Main problems in constructing quantum theory based on finite mathematics

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Abstract

As shown in our publications, quantum theory based on a finite ring of characteristic p (FQT) is more general than standard quantum theory (SQT) because the latter is a degenerate case of the former in the formal limit $p \to \infty$. One of the main differences between SQT and FQT is the following. In SQT, elementary objects are described by irreducible representations (IRs) of a symmetry algebra in which energies are either only positive or only negative and there are no IRs where there are states with different signs of energy. In the first case, objects are called particles, and in the second - antiparticles. As a consequence, in SQT it is possible to introduce conserved quantum numbers (electric charge, baryon number, etc.) so that particles and antiparticles differ in the signs of these numbers. However, in FQT, all IRs necessarily contain states with both signs of energy. The symmetry in FQT is higher than the symmetry in SQT because one IR in FQT splits into two IRs in SQT with positive and negative energies at $p \to \infty$. Consequently, most fundamental quantum theory will not contain the concepts of particle-antiparticle and additive quantum numbers. These concepts are only good approximations at present since at this stage of the universe the value p is very large but it was not so large at earlier stages. The above properties of IRs in SQT and FQT have been discussed in our publications with detailed technical proofs. The purpose of this paper is to consider models where these properties can be derived in a much simpler way.

Keywords: finite mathematics; standard mathematics; finite quantum theory; standard quantum theory

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List of Abbreviations

FM: finite mathematics

SM: standard mathematics

SR: special relativity

NM: nonrelativistic mechanics

QT: quantum theory

CT: classical theory

FQT: Quantum theory based on finite mathematics

SQT: Standard quantum theory

IR: irreducible representation

QFT: Quantum Field Theory

NQT: Nonrelativistic Quantum Theory

RQT: Relativistic Quantum Theory

dS: de Sitter

AdS: Anti de Sitter

dSQT: de Sitter Quantum Theory

AdSQT: Anti de Sitter Quantum Theory

1 The main goal of this paper

One of the key problems of QFT is the problem of divergences: the theory gives divergent expressions for the S-matrix. While in renormalized theories, the divergences can be eliminated by renormalization, in non-renormalized QFTs, they cannot be eliminated and this is a great obstacle for constructing quantum gravity based on QFT.

The problem of divergences has been considered by many physicists, and there has long been an idea in the air that this problem can only be solved within the framework of a discrete and finite quantum theory. It would seem natural to think that such a theory should proceed from discrete and finite mathematics. However, most mathematicians and physicists believe that SM (with infinities and continuities) is fundamental while discrete and finite mathematics is a science of a lower rank which is only needed for applications in some models. This point of view has developed for historical reasons (because more than 300 years ago Newton and Leibniz proposed the calculus of infinitesimals) and due to the fact that SM has achieved many impressive successes in describing experimental data.

The calculus of infinitesimals seemed natural when people did not know about elementary particles and thought that any substance could be divided into any arbitrarily large number of arbitrarily small objects. But now we know that at the level of elementary particles there are no arbitrarily small parts and no continuity.

Also, history tells us that if a theory successfully describes many experimental data, this is not yet a guarantee that this theory is the most fundamental. For example, NM successfully describes a lot of experimental data and before the creation of SR it was believed that NM was a fundamental theory. SR did not refute NM, but showed that the latter is a degenerate case of the former in formal limit $c \to \infty$ where c is usually treated as the speed of light. As shown in our works [1, 2, 3, 4], FM is more general (fundamental) than SM: the latter is a degenerate case of the former in formal limit $p \to \infty$ where p is a characteristic of a ring in FM.

Several famous physicists (e.g., Gross, Nambu, Schwinger and Weyl) discussed approaches when QT involves FM (see e.g., [5]). They are called hybrid quantum systems and described in [6]. The reason is that here physical quantities belong to a finite ring but quantum states are elements of standard Hilbert spaces. On the other hand, in [1, 2, 3], we have proposed an approach called finite quantum theory (FQT) where not only physical quantities but also quantum states are described by finite rings. We have shown that FQT is more general (fundamental) than SQT: SQT is a degenerate case of FQT in formal limit $p \to \infty$ where p is the characteristic of the ring in FQT.

In SQT, elementary objects are described by IRs of symmetry algebras in which energies can be either ≥ 0 or ≤ 0 and there are no IRs with both positive and negative energies. In the first case, objects are called particles and in the second - antiparticles, and after secondary quantization, the energies of antiparticles also become positive. In SQT, particles and antiparticles are characterized by additive quantum number, e.g., the electric charge, the baryon number and others. If particle A is characterized by some additive quantum numbers and antiparticle B has the same mass and spin as A, but additive quantum numbers of B are equal to the corresponding additive quantum numbers of A with the opposite sign, then B is called the antiparticle for A. In SQT there are superselection rules that prohibit the superposition of a particle and its antiparticle. For example, electron-positron or proton-antiproton superpositions are prohibited, and this is interpreted as a consequence of the conservation of electric charge and baryon quantum number.

However, in FQT, one IR necessarily contains states with both positive and negative energies. Since such states belong to the same IR, their superpositions are allowed and there are no superselection rules. One can *formally* call states with positive energies particles and assign some additive quantum numbers to them, and call states with negative energies antiparticles and assign opposite quantum numbers to them. Then it turns out that there are no conservation laws for such quantum numbers, and, for example, electron-positron or proton-antiproton superpositions are allowed. It is clear that this completely contradicts the basic concepts of SQT. In other words, in FQT, the standard concepts of electric charge, baryon quantum number and other additive quantum numbers do not work.

This situation may prompt physicists to declare that FQT is not a physical theory and should be rejected. However, symmetry in FQT is higher than symmetry in SQT since one IR in FQT splits into two IRs in SQT with positive and negative energies in the formal limit $p \to \infty$ when FM goes to SM. At the popular level, this situation can be described as follows.

Suppose there are two theories, Theory 1 and Theory 2. In Theory 1, energies in IRs are represented by points on a circle so that the energies on the right semicircle are called positive, and on the left semicircle negative. Since states with positive and negative energies belong to the same IR, their superpositions are allowed. Now let's suppose that in Theory 2 there are two types of IRs: in IRs of the first type, energies can only be positive, and in IRs of the second type - only negative. Then superpositions of states with positive and negative energies are prohibited since such states belong to different IRs. Then Theory 1 in which there is one IR describing the entire circle has higher symmetry than Theory 2 in which there are two IRs describing the right and left semicircles independently.

In such a scenario, the fact that at the present stage of the evolution of the universe, SQT describes experiments with very high accuracy follows the fact that at this stage, the quantity p is very large. As shown in [1, 2], within the framework of semiclassical approximation to FQT, it is possible to derive the law of universal gravitation where the gravitational constant G is proportional to $1/ln(p)$. By comparing this result with the experimental value of G, one gets that $ln(p)$ is of the order of 10^{80} or more, and therefore p is a huge number of the order of $exp(10^{80})$ or more. However, p cannot be treated as an infinite number because, since $ln(p)$ is "only" of the order of 10^{80}, gravity is observable. At the same time, in [1, 2] we have made arguments that in early stages of the universe, the value of p was much smaller than now and so in these stages only FQT may be reliable for describing different experimental data. In [1, 2] we considered several other phenomena where it is important that p is finite and not infinitely large.

There is an analogy here with the fact that when speeds are much less than c one can consider c infinitely large and then NM describes these phenomena with great accuracy. However, when speeds are comparable to c, c cannot be considered an infinitely large value and then only SR can be reliable.

These remarks indicate that the construction of a fundamental quantum theory based on a finite p will be a problem based on fundamentally new concepts: since the concepts of particle-antiparticle, electric charge and baryon quantum number have a physical meaning only for very large values of p, then in such a theory, in the most general case, there should be no such concepts. However, from the point of view of the development of science, the fundamental quantum theory at finite p must be constructed.

When we compare two theories A and B, a question arises what criteria should be used to prove that, for example, theory A is more general than theory B and B is a degenerate case of A. In [1, 2, 3] we have proposed the following criteria

Definition: *Let theory A contain a finite nonzero parameter and theory B be obtained from theory A in the formal limit when the parameter goes to zero or infinity. Suppose that, with any desired accuracy, A can reproduce any result of B by choosing a value of the parameter. On the contrary, when, the limit is already taken, one cannot return to A and reproduce all results of A. Then A is more general than B and B is a degenerate case of A.*

The proofs in [1, 2] of fundamental facts that FM is more general (fundamental) than SM and FQT is more general (fundamental) than SQT contain a lot of technical details and, as a result, these works are quite long (291 and 293 pages, respectively). In this paper, we present two simple examples that require a minimum of prior knowledge and which we hope will stimulate readers to explore these fundamental facts in more depth.

The paper is organized as follows. In Secs. 2 and 4 we describe basic facts about finite rings and quantum theory based on finite mathematics. In Secs. 5 and 6 we describe supersymmetry in AdS theory and consider a simple model example demonstrating the difference between SQT and FQT. In Secs. 7 and 8 this difference is demonstrated on the example of Dirac supersingletons.

2 Basic facts about finite rings

In contrast to SM which starts from the infinite ring $Z = (-\infty, ...-2, -1, 0, 1, 2, ...\infty)$, FM starts from the finite ring $R_p = (0, 1, 2, ...p-1)$ where addition, subtraction and multiplication are defined as usual but modulo p. We believe that the notation Z/p for R_p is not adequate because it may give a wrong impression that FM starts from the infinite set Z and that Z is more general than R_p. However, although Z has more elements than R_p, Z cannot be more general than R_p because Z does not contain operations modulo a number. If p is prime then R_p becomes the Galois field F_p but in this paper we consider only finite rings. The theory of such rings is described in textbooks (see e.g., [7, 8, 9]). The number p is called the characteristic of the ring R_p. For example, if $p = 5$ then 3+1=4 as usual but 3·2=1, 4·3=2, 4·4=1 and 3+2=0. Therefore -2=3, -4=1 etc.

One might say that the above examples have nothing to do with reality since 3+2 always equals 5 and not zero. However, since operations in R_p are modulo p, one can represent R_p as a set $\{0, \pm 1, \pm 2, ..., \pm(p-1)/2)\}$ if p is odd or as a set $\{0, \pm 1, \pm 2, ..., \pm(p/2-1), p/2\}$ if p is even. Let f be a function from R_p to Z such that $f(a)$ has the same notation in Z as a in R_p.

If elements of Z are depicted as integer points on the x axis of the xy plane then, if p is odd, the elements of R_p can be depicted as points of the circumference in Figure 1 and analogously if p is even.

Formally, we can call the element $a \in R_p$ positive if $f(a) > 0$ and negative if $f(a) < 0$. In other words, the element $a \in R_p$ is positive if it is in the right half-plane of Figure 1 and negative if in the left half-plane. While in SM, a sum of two positive numbers is always positive and greater than both original numbers, in FM (where calculations are carried out modulo p), it is even possible that a sum of two positive numbers is negative. For example, $(p-1)/2 + 1 = (p+1)/2 = -(p-1)/2$. However, for numbers a such that $|f(a)|$ is much less than p, the results of all the operations are the same as in Z, i.e., for such numbers we do not notice the existence of p.

When $p \to \infty$, a vicinity of zero in R_p becomes the infinite set Z. Therefore *even from pure mathematical point of view*, the concept of infinity cannot be fundamental because, as soon as we replace R_p by Z, we automatically obtain a

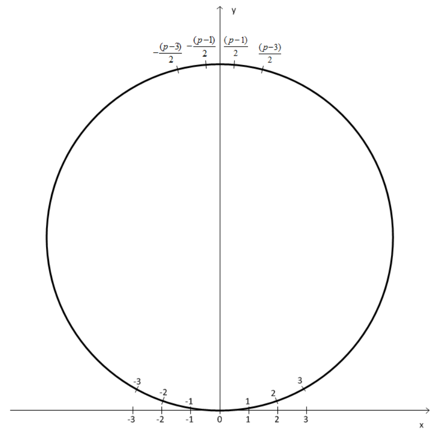

Figure 1: Relation between R_p and Z

degenerate theory because in Z there are no operations modulo a number.

In FQT, states are elements of linear spaces over R_p. One might think that SQT is more general than FQT because in SQT one can work not only with integers but also with rational and real numbers. However, as noted in [1, 2, 3, 4] and Sec. 4, since in SQT the states are projective, for describing wave functions with any desired accuracy it suffices to use only integers.

3 Do we need spacetime background in quantum theory?

Historically, the concepts of background space and fields in this space arose from classical electrodynamics and then they were further developed in General Relativity which is also a classical (i.e., non-quantum) theory. For example, now we know that the electromagnetic field consists of photons but, at the classical level, the theory does not describe the state of each photon. The classical electromagnetic fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ describe the *effective* contribution of all photons at the point $x = (\mathbf{r}, t)$ of Minkowski space, and in classical (non-quantum) theory it is assumed that the parameters $x = (\mathbf{r}, t)$ can be measured with any desired accuracy. This is similar to the situation in statistical physics, where systems of many particles are considered, but the theory does not describe each particle individually, but introduces concepts that make sense only for ensembles of a large number of particles (for example, temperature and pressure).

In quantum theory, every physical quantity must be described by an operator. For example, one can talk about coordinates of a certain particle only if the position operator for this particle is defined. In QFT, particles are described by field operators $\Psi(x)$ where x is a point in the background space (e.g., in Minkowski space). When there are many particles, it may seem that they are in some space. However, such space is only a mathematical and not a physical object. It is not related to certain particles and there are no operators for the coordinates in this space. The goal of QFT is to construct the S-matrix and when the theory is already constructed one can forget about Minkowski space because x is only an integration parameter in the expression for the S-matrix and no physical quantity depends on x. This is in the spirit of the Heisenberg S-matrix program according to which in RQT it is impossible to describe the state of the system at each moment in time and it is possible to describe only transitions of states from the infinite past when $t \to -\infty$ to the distant future when $t \to +\infty$.

Note that the fact that the S-matrix is the operator in momentum space does not exclude a possibility that in some situations it is possible to have a spacetime description with some accuracy but not with absolute accuracy. First of all, as noted by Pauli [10], the problem of time is one of the most important unsolved problems of quantum theory because there is no time operator. Also, the position operator in momentum representation usually exists not only in nonrelativistic theory but in relativistic theory as well. In this case it is known as the Newton- Wigner position operator [11] or its modification. However, the coordinate description of elementary particles can be only approximate. For example, coordinates of a particle with the mass m cannot be measured with the accuracy better than the particle Compton wave length $\hbar/(mc)$ [12].

As noted in the extensive literature on QFT (see, e.g., [13]), the use of field functions $\Psi(x)$ also leads to the following mathematical problem. Quantum interacting local fields can be treated only as operator distributions. A known fact from the theory of distributions is that their product at the same point is not a correct mathematical operation. Hence if $\Psi_1(x)$ and $\Psi_2(x)$ are two local operator fields then the product $\Psi_1(x)\Psi_2(x)$ is not well defined. Physicists often ignore this problem: they think that such products are needed to preserve locality (although the operator of the quantity x does not exist). As a consequence, representation operators of interacting systems constructed in QFT are not well defined and the theory contains anomalies and infinities. A detailed discussion of other problems of QFT can be found, for example, in [14].

Let us also note that so far the approaches to the spacetime background come from SM in which, as is known from Gödel's incompleteness theorems and other results, there are foundational problems. There is an extensive literature which conjectures that foundational problem of quantum theory will be solved in the framework of approaches with fundamental length (see e.g., [15] and references therein) because here it will be possible to circumvent infinitesimals. However, since this literature is based on SM, it involves infinitesimals implicitly.

Despite the noted mathematical problems, QFT is very popular among physicists due to the following. In renormalizable theories, divergences can be eliminated and, although from a mathematical point of view, renormalization (when operations with singularities yield non-singular expressions) is not a correct mathematical operation, in some cases it leads to very impressive agreements between theory and experiment. However, in non-renormalizable theories, singularities cannot be eliminated, and this is the main obstacle to the construction of quantum gravity.

4 Quantum theory based on finite mathematics

In this section, following [1, 2, 4], we briefly describe why SQT is a degenerate case of FQT in the limit $p \to \infty$.

In SQT, physical states are described by elements of complex Hilbert spaces, and operators of physical quantities are self-adjoint operators in such spaces. By analogy, in FQT, physical states are elements of linear spaces over the ring R_{p^2} which is a quadratic generalization of R_p and contains p^2 elements: any element of R_{p^2} can be represented as $a + bi$ where $a, b \in R_p$ and i is a formal element such that $i^2 = -1$. Then the definition of addition, subtraction and multiplication in R_{p^2} is obvious and R_{p^2} is a ring regardless whether p is prime or not.

In both, SQT and FQT, dS symmetry is defined by the operators M^{ab} ($a, b = 0, 1, 2, 3, 4$, $M^{ab} = -M^{ba}$) satisfying the commutation relations

$$[M^{ab}, M^{cd}] = -i(\eta^{ac}M^{bd} + \eta^{bd}M^{ac} - \eta^{ad}M^{bc} - \eta^{bc}M^{ad}) \tag{1}$$

where η^{ab} is the diagonal tensor such that $\eta^{00} = -\eta^{11} = -\eta^{22} = -\eta^{33} = -\eta^{44} = 1$. As discussed in detail in [1, 2, 4, 16], this definition does not involve the fact that the dS group is the group of motions of dS space.

The *definition* of AdS symmetry is given by the same expressions but $\eta^{44} = 1$. By analogy with the dS case, this definition does not involve the fact that the AdS group is the group of motions of AdS space. At the same time, there is an extensive literature on AdS/CFT correspondence in which AdS symmetry essentially involves the properties of classical AdS space (see e.g. [17, 18, 19] and references therein). This literature yields interesting results but currently AdS/CFT correspondence is a conjectured relationship between two kinds of physical theories. As argued in [1, 2, 4], at the most fundamental level, quantum theory should not involve such classical concepts as AdS space and, in this paper, following [1, 2, 3] and other our publications, we argue that quantum theory should be based on FM.

In SQT, operators of physical quantities act in Hilbert spaces supplied by a scalar product (...,...), and these operators are selfadjoint. In particular, the operators in Eqs. (1) are selfadjoint. However, in spaces over R_{p^2} it is impossible to introduce a scalar product that satisfies the condition that $(x, x) > 0$ for all non-zero elements x in such spaces. The matter is that, as explained in Sec. 2, in FM, the concepts $>$ and $<$ have their usual meaning only for those elements $a \in R_p$ for which $|f(a)|$ is much less than p. Therefore, in spaces over R_{p^2} the concept of Hermitian conjugation has limited applicability: it makes sense only for the actions of operators on elements of spaces for which the expansion coefficients with respect to the basis elements are much less than p.

Let us define $\tilde{M}^{ab} = M^{ab}$ if $a, b \neq 4$ and $\tilde{M}^{ab} = iM^{ab}$ if $a \neq 4$, $b = 4$. Then a direct check shows that the set of operators $\tilde{M}^{ab}$ satisfies Eq. (1) if η^{44} is replaced by $-\eta^{44}$. Therefore, if the set of operators M^{ab} satisfies the conditions (1) for the dS algebra, then the set of operators $\tilde{M}^{ab}$ satisfies the conditions (1) for the AdS algebra and *vice versa*.

Therefore in FQT, the dS and AdS theories are equivalent. However, in SQT they are not equivalent for the following reason. Here it is required that the operators M^{ab} should not only satisfy Eq. (1) but additionally they should be selfadjoint (as explained above, in FQT such a requirement cannot be imposed). However, if the operators M^{a4} are Hermitian then the operators $\tilde{M}^{a4} = iM^{ab}$ are anti-Hermitian.

When in SQT the operators in Eq. (1) are selfadjoint then, as described in a wide literature, IRs of the dS and AdS algebras are infinite-dimensional. Representations in spaces over a ring of nonzero characteristic are called modular representations. According to the Zassenhaus theorem (see e.g., [20, 21]), all modular IRs are finite-dimensional. In [22, 23] we constructed modular IRs of the algebras defined by Eq. (1).

In SQT, all Hilbert spaces are separable, i.e., they contain a countable dense subset. In such spaces it is always possible to choose a basis $(e_1, e_2, ...e_n, ...)$ such that the norm of each e_j is an integer. The elements of such spaces can be denoted as $(c_1, c_2, ...c_n, ...)$ where all the expansion coefficients c_j for such a basis are complex numbers and can be represented as $c_j = a_j + ib_j$. As explained in [4], since spaces in quantum theory are projective, it follows from the results of the textbook [24] that:

In SQT, each element of a separable Hilbert space can be approximated with any desired accuracy by a finite linear combination

$$x = \sum_{j=1}^{n} c_j e_j \tag{2}$$

where all the numbers a_j and b_j are integers, i.e., belong to Z.

In FQT, quantum states also can be represented in the form (2) but here the $c_j = a_j + ib_j$ are elements of R_{p^2}. As shown in [1, 2, 3, 4], by using **Definition** and the above results one can prove that FQT is more general (fundamental) than SQT and the latter is a special degenerate case of the former in the formal limit $p \to \infty$: when the numbers (a_j, b_j) are such that $\forall j$, $|f(a_j)|$ and $|f(b_j)|$ are much less than p then FQT reproduces all results of SQT but SQT cannot reproduce all results of FQT if some of the numbers (a_j, b_j) are comparable to p.

5 Supersymmetry

In SQT, supersymmetry is valid in the AdS case but is not valid in the dS one. As shown in [25], in SQT, dS symmetry is more general than AdS one, and it may be a reason why supersymmetry has not been discovered yet. However, as shown in [1, 2, 3, 4], SQT is a degenerate case of FQT in the formal limit $p \to \infty$, and, as shown in Sec. 4, in FQT, dS and AdS symmetries are equivalent.

Representations of the osp(1,4) superalgebra are described by 14 operators, as well as representations of the Poincare superalgebra. However, effectively, representations of the osp(1,4) superalgebra can be described only by four fermionic operators. The matter is that ten bosonic operators of the osp(1,4) superalgebra are the anticommutators of the four fermionic operators. This is not the case for the Poincare superalgebra since the Poincare algebra operators are obtained from the so(2,3) ones by contraction. One can say that the representation of the osp(1,4) superalgebra is the implementation of the idea that supersymmetry is the extraction of the square root from the usual symmetry (by analogy with the treatment of the Dirac equation as a square root from the Klein-Gordon equation).

Let $(d_1', d_2', d_1'', d_2'')$ be the fermionic operators of the osp(1,4) superalgebra. They should satisfy the following relations. If (A, B, C) are any fermionic operators, [...,...] is used to denote a commutator and $\{..., ...\}$ to denote an anticommutator then

$$[A, \{B, C\}] = F(A, B)C + F(A, C)B \tag{3}$$

where the form $F(A, B)$ is skew symmetric, $F(d_j', d_j") = 1$ $(j = 1, 2)$ and the other independent values of $F(A, B)$ are equal to zero.

As shown by various authors (see e.g., [1, 2, 26]), the operators M^{ab} in Eqs. (1) can be expressed through bilinear combinations of the fermionic operators as follows:

$$\begin{aligned}
&h_1 = \{d_1', d_1''\},\ h_2 = \{d_2', d_2''\},\ M_{04} = h_1 + h_2,\ M_{12} = L_z = h_1 - h_2 \\
&L_+ = \{d_2', d_1''\},\ L_- = \{d_1', d_2''\},\ M_{23} = L_x = L_+ + L_- \\
&M_{31} = L_y = -i(L_+ - L_-),\ M_{14} = (d_2'')^2 + (d_2')^2 - (d_1'')^2 - (d_1')^2 \\
&M_{24} = i[(d_1'')^2 + (d_2'')^2 - (d_1')^2 - (d_2')^2] \\
&M_{34} = \{d_1', d_2'\} + \{d_1'', d_2''\},\ M_{30} = -i[\{d_1'', d_2''\} - \{d_1', d_2'\}] \\
&M_{10} = i[(d_1'')^2 - (d_1')^2 - (d_2'')^2 + (d_2')^2] \\
&M_{20} = (d_1'')^2 + (d_2'')^2 + (d_1')^2 + (d_2')^2
\end{aligned} \tag{4}$$

where $\mathbf{L} = (L_x, L_y, L_z)$ is the standard operator of three-dimensional rotations.

We require the existence of the generating vector e_0 satisfying the conditions :

$$d_j' e_0 = d_2' d_1'' e_0 = 0, \quad d_j' d_j'' e_0 = q_j e_0 \quad (j = 1, 2) \tag{5}$$

The full representation space can be obtained by successively acting by the fermionic operators on e_0 and taking all possible linear combinations of such vectors. The theory of self-adjoint IRs of the osp(1,4) algebra has been developed by several authors (see e.g., [26]), and in [1, 2] this theory has been generalized to the case of FQT.

6 Model example

In this section we consider a simple model example when in Eq. (5) there are only two fermionic operators $(d', d")$ and one bosonic operator h such that

$$h = \{d', d''\}, \quad [h, d'] = -d', \quad [h, d''] = d'' \tag{6}$$

Here the first expression shows that the relations (6) can be formulated only in terms of the fermionic operators. We will consider IRs of the superalgebra (6) in SQT and FQT.

6.1 IRs of the superalgebra (6) in SQT

Consider an IR of the algebra (6) generated by a vector e_0 such that

$$d'e_0 = 0, \quad he_0 = q_0 e_0, \quad (q_0 > 1/2) \tag{7}$$

and define $e_n = (d'')^n e_0$ $(n = 1, 2, ...)$. Then $d'e_n = a(n)e_{n-1}$ where, as follows from Eq. (7), $a(0) = 0$, $a(1) = q_0$ and

$$a(n) = q_0 + n - 1 - a(n-1) \tag{8}$$

The solution of this equation is

$$a(n) = n/2 + (q_0 - 1/2)[1 - (-1)^n]/2 \tag{9}$$

Therefore, $a(n) > 0\ \forall n$ and, as follows from Eq. (6), $he_n = (n + q_0)e_n$. So, we have obtained an infinite-dimensional IR of the algebra (6) with the basis $(e_0, e_1, e_2...)$ where all basis vectors are eigenvectors of the operator h with positive eigenvalues.

Consider now an IR of the algebra (6) generated by a vector f_0 such that

$$d''f_0 = 0, \quad hf_0 = -q_0 e_0, \quad (q_0 > 1/2) \tag{10}$$

and define $f_n = (d')^n f_0$ $(n = 1, 2, ...)$. Then $d''f_n = b(n)f_{n-1}$ where, as follows from Eq. (10), $b(0) = 0$, $b(1) = -q_0$ and

$$b(n) = (-n - q_0 + 1) - b(n-1) \tag{11}$$

The solution of this equation is

$$b(n) = -n/2 - (q_0 - 1/2)[1 - (-1)^n]/2 \tag{12}$$

Therefore, $b(n) < 0\ \forall n$ and, as follows from Eq. (6), $hf_n = -(n + q_0)f_n$. So, we have obtained an infinite-dimensional IR of the algebra (6) with the basis $(f_0, f_1, f_2...)$ where all basis vectors are the eigenvectors of the operator h with negative eigenvalues.

6.2 IRs of the superalgebra (6) in FQT

In FQT we can also consider an IR of the algebra (6) generated by a vector e_0 such that

$$d'e_0 = 0, \quad d'd''e_0 = q_0 e_0 \tag{13}$$

where now $q_0 \in R_p$. As in SQT, we define $e_n = (d'')^n e_0$. Then $d'e_n = a(n)e_{n-1}$ where, as follows from Eq. (13), $a(0) = 0$, $a(1) = q_0$ and for $a(n)$ we have the same equation as in (8):

$$a(n) = q_0 + n - 1 - a(n-1) \tag{14}$$

We assume that p is odd and then, instead of the solution (9), the solution is

$$a(n) = \frac{p+1}{2}n + \frac{p+1}{2}(q_0 - \frac{p+1}{2})[1-(-1)^n] \quad (15)$$

Then, unlike the situation in SQT where $a(n) > 0$ at $(n = 1, 2, ...\infty)$, we have that, since in R_p the results should be taken modulo p, $a(n) = 0$ if $n = 2p + 1 - 2q_0$. Therefore, the IR under consideration is finite-dimensional, the basis of this IR consists of vectors $(e_0, e_1, ...e_N)$ where $N = n_{max} = 2p - 2q_0$ and the dimension of the IR is $2p + 1 - 2q_0$.

Just like in SQT, we have that $he_n = (n + q_0)e_n$ at $n = 0, 1, ...N$, i.e., the basis vectors e_n are eigenvectors of the operator h with the eigenvalues $\lambda_n = n + q_0$. Since all the λ_n should be different in R_p, it should be $N \leq (p - 1)$.

We choose $q_0 = (p+1)/2 + a$ where $a \in R_p$ and a can be one of the values $(0, 1, ..(p - 3)/2)$. Then $f(q_0) < 0$, i.e., q_0 is in the left half-plane of Figure 1. Then, unlike the situation in SQT where $\lambda_n > 0$ at all $n = 0, 1, ...\infty$, we have that in FQT $f(\lambda_n) < 0$ at $n = (0, 1..., (p - 3)/2 - a)$, $f(\lambda_n) = 0$ at $n = (p - 1)/2 - a$ and finally $f(\lambda_n) > 0$ at $n = ((p + 1)/2) - a, ..., n_{max})$.

Thus, the construction of the basis begins with the basis vector e_0 with the eigenvalue of h equal to $\lambda_0 = q_0$ and ends with the basis vector e_N with the eigenvalue $\lambda_N = -q_0$. **Thus, in contrast to SQT where there are two different IRs with positive and negative eigenvalues of h, respectively, in FQT there is only one IR which contains the analogs of negative and positive IRs in SQT and contains a basis vector with the eigenvalue of h equal to zero. In addition, if in SQT, negative and positive IRs are infinite-dimensional, then in FQT, the only IR is finite-dimensional with dimension $p - 2a$.**

7 Dirac supersingleton

When describing elementary particles within the framework of AdS symmetry, the following problems arise.

If m is the mass of a particle in Poincare invariant theory then its mass μ in AdS theory is dimensionless and the relation between μ and m is $\mu = mR$ where R is the contraction parameter for the transition from AdS to Poincare symmetry. As explained in [25], the data on cosmological acceleration show that, at the present stage of the universe, R is of the order of $10^{26}m$ [25]. Therefore, even for elementary particles, the AdS masses are very large. For example, the AdS masses of the electron, the Earth and the Sun are of the order of 10^{39}, 10^{93} and 10^{99}, respectively. The fact that even the AdS mass of the electron is so large might be an indication that the electron is not a true elementary particle. In addition, the present upper level for the photon mass is $10^{-17}ev$ or less. This value seems to be an extremely tiny quantity. However, the corresponding AdS mass is of the order of 10^{16} and so, even the mass which is treated as extremely small in Poincare invariant theory might be very large in AdS invariant theory.

As shown in [25], in SQT, dS symmetry is more general than AdS one but in the framework of dS symmetry it is not possible to describe neutral elementary particles, i.e., particles which are equivalent to the their antiparticles. As shown in Sec. 4, in FQT, dS and AdS symmetries are equivalent and, as shown in [1, 2], in this theory also there are no neutral elementary particles. In particular, even the photon is not elementary.

This problem has been discussed by several authors. In Standard Model (based on Poincare invariance) only massless particles are treated as elementary. However, as shown in the seminal paper by Flato and Fronsdal [27] (see also [28]), in standard AdS theory, each massless IR can be constructed from the tensor product of two singleton IRs discovered by Dirac in his paper [29] titled "A Remarkable Representation of the 3 + 2 de Sitter group", and the authors of [27] believe that this is indeed a truly remarkable property.

The IR describing the supersingleton is constructed as follows. In Eq. (5), we choose q_1 and q_2 the same and equal q_0 where $q_0 = 1/2$ in standard theory over complex numbers and $q_0 = (p+1)/2$ in FQT, where p is the characteristic of the ring and p is odd.

The authors of [27] and other publications treat singletons as true elementary particles because their weight diagrams has only a single trajectory (that's why the corresponding IRs are called singletons). However, one should answer the following questions:

- a) Why singletons have not been observed yet.
- b) Why such massless particles as photons and others are stable and their decays into singletons have not been observed.

There exists a wide literature (see e.g., [30, 31] and references therein) where this problem is investigated from the point of view of standard AdS QFT. For example, in AdS QFT, singleton fields live on the boundary at infinity of the AdS bulk (boundary which has one dimension less than the bulk). However, as noted in Sec. 4, the explanation in the framework of quantum theory should not involve classical spaces.

On the other hand, as argued in [1, 2], in FQT, the properties of Dirac singletons are even more remarkable than in standard theory and here the properties a)-b) have a natural explanation.

In standard AdS theory, there exist four Dirac singletons which in the literature are called Di singleton, Rac singleton and their antiparticles. In the case of supersymmetry, Di and Rac singletons are combined into one superparticle - the Dirac supersingleton, so that there are two supersingletons - the Dirac supersingleton and its antiparticle. However, in FQT those supersingletons are combined into one object and so there is only one supersingleton. Here, one of the remarkable properties of supersingletons is the following. The physical meaning of division comes from classical physics, which assumes that every object can be divided into any arbitrarily large number of arbitrarily small parts. However, standard division loses its standard physical meaning when we reach the level of elementary particles since, for example, the electron cannot be divided into two, three, and so on parts. As shown in [1, 2], in FQT, the theory of singletons can be built over a ring in which there is no division, but only addition, subtraction and multiplication.

As shown in [1, 2, 4], the important property of supersingletons is that the above results can be immediately generalized to the case of higher dimensions, and in this case it is interesting to explore the possibility that spatial and internal quantum numbers are on equal footing. The fact that singleton physics can be directly generalized to the case of higher dimensions has been indicated by several authors (see e.g., [30] and references therein).

8 Supersingleton IRs in SQT and FQT

As shown in [1, 2], $[d_1'', d_2''] = 0$ in the space of the supersingleton IR. For this reason, the results for supersingleton IRs can be obtained by directly generalizing the results of Subsecs. 6.1 and 6.2.

8.1 Supersingleton IRs in SQT

In this subsection it will be shown that in SQT, there is only one supersingleton IR with positive energies and only one supersingleton IR with negative energies.

Consider first the representation generated by a vector e_0 such that

$$d_j' e_0 = 0, \quad h_j e_0 = \frac{1}{2} e_0, \quad j = 1, 2. \tag{16}$$

The basis of the IR consists of the vectors $e_{jk} = (d_1'')^j (d_2'')^k e_0$. Then, as follows from Eq. (9)

$$h_1 e_{jk} = (j + \frac{1}{2}) e_{jk},\ h_2 e_{jk} = (k + \frac{1}{2}) e_{jk},\ d_1' e_{jk} = \frac{j}{2} e_{j-1,k},\ d_2' e_{jk} = \frac{k}{2} e_{j,k-1} \tag{17}$$

where $j, k = 0, 1, 2, ...\infty$. As shown in [1, 2], M_{04} is the AdS analog of the energy operator because M_{04} becomes the Poincare energy upon contraction of the AdS algebra to the Poincare algebra. Then, as follows from Eqs. (4) and (17),

$$M_{04} e_{jk} = (1 + j + k) e_{jk} \tag{18}$$

and therefore, the positive energy IR is infinite-dimensional.

Consider now the representation generated by a vector f_0 such that

$$d_j'' f_0 = 0, \quad h_j f_0 = -\frac{1}{2} e_0, \quad j = 1, 2. \tag{19}$$

The basis of the IR consists of the vectors $f_{jk} = (d_1')^j (d_2')^k f_0$. Then, as follows from Eq. (12)

$$h_1 f_{jk} = -(\frac{1}{2} + j) f_{jk},\ h_2 jk = -(\frac{1}{2} + k) f_{jk},\ d_1'' f_{jk} = -\frac{j}{2} f_{j-1,k},\ d_2'' f_{jk} = -\frac{k}{2} f_{j,k-1} \tag{20}$$

where $j,k = 0,1,2,...\infty$. As follows from Eqs. (4) and (20),

$$M_{04}f_{jk} = -(1+j+k)e_{jk} \tag{21}$$

and therefore, the negative energy IR is infinite-dimensional.

8.2 Supersingleton IRs in FQT

In this subsection, it will be shown that, in contrast to the situation in SQT, in FQT there exists only one supersingleton IR. For definiteness, we assume that p is odd. Then the FM analog of Eq. (16) is

$$d'_j e_0 = 0, \quad h_j e_0 = \frac{p+1}{2}e_0, \quad j = 1,2. \tag{22}$$

and the FQT analog of Eq. (17) is

$$\begin{aligned} &h_1 e_{jk} = (j + \frac{p+1}{2})e_{jk},\ h_2 e_{jk} = (k + \frac{p+1}{2})e_{jk} \\ &d'_1 e_{jk} = \frac{j(p+1)}{2}e_{j-1,k},\ d'_2 e_{jk} = \frac{k(p+1)}{2}e_{j,k-1} \end{aligned} \tag{23}$$

Since now the results should be taken modulo p, it follows from these expressions that $d'_1 e_{jk} = 0$ at $j = p$ and $d'_2 e_{jk} = 0$ at $k = p$. Therefore $e_{jk} \neq 0$ at $j,k = 0,1,..p-1$.

We conclude that, in contrast to the situation in SQT, where there are two infinite-dimensional IRs (one positive-energy IR and one negative-energy IR), in FQT there is only one IR which is finite-dimensional with the dimension p^2.

As follows from Eq. (23), since now the results should be taken modulo p, we have for the eigenvalues of the operator M_{04} formally the same result as in Eq. (18) that the elements e_{jk} are the eigenvectors of the operator $M_{04} = h_1 + h_2$ with the eigenvalues $(j+k+1)$. Therefore when $j,k \ll p$ we have an analog of positive energy IR. On the other hand, if $j = p-1-j'$ and $k = p-1-k'$ then, since the results should be taken modulo p, we have that in terms of j' and k' the eigenvalues of M_{04} are equal to $-(1+j'+k')$ by analogy with Eq. (21). Therefore when $j',k' \ll p$ we have an analog of negative energy IR.

9 Conclusion

In this paper we note that, as shown in our previous works, fundamental quantum theory should be based on finite mathematics in which it is not assumed that the characteristic p of the ring used in this mathematics is anomalously large. In this theory there are no concepts of particle-antiparticle and conserved additive quantum numbers such as electric charge, baryon quantum number etc.

The above properties have been discussed in our publications with detailed technical proofs. The purpose of this paper is to consider models where differences between SQT and FQT can be described in a much simpler way. In Sec. 6 we consider a model where a superalgebra is defined by only two operators and in Sec. 8 we consider the model of Dirac supersingleton.

Our publications and this paper illustrate that the standard concepts of particle-antiparticle and conserved additive quantum numbers are not fundamental. As noted in [1, 2, 3, 4], our results on universal law of gravity indicate that at the present state of the universe the value p is very large (of the order of $exp(10^{80})$ or more) and that is why these concepts work approximately with very high accuracy. However, there is no reason to think that p is a fundamental quantity that had the same value at all stages of the universe.

Each computer can carry out calculations only modulo a certain number, which depends on the maximum number of bits with which this computer can work. The literature discusses the possibility that the universe can be treated as a computer (see e.g., [32]). From this point of view, the value p is not some fundamental constant but is determined by the state of the universe at a given stage. And, since the state of the universe is changing, it is natural to expect that the number p describing physics at different stages of the evolution of the universe will be different at different stages.

There are several reasons to think that at early stages of the universe the value p was much less than now. One of the reasons is the problem of the baryon asymmetry of the universe (BAU). Modern cosmological theories state that the numbers of baryons and antibaryons in the early stages of the universe were the same. Then, since the baryon number is the conserved quantum number, those numbers should be the same at the present stage. However, at this stage the number of baryons is much greater than the number of antibaryons. However, if the value p at early stages of the universe was much less than now then the statement that the numbers of baryons and antibaryons were the same, does not have a physical meaning and the BAU problem does not arise (see e.g., [1, 2, 4] for more details).

Another reason is the problem of time in quantum theory. As noted in Sec. 3, this problem was posed by Pauli in view of the fact that in this theory there is no time operator. In [1, 2] we discussed a conjecture that standard classical time t manifests itself because the value p changes, i.e., t is a function of p. We do not say that p changes over time because classical time t cannot be present in quantum theory; we say that we feel changing time because p changes. In [33] we discussed a model where, in semiclassical approximation, the variations of t and p at the present stage of the universe, are related as

$$\Delta t = \frac{R}{c}\frac{\Delta ln(p)}{ln(p)}$$

where R is the contraction parameter from the dS to the Poincare algebra. In this model, the quantities t and p increase during the evolution of the universe. However, since we do not know how the states of the universe were described in its early stages, we cannot say what the magnitudes of p were at these stages.

In summary, the concepts of particle-antiparticle and additive quantum numbers should not be present in the ultimate quantum theory which should not assume that p is necessarily very large. Thus, the main problem is what principles this theory should be based on.

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