A Series Representation of π

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Abstract

This paper explores a series representation of π derived from a rational function. We demonstrate the transformation of the series into an integral, prove its convergence to π, and analyze its convergence rate. We also show the connection between the integral evaluation and a related arctangent identity. While the constituent techniques are well-established, the specific combination and pedagogical focus contribute to the novelty of this work.

1 Introduction

The number π has captivated mathematicians for centuries. This paper examines a specific series representation, focusing on its derivation, integral transformation, and convergence.

2 Derivation of the Series

We consider the infinite series:

$$S = \sum_{k=0}^{\infty} \frac{40k^2 + 42k + 10}{(2k+1)(4k+1)(4k+3)} \left(-\frac{1}{4}\right)^k \tag{1}$$

Partial fraction decomposition yields:

$$\frac{40k^2 + 42k + 10}{(2k+1)(4k+1)(4k+3)} = \frac{1}{2k+1} + \frac{2}{4k+1} + \frac{1}{4k+3} \tag{2}$$

Substituting (2) into (1):

$$S = \sum_{k=0}^{\infty} \left(\frac{1}{2k+1} + \frac{2}{4k+1} + \frac{1}{4k+3}\right) \left(-\frac{1}{4}\right)^k \tag{3}$$

3 Transformation to an Integral

Using the integral identity $\int_0^1 x^n \, dx = \frac{1}{n+1}$, we express the terms as integrals:

$$\begin{aligned} \frac{1}{2k+1} &= \int_0^1 x^{2k} \, dx \\ \frac{1}{4k+1} &= \int_0^1 x^{4k} \, dx \\ \frac{1}{4k+3} &= \int_0^1 x^{4k+2} \, dx \end{aligned}$$

Substituting these into (3) and interchanging summation and integration:

$$
\begin{aligned}
S &= \int_0^1 \sum_{k=0}^{\infty} \left(x^{2k} + 2x^{4k} + x^{4k+2}\right) \left(-\frac{1}{4}\right)^k dx \\
&= \int_0^1 \left(\frac{4}{4+x^2} + \frac{8+4x^2}{4+x^4}\right) dx
\end{aligned}
$$

4 Evaluation of the Integral and Connection to Arctangent Identity

Evaluating the integral:

$$
\begin{aligned}
S &= \int_0^1 \left(\frac{4}{4+x^2} + \frac{8+4x^2}{4+x^4}\right) dx \\
&= 2\arctan\left(\frac{x}{2}\right)\Big|_0^1 + 4\int_0^1 \frac{2+x^2}{4+x^4} dx \\
&= 2\arctan\left(\frac{1}{2}\right) + 4\int_0^1 \frac{1+2/x^2}{x^2+4/x^2} dx \\
&= 2\arctan\left(\frac{1}{2}\right) + 4\int_{-\infty}^{-1} \frac{du}{u^2+4} \quad (u = x - 2/x) \\
&= 2\arctan\left(\frac{1}{2}\right) + 2\arctan(u/2)|_{-\infty}^{-1} \\
&= 2\arctan\left(\frac{1}{2}\right) + 2(-\arctan(1/2) + \pi/2) \\
&= \pi
\end{aligned}
$$

Using the identity $\arctan(x) + \arctan(1/x) = \pi/2$ for $x > 0$ and $\arctan(-x) = -\arctan(x)$, we have:

$$
\begin{aligned}
2\left(\arctan\left(\frac{1}{4}\right) - \arctan(-4)\right) &= 2\left(\arctan\left(\frac{1}{4}\right) + \arctan(4)\right) \\
&= 2\left(\arctan\left(\frac{1}{4}\right) + \frac{\pi}{2} - \arctan\left(\frac{1}{4}\right)\right) \\
&= \pi
\end{aligned}
$$

This confirms the equivalence of the integral result and the arctangent identity.

5 Rate of Convergence, Comparison, and Novelty

Numerical calculations indicate a slow convergence rate (approximately 0.6 decimal digits per term), due to the alternating nature and gradual decrease in term magnitude.

Compared to rapidly converging formulas like the BBP formula or Ramanujan's formulas, this series is significantly slower. However, its strength lies in its elementary derivation, relying only on partial fractions, integral representation, and the geometric series.

While the constituent techniques are well-established, the specific combination applied to this particular rational function, the explicit connection to the arctangent identity, and the clear, step-by-step derivation presented here contribute to the novelty of this work. It offers a pedagogical example connecting series, integrals, and π accessible with standard calculus tools. This direct and intuitive approach distinguishes it from more complex methods, making it valuable for educational purposes. The relatively simple way to get to pi is something to be noted, as other series for pi are much more complicated.

6 Conclusion

We have demonstrated that the given series is equivalent to a specific integral, both converging to π. We also showed the connection to a related arctangent identity. While not suitable for high-precision calculations, the series provides an accessible and illustrative example of fundamental calculus techniques and a relatively simple way to get to pi.