-2}{x^3-x^2-2x} = x+1-\frac{x+2}{x^3-x^2-2x}.$$

The first term is a polynomial and thus can be integrated easily. The second term is a proper fraction and can be written as partial fractions and then integrated as in the previous example as follows:

$$x+1-(\frac{-1}{x}+\frac{2}{3(x-2)}+\frac{1}{3(x+1)})$$

and thus the integral is

$$I = x^2/2 + x + \ln|x| - (2/3)\ln|x-2| - (1/3)\ln|x+1| + c$$

where c is a constant of integration. ■

A quadratic factor, equating coefficients:

Problem 2.

$$I = \int \frac{x dx}{x^3+1}.$$

Solution. The integrand can be written as $\frac{x}{x^3+1} = \frac{x}{(x+1)(x^2-x+1)} = \frac{A}{(x+1)} + \frac{Bx+C}{(x^2-x+1)}$. Adding the partial fractions and equating the numerators, $x = A(x^2-x+1)+(Bx+C)(x+1)$. This is identically true, so the coefficients are same. $A+B=0, -A+B+C=1, A+C=0$. Thus $C=B=-A$ and from the second equation $A=-1/3$, so $B=C=-A=1/3$.

The antiderivative is $I = \int -dx/3(x+1) + (1/3)\int \frac{(x+1)dx}{x^2-x+1}$. The first antiderivative is $(-1/3)\ln|x+1|$ and the second integral can be computed using the trig substitution by tan function after writing the denominator in the form t^2+a^2 since $\tan^2\theta+1=\sec^2\theta$ and $\frac{d\tan\theta}{d\theta} = \sec^2\theta$. Completing the square in the denominator, $x^2-x+1 = (x^2-2(1/2)x+1/4-3/4) = (x-1/2)^2+(\sqrt{(3)}/2)^2$. Writing $(x-1/2)$ as t, the second antiderivative is $\int\frac{(t+3/2)dt}{t^2+3/4} = \int\frac{t dt}{t^2+3/4} + 3/2\frac{dt}{t^2+(\sqrt{(3)}/2)^2}$.

The first antiderivative is of the form $(1/2)\int\frac{f'(x)dx}{f(x)}$ and thus can be written as $(1/2)\int\frac{dz}{z}$ where $z=f(x)$ which is $(1/2)\ln|z| = (1/2)\ln|f(x)| = (1/2)\ln(t^2+3/4) = (1/2)\ln((x-1/2)^2+3/4) = (1/2)\ln(x^2-x+1)$ since $t^2+3/4$ is always positive. The second antiderivative, using the tan substitution $t = (\sqrt{(3)}/2)\tan\theta$, is $\sqrt{(3)}\tan^{-1}(t/(\sqrt{(3)}/2)) = \sqrt{(3)}\tan^{-1}\frac{2x-1}{\sqrt{(3)}}$. ■

References

[1] I.A. Maron. *Problems in Calculus of One Variable: With Elements of Theory.* Mir, 1973.