Cycloid, Semi-cycloid, Elliptic Cycloid and Elliptic Semi-cycloid

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Abstract

Cycloid, semi-cycloid, elliptic cycloid, and elliptic semi-cycloid are all types of trochoids, and parts of roulette. They all refer to curves that trace their paths as they roll along a straight line, a circular orbit, or an elliptical orbit. Unlike cycloids, semi-cycloids are the curves traced by a point on a bicycle wheel as it rolls around the bicycle axle. Elliptic cycloids and elliptic semi-cycloids are the curves traced by ellipses as they roll along a straight line, but ellipses do not roll as smoothly as circles on a straight line. I also investigated the curves traced by circles and ellipses as they roll along circular or elliptical paths.

Keywords: Cycloid, Epicycloid, Hypocycloid, Semi-cycloid, Elliptic Cycloid, Elliptic Semi-cycloid, Trochoid, Cardioid, Elliptic Cardioid, Astroid, Elliptic Astroid

A. Trochoid and centered trochoid

There is a circle that rolls along a straight line without slipping. If there is a fixed line that extends from the center of the circle through the radius to the outside of the circle, a point b on the line rolls along a curve with the circle as the circle rolls. The curve drawn by a point b on the line is called trochoid.

The parametric equation of a trochoid is as follows,

$$x = a\theta - b\sin\theta$$
$$y = a - b\cos\theta. \tag{1}$$

If the point is outside the circle, $b > a$, it is called a prolate trochoid, if the point is inside the circle, $b < a$, it is called a curtate trochoid, and if $b = a$, it is called a cycloid.

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A cycloid is given as

$$x = r\theta - r\sin\theta$$
$$y = r - r\cos\theta. \tag{2}$$

The arc length of a cycloid is $8r$, and the area $3\pi r^2$.

A centered trochoid is the trajectory that a circle draws as it rolls around another circle. The term includes both epitrochoid and hypotrochoid. The curve traced by a circle that rolls along the outside of another circle without slipping is called the epitrochoid. The parametric equation for an epitrochoid that starts at $(0, R)$ and rolls clockwise can be written as

$$x = (R + r)\sin\left(\frac{r\theta}{R}\right) - b\sin\left(\theta + \frac{r\theta}{R}\right)$$
$$y = (R + r)\cos\left(\frac{r\theta}{R}\right) - b\cos\left(\theta + \frac{r\theta}{R}\right), \tag{3}$$

where, R is the radius of the fixed circle, r is the radius of the rolling circle, b is a fixed point on the rolling circle.

If $b = r$, it is an epicycloid,

$$x = (R + r)\sin\left(\frac{r\theta}{R}\right) - r\sin\left(\theta + \frac{r\theta}{R}\right)$$
$$y = (R + r)\cos\left(\frac{r\theta}{R}\right) - r\cos\left(\theta + \frac{r\theta}{R}\right). \tag{4}$$

The length and area of the epicycloid drawn by one rotation of the rolling circle are

$$l = 8r\left(1 + \frac{r}{R}\right)$$
$$A = \pi r(R + r)\left(1 + \frac{2r}{R}\right) + \frac{1}{4}R^2\sin\left(\frac{4\pi r}{R}\right). \tag{5}$$

A cardioid rolling clockwise is obtained when $R = r$ in Eq. (4),

$$x = 2r\sin\theta - r\sin(2\theta)$$
$$y = 2r\cos\theta - r\cos(2\theta). \tag{6}$$

A cardioid rolling anticlockwise around the circle centered at $(-r, 0)$ is given by

$$x = 2r\cos\theta(1-\cos\theta)$$
$$y = 2r\sin\theta\,(1-\cos\theta). \tag{7}$$

The arc length and area of a cardioid is given from the Eq. (5),

$$l = 16r,$$
$$A = 6\pi r^2. \tag{8}$$

And, the curve drawn by a fixed point at a distance b from the center of a circle that rolls while rolling inscribed the fixed circle is called a hypotrochoid, which parametric equation is given as

$$x = (R-r)\sin\left(\frac{r\theta}{R}\right) - b\sin\left(\theta - \frac{r\theta}{R}\right), \qquad r < R,$$
$$y = (R-r)\cos\left(\frac{r\theta}{R}\right) + b\cos\left(\theta - \frac{r\theta}{R}\right). \tag{9}$$

If $b = r$, then we get a hypocycloid rolling clockwise starting from $(0,\ R)$,

$$x = (R-r)\sin\left(\frac{r\theta}{R}\right) - r\sin\left(\theta - \frac{r\theta}{R}\right), \quad r < R,$$
$$y = (R-r)\cos\left(\frac{r\theta}{R}\right) + r\cos\left(\theta - \frac{r\theta}{R}\right). \tag{10}$$

The arc length and area of the hypocycloid drawn by one rotation of the rolling circle that rolls inside the fixed circle are as follows,

$$l = 8r\left(1 - \frac{r}{R}\right),$$
$$A = \pi r(R-r)\left(1 - \frac{2r}{R}\right) + \frac{1}{4}R^2\sin\left(\frac{4\pi r}{R}\right). \tag{11}$$

One may get an astroid when $R = 4r$ from Eq. (10),

$$x = 3r\sin\left(\frac{\theta}{4}\right) - r\sin\left(\frac{3\theta}{4}\right) = 4r\sin^3\left(\frac{\theta}{4}\right),$$
$$y = 3r\cos\left(\frac{\theta}{4}\right) + r\cos\left(\frac{3\theta}{4}\right) = 4r\cos^3\left(\frac{\theta}{4}\right). \tag{12}$$

And the arc length and area of an astroid are given from Eq. (11)

$$l = 6r,$$
$$A = \frac{3}{2}\pi r^2. \tag{13}$$

B. Semi-trochoid

When a circle rolls on a straight line without slipping, the path drawn by a point on the circle is trochoid. At this time, the center of the circle moves along the axis parallel to the straight line. In other words, the center of the rolling circle moves in translation. While the axis moves in translation, the curve drawn by a point on the circle rolling around the axis is a curve that repeatedly goes up and down the axis. This curve is called semi-trochoid or axial trochoid. The parametric equation of a semi-trochoid is given as follows,

$$x = a\theta + a - b\cos\theta$$
$$y = b\sin\theta. \tag{14}$$

where a represents the radius of the rolling circle and b a point on the circle.

If $a = b$, we get a semi-cycloid,

$$x = r\theta + r - r\cos\theta$$
$$y = r\sin\theta. \tag{15}$$

The length of a semi-cycloid's trajectory is $8r$, which is the same as that of a cycloid, but its area is πr^2, which is the same as the area of the rolling circle. However, the area of the first half drawn by the semi-cycloid is $\left(\frac{1}{2}\pi r^2 + 2r^2\right)$, which is larger than that of the second half $\left(2r^2 - \frac{1}{2}\pi r^2\right)$.

Unlike the epitrochoid, the semi-epitrochoid is the trajectory traced by a point on a rolling circle as the center of the rolling circle rolls along a circular or elliptical orbit without slipping. If the distance traveled by a point on a rolling circle is equal to the distance traveled by the fixed circle, the trajectory of a semi-epicycloid that rolls clockwise is expressed as a parametric equation as follows,

$$x = R\sin\left(\frac{r\theta}{R}\right) - r\sin\left(\theta + \frac{r\theta}{R}\right)$$

$$y = R\cos\left(\frac{r\theta}{R}\right) - r\cos\left(\theta + \frac{r\theta}{R}\right), \tag{16}$$

where R represents the radius of the fixed circle and r the radius of the rolling circle.

The area of a semi-epicycloid by one rotation is

$$\begin{aligned} A = &\int_0^{2\pi} (Rr)\cos^2\left(\frac{r\theta}{R}\right)d\theta + \int_0^{2\pi}\left(\frac{r^3}{R} + r^2\right)\cos^2\left(\theta + \frac{r\theta}{R}\right)d\theta \\ &- \int_0^{2\pi}(Rr + 2r^2)\cos\left(\theta + \frac{r\theta}{R}\right)\cos\left(\frac{r\theta}{R}\right)d\theta \\ = &\ \pi r\left(R + r + \frac{r^2}{R}\right) + \frac{1}{4}(R - r)^2\sin\left(\frac{4\pi r}{R}\right). \end{aligned} \tag{17}$$

C. Elliptic trochoid

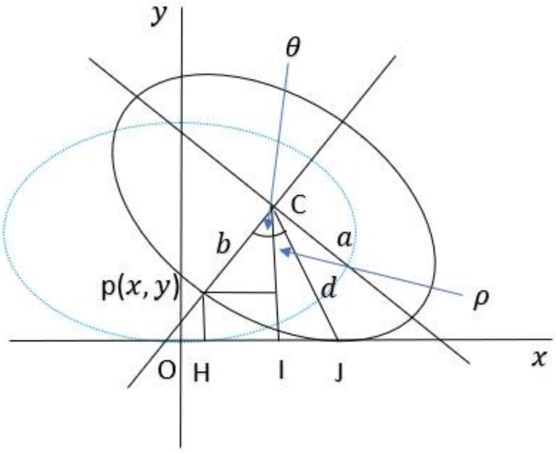
Unlike a trochoid, the trajectory drawn by a fixed point on an axis of an ellipse as it rolls along a straight line without slipping is called the elliptic trochoid.

The greater the eccentricity of the ellipse, the less smoothly it rolls, but the trajectory is drawn incessantly with a cusp as a trochoid.

If one rides a bicycle with elliptical wheels, the ride may not be satisfactory, but there is no problem moving forward.

The trajectory drawn by an ellipse with a fixed point at the vertex of one axis as it rolls along a straight line is called elliptic cycloid.

An elliptic cycloid generates different trajectories when its major axis is parallel to the x-axis and when its major axis lies on the y-axis. In the former case, as shown in the diagram below, the line segment OJ drawn by the elliptic cycloid is equal to the arc length pJ that the ellipse has rolled by $(\theta + \rho)$.



Elliptic Cycloid

Arranging by a parametric equation, we have

$$OJ = pJ = a(\theta + \rho)E(k)$$
$$x = OJ - b\sin\theta - d\sin\rho \qquad (18)$$
$$y = -b\cos\theta + d\cos\rho$$
$$d = \sqrt{a^2\cos^2\left(\frac{\pi}{2} - (\theta + \rho)\right) + b^2\sin^2\left(\frac{\pi}{2} - (\theta + \rho)\right)}.$$

Since the deviation ρ is due to de La Hire's point construction of an ellipse, it is also valid when $\rho = 0$. Therefore, the parametric equation for θ alone[2] can be written as follows,

$$x = a\theta E(k) - b\sin\theta \qquad (19)$$
$$y = \sqrt{a^2\sin^2\theta + b^2\cos^2\theta} - b\cos\theta,$$

where a represents the semi-major axis and b the semi-minor axis of an ellipse.

Here, $E(k)$ is the function that determines the arc length of an ellipse, which is the sum of the following power series[3] for the eccentricity k of an ellapse,

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[2] The deviation ρ becomes 0 at starting point when $\theta = 0$, reaches a maximum at $\theta = \frac{\pi}{4}$, and becomes 0 again at $\theta = \frac{\pi}{2}$ periodically. Refer to de La Hire's point construction (https://en.wikipedia.org/wiki/Ellipse).

[3] Refer to "On the Arc Length of an Ellipse" (https://vixra.org/abs/2407.0173)

$$E(k) = 1 - \frac{1}{4}k^2 - \frac{3}{64}k^4 - \frac{5}{256}k^6 - \frac{175}{2^{14}}k^8 - \frac{441}{2^{16}}k^{10} - \frac{4851}{2^{20}}k^{12} - \cdots$$
$$= 1 - \sum_{n=1}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2 \frac{k^{2n}}{2n-1}, \tag{20}$$

where, $k = \sqrt{1 - \frac{b^2}{a^2}}$ or $k = \frac{c}{a}$ represents the eccentricity of an ellipse.

Also, $(2n-1)!!$ represents the double factorial for odd numbers, and $(2n)!!$ represents the double factorial for even numbers.

When an ellipse rolls on a straight line, the center of the rolling ellipse moves up and down in the order of semi-minor axis → semi-major axis → semi-minor axis or semi-major axis → semi-minor axis → semi-major axis.

This is the same as the trajectory drawn when the fixed point is the center of the ellipse among elliptic trochoids. From Eq. (19), the curve drawn by the center of an ellipse is as follows,

$$x = a\theta E(k) - b\sin\theta$$
$$y = \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}. \tag{21}$$

The area drawn by the elliptic cycloid in Eq. (19) is as follows,

$$A = \int_0^{2\pi}(-Eab\cos\theta + b^2\cos^2\theta)\,d\theta$$
$$+ \int_0^{2\pi}(aE - b\cos t)\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\,d\theta \tag{22}$$
$$= 2\pi E^2a^2 + \pi b^2.$$

The elliptic cycloid whose major axis lies on y-axis can be written as follows.

$$x = a\theta E(k) - a\sin\theta$$
$$y = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} - a\cos\theta. \tag{23}$$

The area of the above elliptic cycloid is $(2\pi E^2a^2 + \pi a^2)$.

D. Elliptic semi-trochoid

An elliptic trochoid is a trajectory traced by an ellipse as it rolls on a plane, and the center of the ellipse is not parallel to the plane and rolls unevenly. However, an elliptic semi-trochoid is a trajectory traced by a point on an ellipse as the axis of the ellipse translates along a straight line. That is, while a point on an ellipse rolls, the center of the ellipse translates. The parametric equation of an elliptic cycloid traced by a point on an ellipse, starting a fixed point at the left vertex of the major axis of the ellipse at $(0,0)$, is

$$x = a\theta E(k) + \sqrt{(a\cos\theta)^2 + (b\sin\theta)^2} - a\cos\theta$$
$$y = a\sin\theta. \tag{24}$$

The area drawn by an elliptic semi-cycloid by one rotation is

$$A = \int_0^{2\pi} (Ea^2\sin\theta + a^2\sin^2\theta)d\theta - \int_0^{2\pi} \frac{a\big((a^2 - b^2)\cos\theta\sin\theta\big)\sin\theta}{\sqrt{(a\cos\theta)^2 + (b\sin\theta)^2}}d\theta$$
$$= \pi a^2. \tag{25}$$

The area drawn by the first half $\left(2Ea^2 + \frac{1}{2}\pi a^2\right)$ is larger than the area drawn by the latter half $\left(-2Ea^2 + \frac{1}{2}\pi a^2\right)$.

When the major axis of an ellipse lies on the y-axis, the coordinates of the trajectory drawn from the left co-vertex of the minor axis as follows,

$$x = a\theta E(k) + \sqrt{(a\sin\theta)^2 + (b\cos\theta)^2} - b\cos\theta$$
$$y = b\sin\theta. \tag{26}$$

In this case, the area drawn by the curve is πb^2.

E. Centered trochoid on an elliptical orbit

An epicycloid on an elliptical orbit is a curve drawn by a rolling circle as it rolls clockwise along the orbit of a central ellipse. Since the arc length of the central ellipse moved by an angle $\emptyset$ is $AE(k)\emptyset$, the parametric equation of a circle rolling along the arc of a central ellipse whose major axis lies on the y-axis is as follows,

$$x = (B + r)\sin\left(\frac{r\theta}{AE}\right) - r\sin\left(\theta + \frac{r\theta}{AE}\right)$$
$$y = (A + r)\cos\left(\frac{r\theta}{AE}\right) - r\cos\left(\theta + \frac{r\theta}{AE}\right), \quad (27)$$

where, A represents the semi-major axis of the central ellipse, B represents the semi-minor axis, and E the sum of eccentric power series of the central ellipse.

The parametric equation of a hypocycloid on an elliptical orbit rolling clockwise inside a fixed ellipse whose major axis lies on the y-axis is as follows,

$$x = (B - r)\sin\left(\frac{r\theta}{AE}\right) - r\sin\left(\theta - \frac{r\theta}{AE}\right)$$
$$y = (A - r)\cos\left(\frac{r\theta}{AE}\right) + r\cos\left(\theta - \frac{r\theta}{AE}\right). \quad (28)$$

F. Semi-epicycloid on a circular orbit or on an elliptical orbit

A semi-epicycloid is a type of semi-epitrochoid, which is the path drawn by a point on the circumference of a rolling circle as the center of the rolling circle moves along the circular orbit of the fixed circle. In other words, the curve of the rolling circle is the same as the curve drawn by the rolling circle as it rolls on an imaginary circle whose radius is the radius of the fixed circle less the radius of the rolling circle. This curve is different from the curve drawn by the epicycle.

A semi-epicycloid in a circular orbit that rolls clockwise is

$$x = R\sin\left(\frac{r\theta}{R}\right) - r\sin\left(\theta + \frac{r\theta}{R}\right)$$
$$y = R\cos\left(\frac{r\theta}{R}\right) - r\cos\left(\theta + \frac{r\theta}{R}\right). \quad (29)$$

The area of the semi-epicycloid by one rotation is as follows

$$A = \int_0^{2\pi} (Rr)\cos^2\left(\frac{r\theta}{R}\right)d\theta + \int_0^{2\pi}\left(\frac{r^3}{R} + r^2\right)\cos^2\left(\theta + \frac{r\theta}{R}\right)d\theta$$
$$- \int_0^{2\pi} (Rr + 2r^2)\cos\left(\theta + \frac{r\theta}{R}\right)\cos\left(\frac{r\theta}{R}\right)d\theta \qquad (30)$$
$$= \pi r\left(R + r + \frac{r^2}{R}\right) + \left(\frac{1}{2}R^2 + \frac{1}{4}r^2 - \frac{1}{2}Rr\right)\sin\left(\frac{4\pi r}{R}\right).$$

A semi-epicycloid on an elliptical orbit is a trajectory drawn by the axis of a circle rolling along an elliptical orbit. The parametric equation for the trajectory that follows an elliptical orbit clockwise with the major axis of the central ellipse as the y-axis starting at $(0,\ (A-r))$ is

$$x = B\sin\left(\frac{r\theta}{AE}\right) - r\sin\left(\theta + \frac{r\theta}{AE}\right)$$
$$y = A\cos\left(\frac{r\theta}{AE}\right) - r\cos\left(\theta + \frac{r\theta}{AE}\right), \qquad (31)$$

where, A represents the semi-major axis of the central fixed ellipse, and B represents the semi-minor axis.

G. Centered elliptic trochoid on a circular orbit or on an elliptical orbit

An elliptic epitrochoid on a circular orbit or on an elliptic orbit is the trajectory drawn by a piont on a rolling ellipse as it rolls around a circular orbit of a fixed circle without slipping, or as it rolls around an elliptical orbit of a fixed ellipse without slipping.

Similarly, an elliptic hypotrochoid inside a fixed circle or inside a fixed ellipse is the trajectory drawn by a point on a rolling ellipse as it rolls inside a fixed circle without slipping, or as it rolls inside a fixed ellipse without slipping.

In the case of a circular orbit, the major axis of the rolling ellipse is parallel to the x-axis and draws the following curve with the starting point at $(0,\ R)$,

$$x = \left(R + \sqrt{a^2\sin^2\theta + b^2 cos^2\theta}\right)\sin\left(\frac{Ea\theta}{R}\right) - b\sin\left(\theta + \frac{Ea\theta}{R}\right) \qquad (32)$$

$$y = \left(R + \sqrt{a^2 \sin^2\theta + \ b^2 a\, cos^2\theta}\right)\cos\left(\frac{Ea\theta}{R}\right) - b\cos\left(\theta + \frac{Ea\theta}{R}\right),$$

where R represents the radius of a fixed circle, a the semi-major axis of a rolling ellipse, b the semi-minor axis and E the function of the eccentricity of the rolling ellipse.

An elliptic epicycloid on an elliptical orbit is a curve where the major axis of the rolling ellipse is parallel to the x-axis and the major axis of the central ellipse lies on the y-axis and the starting point is $(0,\ A)$, is given as follows,

$$\begin{aligned} x &= \left(B + \sqrt{a^2 \sin^2\theta + \ b^2\, cos^2\theta}\right)\sin\left(\frac{E_2 a\theta}{AE_1}\right) - b\sin\left(\theta + \frac{E_2 a\theta}{AE_1}\right) \\ y &= \left(A + \sqrt{a^2 \sin^2\theta + \ b^2\, cos^2\theta}\right)\cos\left(\frac{E_2 a\theta}{AE_1}\right) - b\cos\left(\theta + \frac{E_2 a\theta}{AE_1}\right), \end{aligned} \tag{33}$$

where, E_1 is the eccentric function of the central ellipse, and E_2 is that of the rolling ellipse.

When the major axes of both lie on y-axis, the parametric equation is given as follows,

$$\begin{aligned} x &= \left(B + \sqrt{a^2 \cos^2\theta + \ b^2 \sin^2\theta}\right)\sin\left(\frac{E_2 a\theta}{AE_1}\right) - a\sin\left(\theta + \frac{E_2 a\theta}{AE_1}\right) \\ y &= \left(A + \sqrt{a^2 \cos^2\theta + \ b^2 \sin^2\theta}\right)\cos\left(\frac{E_2 a\theta}{AE_1}\right) - a\cos\left(\theta + \frac{E_2 a\theta}{AE_1}\right). \end{aligned} \tag{34}$$

An elliptic cardioid is the trajectory of a curve that is drawn when an ellipse rolls over the curve of another ellipse with the major axes of the same size aligned, and is a curve that is closer to the heart than a cardioid. From (34), an elliptic cardioid rolls clockwise from the starting point $(0, a)$, which point is the only one cusp, is given as

$$\begin{aligned} x &= 2b\sin\theta - a\sin(2\theta) \\ y &= 2a\cos\theta - a\cos(2\theta). \end{aligned} \tag{35}$$

An elliptic cardioid has the area of $(4ab\pi + 2a^2\pi)$. If $b = a$, then we have that of a cardioid.

An elliptic hypocycloid drawn by a point on an ellipse inscribed in a circle is the curve in case of an ellipse whose major axis is parallel to the x-axis starting at $(0, R)$

$$x = \left(R - \sqrt{a^2 \sin^2 \theta + \ b^2 \, cos^2 \, \theta}\right) \sin\left(\frac{Ea\theta}{R}\right) - b \sin\left(\theta - \frac{Ea\theta}{R}\right)$$
$$y = \left(R - \sqrt{a^2 \sin^2 \theta + \ b^2 a \, cos^2 \, \theta}\right) \cos\left(\frac{Ea\theta}{R}\right) + b \cos\left(\theta - \frac{Ea\theta}{R}\right). \tag{36}$$

In case of an elliptic hypocycloid in an ellipse, the major axes of both the rolling ellipse and the central ellipse are located on the y-axis, we have

$$x = \left(B - \sqrt{a^2 \cos^2 \theta + \ b^2 \sin^2 \theta}\right) \sin\left(\frac{E_2 a\theta}{AE_1}\right) - a \sin\left(\theta - \frac{E_2 a\theta}{AE_1}\right)$$
$$y = \left(A - \sqrt{a^2 \cos^2 \theta + \ b^2 \sin^2 \theta}\right) \cos\left(\frac{E_2 a\theta}{AE_1}\right) + a \cos\left(\theta - \frac{E_2 a\theta}{AE_1}\right). \tag{37}$$

An elliptic astroid can be derived from the elliptic hypocycloid inscribed in a fixed ellipse. It is the trajectory that an ellipse of the same shape and a quarter of the size is inscribed in a fixed ellipse and rolls without slipping. It is similar to the closed curve known as the evolute of an ellipse which is sometimes known as the Lamé curve.

Parametric equation from (37) is as follows,

$$x = \left(4b - \sqrt{a^2 \cos^2 \theta + \ b^2 \sin^2 \theta}\right) \sin\left(\frac{\theta}{4}\right) - a \sin\left(\frac{3\theta}{4}\right)$$
$$y = \left(4a - \sqrt{a^2 \cos^2 \theta + \ b^2 \sin^2 \theta}\right) \cos\left(\frac{\theta}{4}\right) + a \cos\left(\frac{3\theta}{4}\right). \tag{38}$$

Or equivalently,

$$x = (4b - a) \sin\left(\frac{\theta}{4}\right) - a \sin\left(\frac{3\theta}{4}\right)$$
$$y = 3a \cos\left(\frac{\theta}{4}\right) + a \cos\left(\frac{3\theta}{4}\right). \tag{39}$$

The area of an elliptic astroid is $A = 3ab\pi - \frac{3}{2}\pi a^2$.

H. Elliptic semi-cycloid on a circular orbit and on an elliptical orbit

An elliptic semi-cycloid on a circular orbit is the trajectory drawn by a point on a rolling ellipse as its central point slides along a circular orbit, and an elliptic semi-cycloid on an elliptical orbit is the trajectory drawn by a point on a rolling ellipse as its central point slides along the orbit of the central ellipse. In the case of a circular orbit, the major axis of the rolling ellipse is parallel to the x-axis, and the parametric equation for the trajectory drawn with the starting point at $(0, R)$ is as follows,

$$x = R\sin\left(\frac{Ea\theta}{R}\right) - \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\sin\left(\theta + \frac{Ea\theta}{R}\right)$$
$$y = R\cos\left(\frac{Ea\theta}{R}\right) - \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\cos\left(\theta + \frac{Ea\theta}{R}\right). \quad (40)$$

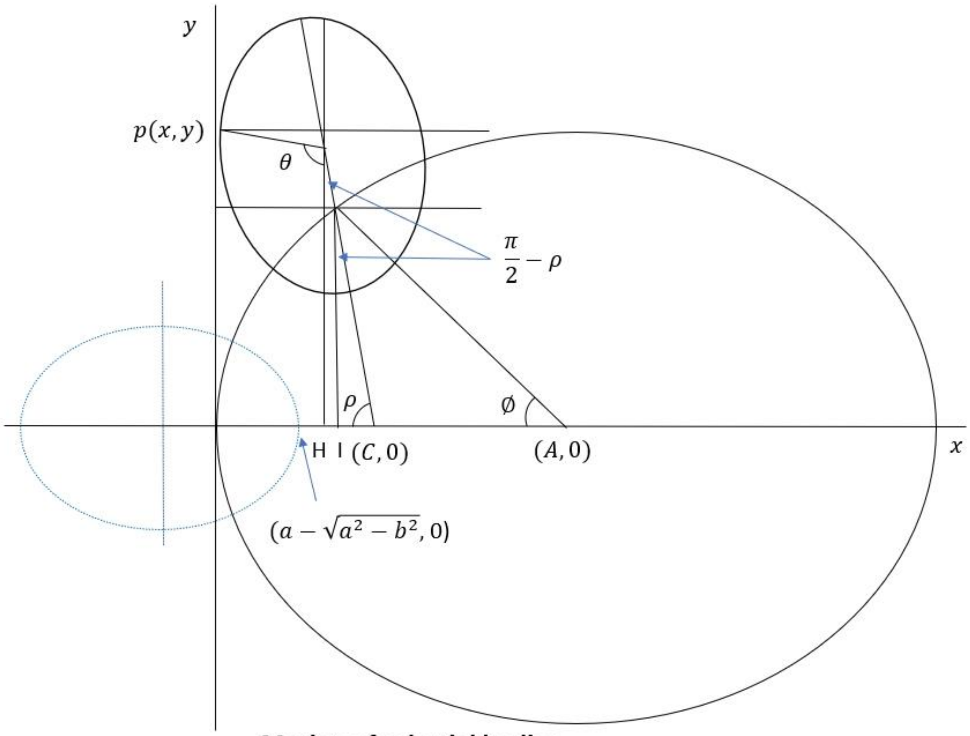
An ellipse onto an elliptical orbit with the major axis of the central ellipse and the major axis of the rolling ellipse on the y-axis at $(0,\ (A-a))$ is as follows.

$$x = B\sin\left(\frac{a\theta E_2}{AE_1}\right) - \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\sin\left(\theta + \frac{a\theta E_2}{AE_1}\right)$$
$$y = A\cos\left(\frac{a\theta E_2}{AE_1}\right) - \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\cos\left(\theta + \frac{a\theta E_2}{AE_1}\right). \quad (41)$$

I. Celestial motion

The elliptic semi-cycloid can be used to describe the motion of celestial bodies. The Moon and Earth orbit a common barycenter. That is, if one point of an elliptical orbit is the moon, one focus of the moon translates as it rolls over the Earth's elliptic orbit. And the Earth moves along the elliptical orbit of the central ellipse. At this time, the center of the central ellipse is the barycenter or the center of gravity.

Since the Sun's barycenter is slightly off center, the Sun also rolls along an elliptical orbit.



Motion of celestial bodies

The angular velocity of the satellite ellipse ($v_2 = a\theta E_2$) and the angular velocity of the central ellipse ($v_1 = A\emptyset E_1$) represent the velocities at their respective central points, but since the focus of the satellite ellipse moves elliptically around the focus of the central ellipse, the parametric equation is obtained by substituting the relationship between the focus of the central ellipse and the center point, as follows,

$$\begin{aligned} x &= A - A\cos\emptyset \\ &\quad - \left(a - \sqrt{a^2 - b^2}\right)\sin\rho \\ &\quad - \sqrt{a^2\cos^2(\rho+\theta) + b^2\sin^2(\rho+\theta)}\,\sin\theta \\ y &= B\sin\emptyset + \left(a - \sqrt{a^2 - b^2}\right)\cos\rho \\ &\quad - \sqrt{a^2\cos^2(\rho+\theta) + b^2\sin^2(\rho+\theta)}\,\cos\theta \\ \rho &= \tan^{-1}\left(\frac{B\sin\emptyset}{A - \sqrt{A^2 - B^2} - A\cos\emptyset}\right), \end{aligned} \tag{42}$$

where A and B are the semi-major axis and semi-minor axis of the central ellipse, and a and b are those of a satellite ellipse.

References

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