PROBLEM THEORY AND APPLICATION TO THE P VS NP PROBLEM

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Abstract. This paper introduces problem theory, a new framework for studying problem and solution spaces through the lens of point-set topology and abstract algebra. We define solution spaces as topological constructs induced by the assignment of solutions to problems and establish their fundamental properties. Key results include the identification of compactness and continuity conditions in solution spaces and their algebraic interpretations within module-theoretic settings. This theory bridges abstract algebra and topology, providing new insights into the interplay between algebraic structures and topological spaces. Potential applications and directions for future research are discussed.

Contents

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1. Introduction

The central theme of this paper is that a problem should not be viewed only as an isolated input-output task, but also as an organized object with internal structure: a family of subproblems, a corresponding family of solutions, and a web of relations linking one problem to another. This perspective is natural in modern complexity theory, where the basic question is not merely whether a solution exists but how the search for a solution is organized, how quickly it can be carried out, and how the act of verification compares with the act of construction. In the classical formulation of the P versus NP problem, the core issue is whether every language whose certificates can be checked efficiently can also be decided efficiently. [3, 7] The reduction-based viewpoint that emerged from the Cook–Levin theorem and the Karp landmark work on reducibility shows that the right notion of hardness is often relational rather than absolute: one problem is informative precisely because it can be transformed into, or encoded by, another. [3, 4]

The theory developed here takes that relational idea seriously and pushes it inward. Instead of treating reductions only as maps between externally given decision problems, the paper builds a structural language in which a problem carries its own induced problem space and solution space. The guiding intuition is that every nontrivial problem comes with a hierarchy of subproblems and that the meaning of a solution is not exhausted by a single answer token but includes the collection of tasks that must be traversed in order for the answer to exist. This shift in perspective is the novelty of the paper. The resulting framework is not merely descriptive; it is designed to allow one to classify problems by the shape of their internal decomposition, to compare problem spaces through equivalence and overlap, and to track how algebraic and topological properties of these spaces constrain solvability and complexity.

The paper begins by defining a problem space $\mathcal{P}_Y(X)$ associated with a problem Y and a solution X, together with the corresponding solution space $\mathcal{S}_Y(X)$. These are not formal decorations: they are the objects from which the later theory is built. Once these spaces are defined, one can assign them numerical invariants such as their size, called the complexity of the problem space, and the index of the solution space. The reciprocal index gives an entropy-like quantity that measures how concentrated a solution space is. This is a deliberate abstraction: the paper does not use entropy in the measure-theoretic or information-theoretic sense but rather as a compact way to encode the idea that richer solution spaces can be thinner or denser depending on how many solutions they contain. In this way, the manuscript introduces a set of basic invariants that are meant to play for problems and solutions a role analogous to cardinal, geometric, and combinatorial invariants in other branches of structural mathematics.

A key conceptual step is the theorem that asserts that a problem belongs to the problem space induced by one of its solutions. This self-inclusion principle makes the theory recursive: a solution does not merely terminate the analysis, but generates a new layer of structure from which further problem and solution spaces can be compared. From that starting point, the paper introduces equivalent problems, subproblems, and induced inclusions of problem spaces. These notions allow the author to express the intuition that solving one problem may implicitly solve a family of related problems, while also making it possible to compare the structural depth of different questions. The theory then separates the problems into reducible and irreducible types. Reducibility is defined through the existence of a proper subproblem with no further proper subproblem, while irreducibility means that every proper subproblem still admits further descent. In the language of the paper, irreducibility captures a kind of structural infinitude: the problem cannot be flattened into a terminal finite layer of substructure. This idea is used to support the theorem that asserts the existence of a problem without a solution and provides the paper with one of its most striking organizing principles.

The later sections refine this picture by studying regular and irregular problems. A regular problem is one whose subproblems can be arranged into a chain, while an irregular problem admits incomparable subproblems. This distinction is important because it isolates the cases in which the internal structure of a problem can be linearized, and therefore analyzed by successive descent. The paper then introduces maximal and minimal subproblems and shows how these extremal objects interact with reducibility and regularity. These notions resonate with familiar ideas from order theory and lattice-like structures, but they are used here in a distinctly problem-theoretic way: a maximal subproblem is not merely an extremal element in a poset, but a structural bottleneck through which the solvability of the larger problem may pass. In this sense, the paper tries to recast the complexity of a problem in terms of the architecture of its internal constraints.

The manuscript also develops a notion of connectedness between problem spaces, together with stronger and weaker notions of overlap. This is a useful extension of the theory because it makes it possible to ask whether the solution of one problem leaves a nontrivial footprint in the space of another. Closely related ideas have already appeared in the study of random constraint satisfaction problems, where the geometry of solution spaces—clustering, frozen variables, and threshold phenomena—has been shown to govern algorithmic behavior. [5, 6] More recently, combinatorial optimization work has begun to directly compute solution-space properties, including counting, sampling, and the enumeration of optimal objects, by using unified algebraic frameworks such as tensor networks. [2] The present paper belongs to this larger intellectual movement, but is more axiomatic in style: rather than studying one particular model of constraints, it proposes a general language in which the geometry of problems and solutions can be discussed at a higher level of abstraction.

Another important ingredient is the treatment of alternative solutions. The paper introduces a replacement principle that shows that solution spaces remain invariant under substitution by alternative solutions. This is conceptually valuable because it emphasizes that the solution space is not tied to a single certificate but to the equivalence class of certificates that accomplish the same task. The same philosophy underlies the definitions of separable and inseparable problem spaces and their corresponding solution spaces. The paper proves that separability on the problem side is equivalent to separability on the solution side, thereby establishing one of the central transfer principles of the theory. This is a meaningful structural result: it says that a decomposition visible at the level of tasks must also be visible at the level of solutions, and conversely. In effect, the paper argues that the problem space and the solution space are two faces of the same underlying organization.

The quotient construction further enriches the theory by introducing principal and ideal subspaces and by linking them to maximal subproblems in regular settings. This is one of the places where the algebraic flavor of the manuscript becomes strongest. The quotient language is not used merely as notation; it allows the author to encode how a subspace behaves when a single problem is adjoined to it and to interpret certain subspaces as structurally privileged. This is followed by the discussion of overlapping, non-overlapping, and symmetric problem spaces. Symmetry is defined through the existence of equivalent problems across spaces, and the paper shows that such symmetry forces equality of the corresponding solution spaces. This gives a particularly clean structural criterion: when problem spaces are equivalent in the appropriate sense, the solutions they admit cannot be distinguished by the theory. Such statements are mathematically appealing because they transform qualitative similarity into an exact identity at the level of solutions.

The final major component of the paper is the analysis of time complexity. Here, the theory returns explicitly to the classical motivation from complexity theory. The paper distinguishes resolution complexity, the time required to generate a solution, from verification complexity, the time required to check it. This separation is closely aligned with the standard interpretation of P versus NP and with the role certificates play in complexity theory. [3, 7] What is distinctive about the present manuscript is that these complexities are studied in problem and solution spaces rather than only in isolated instances. The paper introduces equilibrium, meaning equality between resolution and verification complexity, and investigates how this equilibrium behaves under descent to subproblems, especially for regular and reducible problems. The resulting transfer principles are intended to show that complexity can be controlled by the structure of the finest subproblem and that a bottom-up analysis may suffice in well-behaved cases. In this way, the theory aims to connect global complexity with local decomposition.

1.1. **Organization of the paper.** The paper is organized to build this language progressively. Section 2 introduces problems, problem spaces, solution spaces, and the first basic invariants. Section 3 develops reducibility and irreducibility and uses them to obtain the first existence result. Section 4 studies regular and irregular problems, together with maximal and minimal subproblems. Section 5 analyzes the connectedness and disconnection between problem spaces, while Section 6 explains the role of alternative solutions.

Section 7 develops separability and proves its equivalence on the problem and solution sides. Section 8 introduces quotient problem and solution spaces and isolates principal and ideal subspaces. Section 9 turns to overlap and symmetry, and Section 10 develops the theory of time complexity and its relevance to the perspective of P versus NP problem. The overall aim is to show that a unified theory of problems can be built from a small number of structural primitives and that once these primitives are in place, one can begin to rigorously speak about decomposition, transfer, and complexity in a way that is both abstract and operational.

2. Problems and solution spaces

In this section, we introduce and develop the notion of problem and the corresponding solution spaces.

Definition 2.1. Let X denote a solution (resp. answer) to the problem Y (resp. question). We call the collection of all problems to be solved to provide a solution X to the problem Y the problem space induced by providing a solution X to the problem Y. We denote this space by $\mathcal{P}_Y(X)$. If K is any subspace of the space $\mathcal{P}_Y(X)$, then we denote this relation by $K \subseteq \mathcal{P}_Y(X)$. If the space K is a subspace of the space $\mathcal{P}_Y(X)$ with $K \neq \mathcal{P}_Y(X)$, then we write $K \subset \mathcal{P}_Y(X)$. We say that problem V is a sub-problem of problem Y if providing a solution to problem Y furnishes a solution to problem V. If V is a sub-problem of the problem Y, then we write $V \leq Y$. If V is a sub-problem of the problem Y and $V \neq Y$, then we write $V < Y$ and call V a proper sub-problem of Y.

Definition 2.2. Let $\mathcal{P}_Y(X)$ be the problem space induced by providing the solution X to the problem Y. We call the number of problems in the space (size) the *complexity* of the space and denote by $\mathbb{C}[\mathcal{P}_Y(X)]$ the complexity of the space. We make the assignment $Z \in \mathcal{P}_Y(X)$ if problem Z is also a problem in this space.

Definition 2.3. Let X denote a solution (resp. answer) to the problem Y (resp. question). We call the collection of all solutions to problems obtained as a result of providing the solution X to problem Y the solution space induced by providing the solution X to problem Y. We denote this space by $\mathcal{S}_Y(X)$. If K is any subspace of the space $\mathcal{S}_Y(X)$, then we denote this relation by $K \subset \mathcal{S}_Y(X)$. We assign $T \in \mathcal{S}_Y(X)$ if the solution T is also a solution in this space. If T is a sub-solution of the solution X, then we write $T \leq X$. If T is a sub-solution of the solution X and $T \neq X$, then we write $T < X$ and call T a proper sub-solution of X.

Proposition 2.4. *Let $\mathcal{S}_Y(X)$ be the solution space induced by providing solution X to problem Y. We have $X \in \mathcal{S}_Y(X)$.*

Proof. The solution space $\mathcal{S}_Y(X)$ contains all the sub-solutions of the solution X. Since $X \leq X$, we deduce the claim. □

Definition 2.5. Let $\mathcal{S}_Y(X)$ be the solution space induced by providing the solution X to the problem Y. We call the number of solutions in the space (size) the *index* of the space and denote by $\mathbb{I}[\mathcal{S}_Y(X)]$ the index of this space.

Definition 2.6. Let $\mathcal{S}_Y(X)$ be the solution space induced by providing the solution X to the problem Y. By the *entropy* of the space, we mean the expression

$$\mathcal{E}[\mathcal{S}_Y(X)] := \frac{1}{\mathbb{I}[\mathcal{S}_Y(X)]}.$$

Theorem 2.7. *Let $\mathcal{P}_Y(X)$ be the induced problem space of providing the solution X to the problem Y. We have $Y \in \mathcal{P}_Y(X)$.*

Proof. The problem space $\mathcal{P}_Y(X)$ contains all the subproblems of problem Y. Since $Y \leq Y$, we deduce the claim. □

Definition 2.8. Let Y and V be any two problems. We say that problem Y is equivalent to problem V if providing a solution to problem Y also provides a solution to problem V and conversely providing a solution to problem V also provides a solution to problem Y. We denote equivalence by $V \equiv Y$.

Here, we present a simple criterion for creating a subspace of a problem space.

Proposition 2.9. *Let $X \in \mathcal{S}_V(U)$ and $Y \in \mathcal{P}_V(U)$. If X is a solution to the problem Y, then*

$$\mathcal{P}_Y(X) \subset \mathcal{P}_V(U).$$

Proof. Under the requirement $Y \in \mathcal{P}_V(U)$, it follows that Y is a sub-problem to be solved to provide a solution U to problem V. Since $X \in \mathcal{S}_V(U)$, it follows that X is a solution obtained by providing a solution U to the problem V. Since X solves Y and $Y \in \mathcal{P}_Y(X)$, it follows that

$$\mathcal{P}_Y(X) \subset \mathcal{P}_V(U).$$

□

We use the following criterion to determine the solubility of a problem.

Proposition 2.10. *Let V be a problem with solution U. If $Y \in \mathcal{P}_V(U)$, then Y must have a solution.*

Proof. Clearly, problem V is solved by U with an induced problem space $\mathcal{P}_V(U)$. Since this space consists of all sub-problems to be solved in order to provide the solution U to problem V and $Y \in \mathcal{P}_V(U)$, Y has a solution. □

3. Reducible and irreducible problems

In this section, we classify problems in a problem space into two main categories. We study the notion of *irreducibility* and *reducibility* of a problem.

Definition 3.1. Let V be a problem. We say V is *reducible* if there exists a proper sub-problem of V with no proper sub-problem. On the other hand, we say that problem V is *irreducible* if every proper sub-problem of V has a proper sub-problem.

It is a well-known problem to determine if every problem has a solution. Using this classification, we can deduce that there must be a problem with no solution. It turns out that irreducible problems satisfy this property.

Theorem 3.2. *There exists a problem with no solution.*

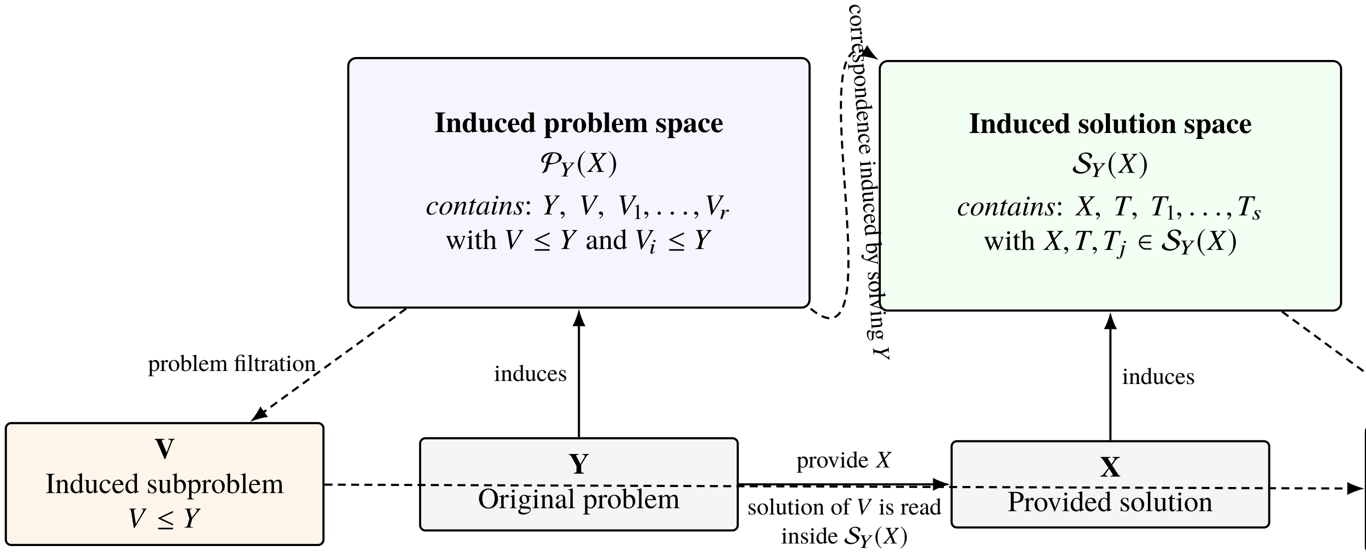


Figure 1. An induced problem space and induced solution space generated by providing a solution X to a problem Y.

Proof. Suppose to the contrary that every problem has a solution. It suffices to argue with only irreducible problems. Now, let V be an irreducible problem with the solution U. Consider the induced problem space $\mathcal{P}_V(U)$. Then by Theorem 2.7, we have $V \in \mathcal{P}_V(U)$. Since V is irreducible, we choose a proper sub-problem Y of V with solution X and construct the problem space $\mathcal{P}_Y(X)$ and solution spaces $\mathcal{S}_Y(X)$. Then $Y \in \mathcal{P}_V(U)$ and $X \in \mathcal{S}_V(U)$ so that

$$\mathcal{P}_Y(X) \subset \mathcal{P}_V(U).$$

Again, V is irreducible, so we can choose a proper sub-problem Z of Y with solution R. Using the same arguments, we have the chain of sub-covers of problem spaces

$$\mathcal{P}_Z(R) \subset \mathcal{P}_Y(X) \subset \mathcal{P}_V(U).$$

Iterating the argument under the same assumption that every problem has a solution, we obtain the infinite chain of sub-covers of smaller problem spaces

$$\cdots \subset \cdots \subset \mathcal{P}_Z(R) \subset \mathcal{P}_Y(X) \subset \mathcal{P}_V(U).$$

This is impossible, and this completes the proof. □

We can now state another important criterion for determining the solubility of a problem, provided that we can put it on par with some category of problems.

Proposition 3.3. *Let V and Y be any two problems such that $V \equiv Y$. If V is irreducible, then Y cannot be solved.*

Proof. Let $V \equiv Y$ and suppose that Y has a solution. Then it follows that V must also have a solution, contradicting the requirement that V is irreducible. □

4. Regular and irregular problems

In this section, we classify problems according to the structure of their sub-problems. We study the notion of *regular* and *irregular* problem.

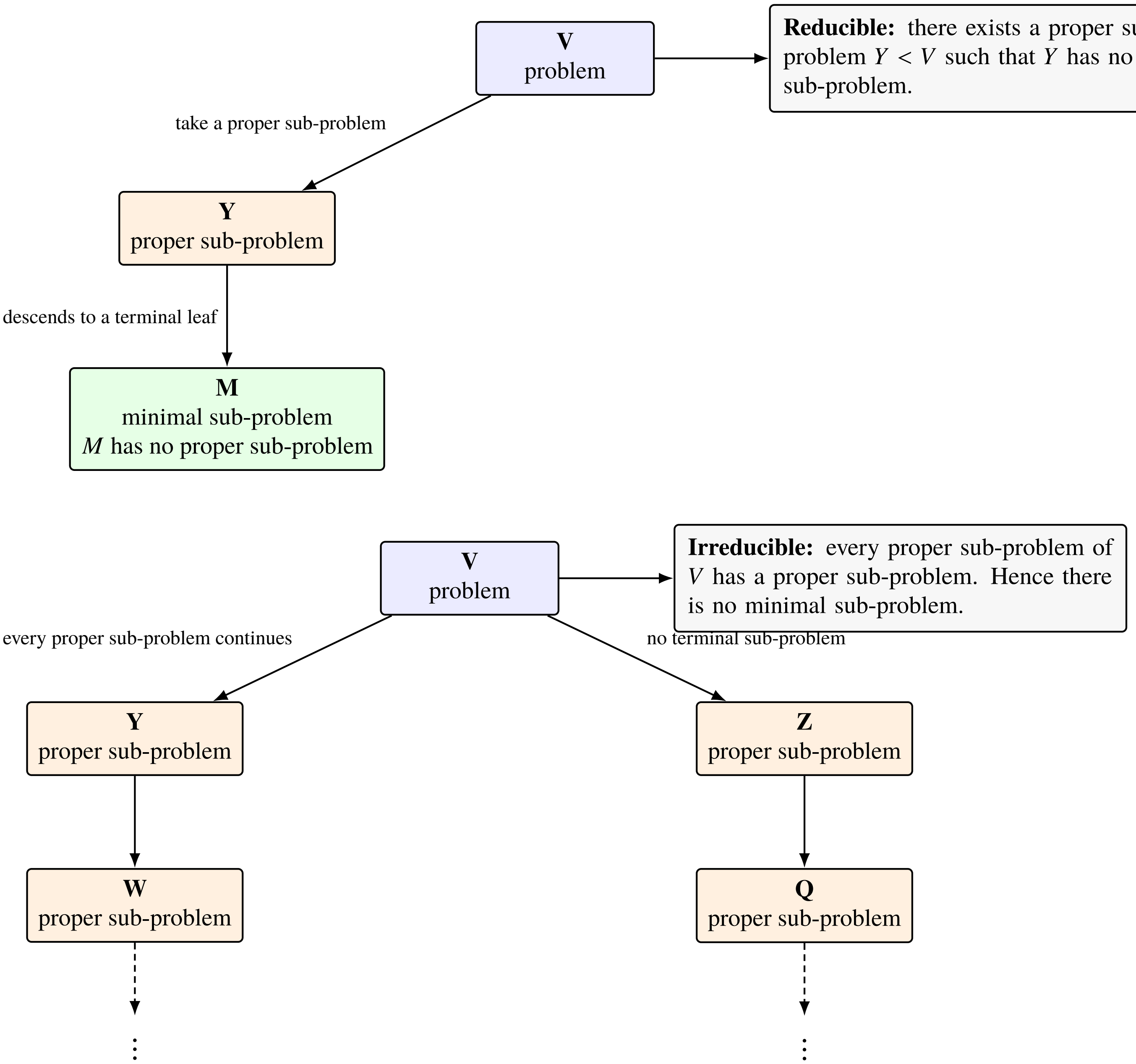
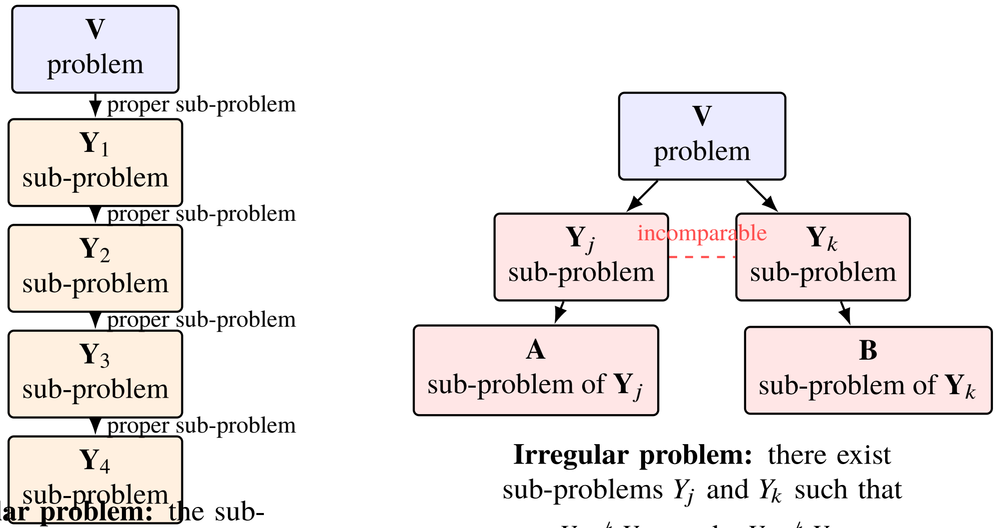


FIGURE 2. Mechanism of reducible and irreducible problems in problem theory. The reducible case terminates at a minimal sub-problem, whereas the irreducible case never terminates and continues to descend through proper sub-problems.

Definition 4.1. Let V be a problem and $\{Y_i\}_{i\geq 1}$ be the sequence of all the sub-problems of V. We say that V is regular if

$$\cdots \leq Y_3 \leq Y_2 \leq Y_1 \leq V.$$

We say it is irregular if there exists sub-problems Y_j and Y_k of V such that $Y_j \nleq Y_k$ and $Y_k \nleq Y_j$.



Regular problem: the sub-problems are totally ordered

$$\cdots \leq Y_4 \leq Y_3 \leq Y_2 \leq Y_1 \leq V.$$

Irregular problem: there exist sub-problems Y_j and Y_k such that

$$Y_j \not\leq Y_k \quad \text{and} \quad Y_k \not\leq Y_j.$$

FIGURE 3. Illustration of regular and irregular problems in problem theory. A regular problem admits a chain of sub-problems, whereas an irregular problem contains incomparable sub-problems.

De facto, regular problem can easily be solved as opposed to irregular problems, where a solution to one sub-problem cannot in anyway be modified and advanced to obtain a solution to other sub-problems. This makes the theory much more tractable with reducible problems.

4.1. Maximal and minimal sub-problems.

Definition 4.2. Let V be a problem and Y a proper sub-problem of V. We say that Y is the *maximal* sub-problem of V if all other proper sub-problems of V are sub-problems of Y. We say it is the *minimal* sub-problem of V if it is a sub-problem of all other sub-problems of V.

Next we relate the notion of minimal sub-problem to the notion of reducibility.

Proposition 4.3. *Let V be a problem. If there exists a minimal sub-problem of V, then V must be reducible.*

Proof. Let Y be the minimal sub-problem of problem V. Then Y has no proper sub-problem. This implies that V must be reducible. □

In a similar fashion, we relate the notion of maximal sub-problem with the notion of regularity.

Theorem 4.4. *Let V be a problem. If every sub-problem of V has a maximal proper sub-problem, then V must be regular.*

Proof. Let Y be the maximal proper sub-problem of V, since $V \leq V$. Then we have the relation $Y < V$ and every other proper sub-problem of V must be a sub-problem of Y. Since every sub-problem of V has a maximal sub-problem, we let Z be the maximal proper sub-problem of Y then $Z < Y$ and every other proper sub-problems of Y are sub-problems of Z. Since the proper sub-problems of V excluding Y are proper sub-problems of Y and the remaining excluding Z are sub-problems of Z, we obtain the chain of sub-problems

$$\cdots < Z < Y < V$$

and thus the chain contains all the sub-problems of V. This shows that V must be a regular problem. □

5. Connected and disconnected problem spaces

In this section, we study the existence of solutions to problems by deriving information about the status of related and analogous problems.

Definition 5.1. Let V be a problem with solution U and Y a problem with solution X. We say that the induced problem spaces $\mathcal{P}_V(U)$ and $\mathcal{P}_Y(X)$ are connected if and only if

$$\mathcal{P}_V(U) \cap \mathcal{P}_Y(X) \neq \emptyset.$$

We say that the connection is high if

$$\frac{|\mathcal{P}_V(U) \cap \mathcal{P}_Y(X)|}{|\mathcal{P}_V(U)|} \geq \frac{1}{2} \quad \textbf{and} \quad \frac{|\mathcal{P}_V(U) \cap \mathcal{P}_Y(X)|}{|\mathcal{P}_Y(X)|} \geq \frac{1}{2}.$$

Otherwise, we say that the connection is low. On the other hand, we say that problem spaces are disconnected if and only if

$$\mathcal{P}_V(U) \cap \mathcal{P}_Y(X) = \emptyset.$$

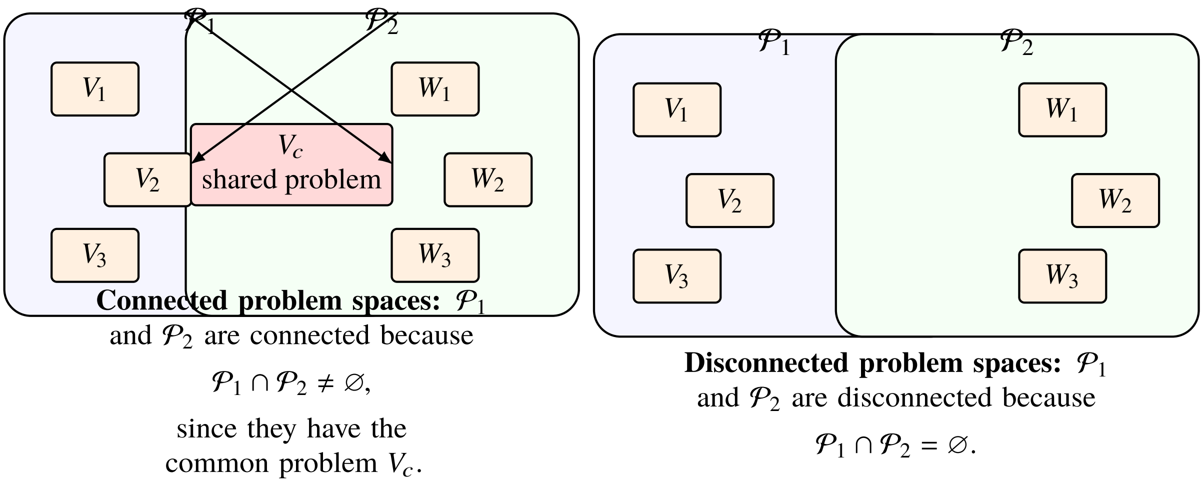


Figure 4. Corrected depiction of connected and disconnected problem spaces. Two problem spaces are connected precisely when they share at least one common problem.

Proposition 5.2. *Let Y be a problem with solution X. If V is also a problem with a maximal proper sub-problem Z such that $Z \in \mathcal{P}_Y(X)$ and V is regular, then V must be solvable and the induced problem space must be connected to $\mathcal{P}_Y(X)$.*

Proof. Since problem Y has solution X, each problem in the space $\mathcal{P}_Y(X)$ has also been solved. The requirement $Z \in \mathcal{P}_Y(X)$ implies that the problem Z has been solved. Since V is regular, we have the chain of all sub-problems of V as

$$\cdots \leq Y_3 \leq Y_2 \leq Y_1 \leq Z$$

since Z is the maximal sub-problem of V. Since Z is solved, it follows that all sub-problems of V are solved and V must have a solution, say T, with induced problem space $\mathcal{P}_V(T)$. The latter claim follows by noting that $Z \in \mathcal{P}_V(T) \cap \mathcal{P}_Y(X)$. □

6. Alternative solutions and their corresponding solution spaces

Definition 6.1. Let Y be a problem. We say that X and U are alternative solutions to Y if and only if U and X both solve Y. We denote this relation by $X \perp U$ or $U \perp X$.

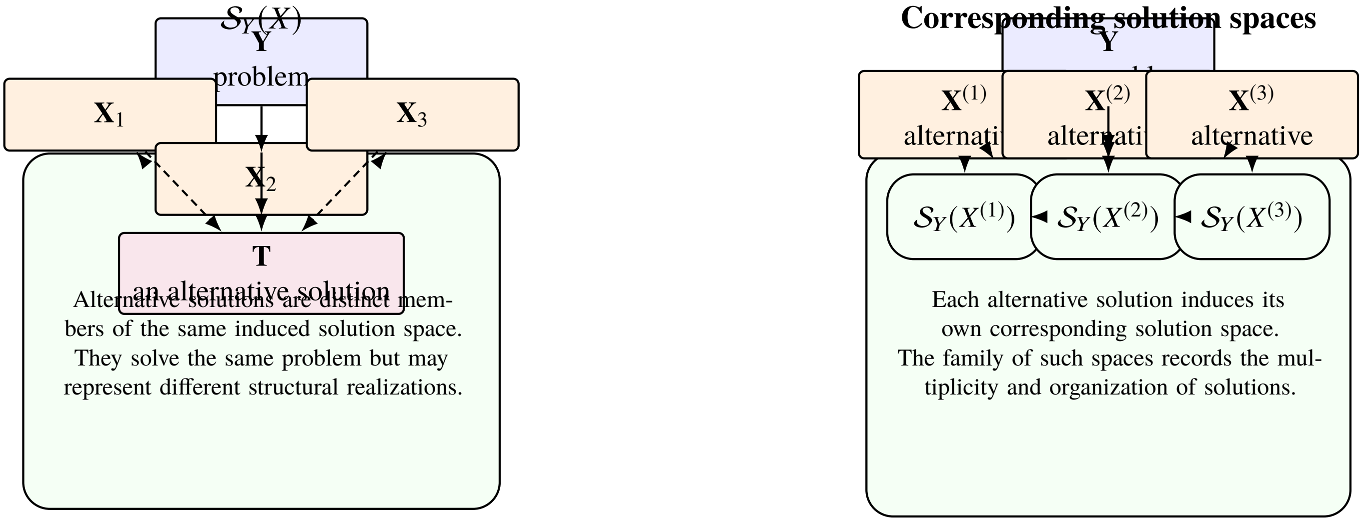


Figure 5. Alternative solutions and corresponding solution spaces in problem theory. The same problem may admit several alternative solutions, each of which can generate a corresponding induced solution space.

Proposition 6.2. *The solution spaces remain invariant under replacement with alternative solutions.*

Proof. Let $\mathcal{P}_Y(X)$ be a problem space with the corresponding solution space $\mathcal{S}_Y(X)$. Suppose that $L \in \mathcal{S}_Y(X)$ with $L \perp K$, then there exists a problem $F \in \mathcal{P}_Y(X)$ that is solved by L. Since $L \perp K$, it follows that K also solves F. Thus, we can replace $L \in \mathcal{S}_Y(X)$ with K. □

7. Separable and inseparable problem and solution spaces

In this section, we introduce and study the notion of *separability* of problems and their corresponding solution spaces.

Definition 7.1. Let $\mathcal{P}_Y(X)$ be a problem space. We say that $\mathcal{P}_Y(X)$ is *separable* if and only if there exist some $\mathcal{P}_V(U) \subset \mathcal{P}_Y(X)$ and $\mathcal{P}_K(L) \subset \mathcal{P}_Y(X)$ such that

$$\mathcal{P}_V(U) \cup \mathcal{P}_K(L) = \mathcal{P}_Y(X)$$

with

$$\mathcal{P}_V(U) \cap \mathcal{P}_K(L) = \emptyset$$

and $F \not\equiv G$ for any $F \in \mathcal{P}_V(U)$ and $G \in \mathcal{P}_K(L)$. Otherwise, we say that the problem space is *inseparable*. Similarly, we say that a solution space $\mathcal{S}_Y(X)$ is separable if and only if there exist some $\mathcal{S}_V(U) \subset \mathcal{S}_Y(X)$ and $\mathcal{S}_K(L) \subset \mathcal{S}_Y(X)$ such that

$$\mathcal{S}_V(U) \cup \mathcal{S}_K(L) = \mathcal{S}_Y(X)$$

with

$$\mathcal{S}_V(U) \cap \mathcal{S}_K(L) = \emptyset$$

and $R \not\perp W$ for any $R \in \mathcal{S}_V(U)$ and $W \in \mathcal{S}_K(L)$. Otherwise, we say that the solution space is inseparable.

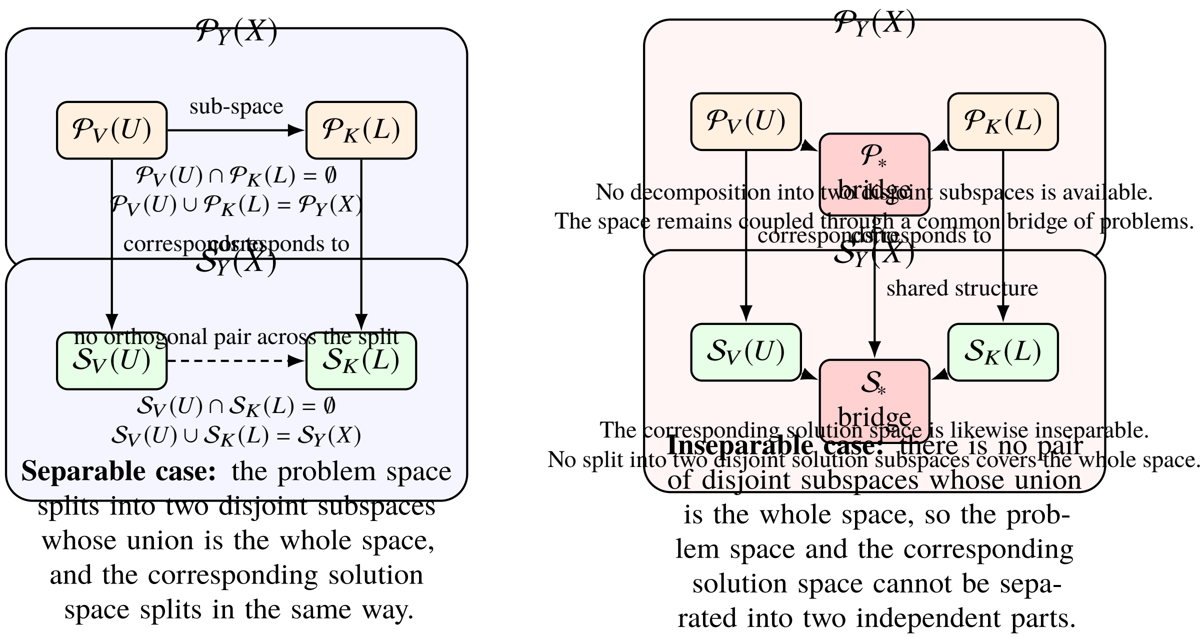


Figure 6. Separable and inseparable problem and solution spaces in problem theory. In the separable case, the problem space and the corresponding solution space decompose into two disjoint parts whose unions recover the whole spaces. In the inseparable case, such a disjoint decomposition does not exist.

We demonstrate that the notion of *separability* can be passed between problems and their corresponding solution spaces. The following result is a formalization of this important concept.

Theorem 7.2. *Let $\mathcal{P}_Y(X)$ be a problem space with the corresponding solution space $\mathcal{S}_Y(X)$. Then $\mathcal{P}_Y(X)$ is separable if and only if $\mathcal{S}_Y(X)$ is separable.*

Proof. Suppose that $\mathcal{P}_Y(X)$ is separable. There exist $\mathcal{P}_V(U) \subset \mathcal{P}_Y(X)$ and $\mathcal{P}_K(L) \subset \mathcal{P}_Y(X)$ such that

$$\mathcal{P}_V(U) \cup \mathcal{P}_K(L) = \mathcal{P}_Y(X)$$

with

$$\mathcal{P}_V(U) \cap \mathcal{P}_K(L) = \emptyset$$

and $F \not\equiv G$ for any $F \in \mathcal{P}_V(U)$ and $G \in \mathcal{P}_K(L)$. For any $F \in \mathcal{P}_V(U)$ there exists some $R \in \mathcal{S}_V(U)$ that solves F and some $W \in \mathcal{S}_K(L)$ that solves G. Since $\mathcal{P}_V(U) \cap \mathcal{P}_K(L) = \emptyset$ and the problems in both spaces are not equivalent, it follows that $R \not\perp W$ and $R \notin \mathcal{S}_K(L)$ and $W \notin \mathcal{S}_V(U)$. Since R and W are arbitrary, it follows that $\mathcal{S}_Y(X)$ must also be separable. Suppose without loss of generality that R solves some problem in the space $\mathcal{P}_K(L)$. In particular, there exists some $T \in \mathcal{P}_K(L)$ that is solved by R. Since R also solves F and there exists some $W \in \mathcal{S}_K(L)$ that solves T, it must be that $W \perp R$, a contradiction. In the case $R \perp W$, we obtain $R \in \mathcal{S}_K(L)$ and $W \in \mathcal{S}_V(U)$ by virtue of Proposition 6.2. Without loss of generality, we examine the case $R \perp W$ and $R \in \mathcal{S}_K(L)$ with $W \notin \mathcal{S}_V(U)$ then $W \in \mathcal{S}_V(U)$ by Proposition 6.2. This is also a contradiction. Conversely, suppose that the solution space $\mathcal{S}_Y(X)$ is separable. Then there exist some $\mathcal{S}_V(U) \subset \mathcal{S}_Y(X)$ and $\mathcal{S}_K(L) \subset \mathcal{S}_Y(X)$ such that

$$\mathcal{S}_V(U) \cup \mathcal{S}_K(L) = \mathcal{S}_Y(X)$$

with

$$\mathcal{S}_V(U) \cap \mathcal{S}_K(L) = \emptyset$$

and $R \not\perp W$ for any $R \in \mathcal{S}_V(U)$ and $W \in \mathcal{S}_K(L)$. Clearly R solves some $G \in \mathcal{P}_V(U)$ and W solves some $T \in \mathcal{P}_K(L)$. We claim $T \not\equiv G$ with

$$\mathcal{P}_V(U) \cup \mathcal{P}_K(L) = \mathcal{P}_Y(X)$$

with

$$\mathcal{P}_V(U) \cap \mathcal{P}_K(L) = \emptyset.$$

Suppose that $T \equiv G$ for some $T \in \mathcal{P}_K(L)$ and $G \in \mathcal{P}_V(U)$, then $R \perp W$, leading to a contradiction. Since

$$\mathcal{S}_V(U) \cup \mathcal{S}_K(L) = \mathcal{S}_Y(X)$$

with

$$\mathcal{S}_V(U) \cap \mathcal{S}_K(L) = \emptyset$$

it follows that

$$\mathcal{P}_V(U) \cup \mathcal{P}_K(L) = \mathcal{P}_Y(X).$$

Suppose to the contrary that

$$\mathcal{P}_V(U) \cup \mathcal{P}_K(L) \subset \mathcal{P}_Y(X)$$

then there exist a problem $A \in \mathcal{P}_Y(X)$ that has no solution in $\mathcal{S}_V(U) \cup \mathcal{S}_K(L)$ but has solution in $\mathcal{S}_Y(X)$. This assertion contradicts the equality

$$\mathcal{S}_V(U) \cup \mathcal{S}_K(L) = \mathcal{S}_Y(X).$$

We note that $\mathcal{S}_V(U) \cap \mathcal{S}_K(L) = \emptyset$ implies $\mathcal{P}_V(U) \cap \mathcal{P}_K(L) = \emptyset$. Suppose that $\mathcal{P}_V(U) \cap \mathcal{P}_K(L) =/\emptyset$. There exists a problem $J \in \mathcal{P}_V(U) \cap \mathcal{P}_K(L)$ so that there exists some $N \in \mathcal{S}_V(U) \cap \mathcal{S}_K(L)$ that solves J. This completes the proof. □

8. Quotient problem and solution spaces

In this section, we introduce and study the notion of the *quotient* problem and their corresponding solution spaces. We launch the following terminologies.

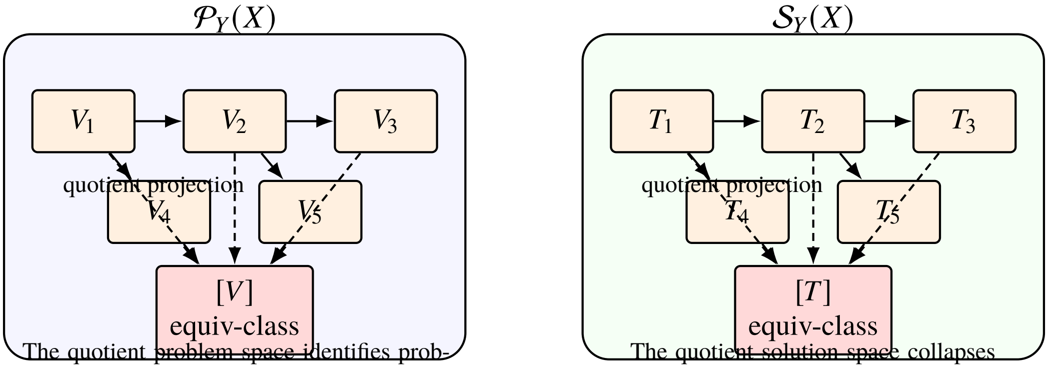
Definition 8.1. Let $\mathcal{P}_Y(X), \mathcal{P}_V(U)$ be problem spaces with

$$\mathcal{P}_V(U) \subset \mathcal{P}_Y(X).$$

We say that the quotient space induced by $\mathcal{P}_V(U)$ in $\mathcal{P}_Y(X)$ regulated by a fixed $T \in \mathcal{P}_Y(X)$, denoted by $\mathcal{P}_Y(X)/_T\mathcal{P}_V(U)$, is the collection of problems

$$\mathcal{P}_Y(X)/_T\mathcal{P}_V(U) := \{T\} \cup \mathcal{P}_V(U).$$

If $\mathcal{P}_Y(X)/_T\mathcal{P}_V(U) := \{T\} \cup \mathcal{P}_V(U) = \mathcal{P}_Y(X)$ for some $T \in \mathcal{P}_Y(X)$, then we say that $\mathcal{P}_V(U)$ is a *principal* subspace of the space $\mathcal{P}_Y(X)$. On the other hand, if $\mathcal{P}_Y(X)/_T\mathcal{P}_V(U) := \{T\} \cup \mathcal{P}_V(U) = \mathcal{P}_V(U)$ for all $T \in \mathcal{P}_Y(X)$ $(T \neq Y)$, then we say that $\mathcal{P}_V(U)$ is an *ideal* sub-space of the problem space $\mathcal{P}_Y(X)$.



The quotient problem space identifies problems that are equivalent under the chosen relation $\sim$, so that each class represents a family of mutually indistinguishable problems.

The quotient solution space collapses alternative solutions into equivalence classes, so that each class represents one structural type of solution.

Figure 7. Quotient problem and quotient solution spaces in problem theory. The quotient construction identifies equivalent problems and equivalent solutions, replacing each equivalence class by a single representative class.

In the sequel, we use the notion of regularity and maximality to find a subspace that is ideal and at the same time principal.

Proposition 8.2. *Let $\mathcal{P}_Y(X), \mathcal{P}_V(U)$ be problem spaces with $\mathcal{P}_V(U) \subset \mathcal{P}_Y(X)$. If Y is a regular problem and V is the maximal sub-problem of Y, then the sub-space $\mathcal{P}_V(U)$ is ideal and principal.*

Proof. Suppose that $\mathcal{P}_V(U) \subset \mathcal{P}_Y(X)$ and assume that Y is a regular problem and V is the maximal sub-problem of Y. It follows that for the sequence of all the sub-problems $\{J_i\}_{i\geq 1}$ of Y except V, we can write

$$\cdots J_n \leq \cdots \leq V \leq Y.$$

Since every problem in the space $\mathcal{P}_V(U)$ is a sub-problem of Y, it follows that for each $T \in \mathcal{P}_Y(X)$ except Y, we must have

$$\{T\} \cup \mathcal{P}_V(U) = \mathcal{P}_V(U)$$

and the space is ideal. Similarly, if we choose $T = Y$, then we have $\{T\} \cup \mathcal{P}_V(U) = \mathcal{P}_Y(X)$ and the space is a principal space. □

9. Overlapping and non-overlapping problem and solution spaces

In this section, we study the notion of *overlapping* and *non-overlapping* problem and solution spaces.

Definition 9.1. Let $\mathcal{P}_Y(X), \mathcal{P}_V(U)$ be problem spaces. We say that they are *overlapping* if and only if

$$\mathcal{P}_Y(X) \cap \mathcal{P}_V(U) \neq \emptyset.$$

Otherwise, we say that they are *non-overlapping*. The same characterization also holds for their corresponding solution spaces.

Proposition 9.2. *Let $\mathcal{P}_Y(X), \mathcal{P}_V(U)$ be problem spaces, with their corresponding solution spaces $\mathcal{S}_Y(X), \mathcal{S}_V(U)$ such that $F \not\equiv G$ for any $F \in \mathcal{P}_Y(X)$ and $G \in \mathcal{P}_V(U)$. Then the problem spaces are non-overlapping if and only if their corresponding solution spaces are non-overlapping.*

Proof. First, suppose that $\mathcal{P}_Y(X) \cap \mathcal{P}_V(U) \neq \emptyset$. Then there exists some $T \in \mathcal{P}_Y(X) \cap \mathcal{P}_V(U)$. Since Y is a problem with solution X and V is a problem with solution U, it follows that T must also be a solved problem. That is, there exist some $K \in \mathcal{S}_Y(X)$ that solves T. Again, $T \in \mathcal{P}_V(U)$ so that there exist some $G \in \mathcal{S}_V(U)$ that solves T. It follows that G and K must be the same solution or $G \perp K$; that is, G and K are alternative solutions to T. Since solutions spaces remain invariant under replacement with alternative solutions, it follows in particular that we can replace $G \in \mathcal{S}_V(U)$ with K and the space $\mathcal{S}_V(U)$ still remains unchanged. Conversely, suppose that $\mathcal{S}_Y(X) \cap \mathcal{S}_V(U) \neq \emptyset$. It follows that for each $F \in \mathcal{S}_Y(X) \cap \mathcal{S}_V(U)$ must be a solution to some problem $T \in \mathcal{P}_Y(X) \cap \mathcal{P}_V(U)$. □

10. Symmetric problem spaces

In this section, we study the notion of symmetry existing among problem spaces. We launch the following languages.

Definition 10.1. Let $\mathcal{P}_Y(X), \mathcal{P}_V(U)$ be problem spaces. We say that the problem spaces are *symmetric* if for each problem $T \in \mathcal{P}_Y(X)$ there exist a problem $L \in \mathcal{P}_V(U)$ such that $K \equiv L$. That is, problem K and problem L are equivalent. We denote the equivalence between the space $\mathcal{P}_Y(X)$ and $\mathcal{P}_V(U)$ by

$$\mathcal{P}_Y(X) \asymp \mathcal{P}_V(U).$$

Proposition 10.2. *Let $\mathcal{P}_Y(X)$ be a problem space with a corresponding solution space $\mathcal{S}_Y(X)$. If $\mathcal{P}_Y(X) \asymp \mathcal{P}_V(U)$, then*

$$\mathcal{S}_Y(X) = \mathcal{S}_V(U).$$

We use the notion of symmetry to justify the assertion that the problems spaces endowed with equivalent problems have indistinguishable solution spaces. In fact, it has consequences that allows us to artificially build solution spaces that can be tweaked without changing the structure.

Proof. Suppose that $\mathcal{P}_Y(X) \asymp \mathcal{P}_V(U)$. Then for each problem $T \in \mathcal{P}_Y(X)$ there exists a problem $K \in \mathcal{P}_V(U)$ such that $K \equiv T$. Since $\mathcal{S}_Y(X)$ is the corresponding solution space for $\mathcal{P}_Y(X)$, there exists some $F \in \mathcal{S}_Y(X)$ that solves T. Since problem T and problem K are equivalent problems, it follows that F also solves $K \in \mathcal{P}_V(U)$. The claim follows by iterating the argument in this manner to build the solution space $\mathcal{S}_V(U)$. □

Proposition 10.3. *Let $\mathcal{S}_Y(X)$ and $\mathcal{S}_V(U)$ be solution spaces. If for each $K \in \mathcal{S}_Y(X)$ there exist some $L \in \mathcal{S}_V(U)$, then*

$$\mathcal{P}_Y(X) \asymp \mathcal{P}_V(U).$$

Proof. Let K and L be arbitrary with $K \in \mathcal{S}_Y(X)$ and $L \in \mathcal{S}_V(U)$. Then there exists a problem $T \in \mathcal{P}_Y(X)$ solved by K and a problem $F \in \mathcal{P}_V(U)$ solved by L. Since $\equiv$ is an equivalence relation and $K \perp L$, it follows that $T \equiv F$, since L also solves T and K also solves F. The claim is deduced by repeating the argument with solutions in the space. □

Remark 10.4. The theory, as developed, is the first phase of the theory to study problems and their generative solutions. The notion of the time complexity of problems and their sub-problems is a notion to be explored in the subsequent section, motivated in part by the P versus NP problem. We suspect the following assertions to be true

Conjecture 10.5. *Let V be a problem. If V has a minimal and a maximal sub-problem, then V must be a regular problem.*

Conjecture 10.6. *Let V be a problem with solution U and Y a problem with solution X. If V be regular and the spaces $\mathcal{P}_V(U)$ and $\mathcal{P}_Y(X)$ are highly connected, then Y must also be regular.*

11. The time complexity

In this section, we study the notion of time complexity of problem and solution spaces.

Definition 11.1. The *resolution* complexity of problem T by providing solution U that solves T is the *algorithmic* time required to generate solution U for problem T. We denote this complexity by $C_r(T, U)$.

Definition 11.2. The *verification* complexity of a solution U to problem T is the *algorithmic* time required to check solution U for correctness. We denote this complexity by $C_v(T, U)$.

Definition 11.3. Let T be a problem with a solution U. We say that the time complexity with respect to problem T with solution U is in *equilibrium* if $C_r(T, U) = C_v(T, U)$.

It is important to emphasize that the time complexity is not unique to problems and solutions. More precisely, it is indeed possible that the resolution time complexity and the verification time complexity may differ quite significantly among equivalent problems and alternative solutions. Consequently, it may not be possible to extend an equilibrium to equivalent problems and alternative solutions. Let us suppose that $C_r(T_1, U_1) < \infty$ and $C_v(T_1, U_1) < \infty$ with $T_1 \equiv T_2$ (equivalent problems) then $U_1 \perp U_2$ (alternative solution). It is possible that

$$C_r(T_1, U_1) \neq C_r(T_2, U_1)$$

and

$$C_v(T_1, U_1) \neq C_v(T_2, U_1)$$

and similarly

$$C_r(T_2, U_2) \neq C_r(T_2, U_1)$$

and

$$C_v(T_2, U_2) \neq C_v(T_2, U_1).$$

Hence, if $C_r(T_1, U_1) = C_v(T_1, U_1)$ and $T_1 \equiv T_2$, then the equilibrium

$$C_r(T_2, U_2) = C_v(T_2, U_2)$$

may only hold under certain condition. We begin to verify that time complexity can be ordered up to sub-problems and sub-solutions of a given problem.

Proposition 11.4. *Let T be a problem with solution U. Let $\{T_i\}_{i\geq 1}$ and $\{U_i\}_{i\geq 1}$ denote the sequence of all sub-problems and sub-solutions of T and U, respectively. If $C_r(T,U) < \infty$ and $C_v(T,U) < \infty$, then we have*

$$C_r(T_i, U_i) < C_r(T, U)$$

and

$$C_v(T_i, U_i) < C_v(T, U)$$

for each $i \geq 1$.

Proof. Since $C_r(T,U) < \infty$ and $C_v(T,U) < \infty$ and

$$C_r(T,U) := \sum_{i\geq 1} C_r(T_i, U_i)$$

and

$$C_v(T,U) := \sum_{i\geq 1} C_v(T_i, U_i)$$

the inequality follows easily. □

Remark 11.5. In cases where we do not want to make a reference to the solution and a problem in the notation of the resolution and the verification time complexity, we will write for simplicity $C_r(T)$ and $C_v(U)$. We will adopt this notation in situations where a reference to a problem or a solution turns out to be irrelevant.

Proving the existence of equilibrium of time complexity of problems is by no means an easy endeavour. In the sequel, we prove that assuming equilibrium in the time complexity can be transferred to sub-problems and sub-solutions. We make these ideas formal in the proposition below.

Proposition 11.6. *Let T be a regular problem with solution U such that for any sub-problems T_i, T_j with $i \neq j$, then $C_r(T_i, U_i) \neq C_v(T_j, U_j)$. If $C_r(T,U) = C_v(T,U)$, then there exists $Q \leq T$ (Q a sub-problem of T) and $L \leq U$ (L a sub-solution of U) that solves Q such that $C_r(Q,L) = C_v(Q,L)$.*

Proof. Suppose that T is a regular problem with solution U. Let $\{T_i\}_{i\geq 1}$ be the sequence of all sub-problems of T with corresponding sequence of solutions $\{U_i\}_{i\geq 1}$. Suppose on the contrary that $C_r(T_i, U_i) = C_v(T_i, U_i)$ for each $i \geq 1$. By virtue of the regularity of T, we can arrange the sequence of sub-problems and sub-solutions in the following way $T_1 \geq T_2 \geq \cdots$ and the corresponding sequence of sub-solutions $U_1 \geq U_2 \geq \cdots$, where each preceding T_i is a sub-problem of T_{i-1} and similarly each U_i is a sub-solution for U_{i-1}. Since problem T is said to be solved by providing a solution to each of the sub-problems, we find under the assumption $C_r(T,U) = C_v(T,U)$, that

$$C_r(T,U) = \sum_{i\geq 1} C_r(T_i, U_i) = \sum_{i\geq 1} C_v(T_i, U_i) = C_v(T,U).$$

Now suppose on the contrary that $C_r(T_1, U_1) \neq C_v(T_1, U_1)$, then under the regularity condition, it follows that

$$\sum_{i\geq 2} C_r(T_i, U_i) \neq \sum_{i\geq 2} C_v(T_i, U_i)$$

since providing a solution to all sub-problems of T_2 solves problem T_2. Under the requirement that $C_r(T_i, U_i) \neq C_v(T_j, U_j)$ for all $i \neq j$, it follows that

$$C_r(T, U) = \sum_{i \geq 1} C_r(T_i, U_i) \neq \sum_{i \geq 1} C_v(T_i, U_i) = C_v(T, U)$$

violating the assumption that $C_r(T, U) = C_v(T, U)$. □

Theorem 11.7. *Let T be a regular problem with a solution K. If M is the maximal sub-problem of T with a solution L and $C_r(M, L) \ll polynomial\ time$ and $C_r(T, K) = C_v(T, K)$, then $C_v(T, K) \ll polynomial\ time$.*

Proof. Suppose T is a regular problem and let $\{T_i\}_{i \geq 1}$ denote the sequence of all sub-problems of T with corresponding sequence of sub-solutions $\{K_i\}_{i \geq 1}$ where each K_i solves T_i. We can arrange the sequence of sub-problems in the following way: $T_1 \geq T_2 \geq \cdots$ where $T_1 := M$ is the maximal sub-problem of T and where each sub-problem T_i is a sub-problem of T_{i-1} for $i \geq 2$. Since problem T is solved by solving each of the sub-problems in the sequence, we can write

$$\begin{aligned} C_r(T, K) &= \sum_{i \geq 1} C_r(T_i, K_i) \\ &= C_r(T_1, K_1) + \sum_{i \geq 2} C_r(T_i, K_i). \end{aligned}$$

By the regularity of problem T, we deduce

$$\sum_{i \geq 2} C_r(T_i, K_i) = C_r(T_1, K_1) \ll polynomial\ time.$$

Thus $C_r(T, K) \ll polynomial\ time$. Under the equality $C_r(T, K) = C_v(T, K)$, we deduce that $C_v(T, K) \ll polynomial\ time$, which completes the proof of the theorem. □

Remark 11.8. Theorem 11.7 is an important ingredient for exploring a deep understanding of the P=NP problem. It purports that once there exist an equilibrium of time complexity of a given problem, it suffices to only investigate the resolution complexity of the maximal sub-problem for a class of well-behaved problems which we refer to as regular problems.

Although the task of proving equilibrium of resolution and verification time complexity can be very hard, we can often achieve this task from bottom-up. That is to say, proving equilibrium of time complexity for sub-problems can be extended to time complexity equilibrium of the actual problem. The following proposition exemplifies this principle.

Proposition 11.9. *Let Y be a problem with solution X and let $\{Y_i\}_{i \geq 1}$ and $\{X_i\}_{i \geq 1}$ denotes the sequence of all proper sub-problems and a solutions to sub-problems of Y. If $C_r(Y_i, X_i) = C_v(Y_i, X_i)$ for each $i \geq 1$, then $C_r(Y, X) = C_v(Y, X)$.*

Proof. The sequences $\{Y_i\}_{i \geq 1}$ and $\{X_i\}_{i \geq 1}$ denotes the sequence of all proper sub-problems and a solutions to sub-problems of Y, respectively. Since the solution to problem Y is furnished solving each of the sub-problems in $\{Y_i\}_{1 \geq 1}$, it follows under the assumption $C_r(Y_i, X_i) = C_v(Y_i, X_i)$ for each $i \geq 1$ that

$$C_r(Y, X) = \sum_{i \geq 1} C_r(Y_i, X_i) = \sum_{i \geq 1} C_v(Y_i, X_i) = C_v(Y, X).$$

□

We now obtain an important characterization of irreducible problems.

Theorem 11.10. *If X is an irreducible problem, then $C_r(X) = \infty$ or X is not solvable.*

Proof. Suppose X is an irreducible problem and assume the contrary that $C_r(X) < \infty$ and that X is solvable. Since X is irreducible, each sub-problem $X_j \leq X$ has a proper sub-problem, and problem X has infinitely many proper sub-problems $X_i < X$. Thus

$$C_r(X) := \sum_{i=1}^{\infty} C_r(X_i) < \infty$$

since the problem X is solved by providing a solution to each of the sub-problems. This implies that for any $\epsilon > 0$, there exists some $N := N(\epsilon)$ such that for all $i \geq N$, we have

$$\sum_{i=N}^{\infty} C_r(X_i) < \epsilon.$$

That is, $C_r(X_i) \longrightarrow 0$ as $i \longrightarrow \infty$. This means the algorithmic time required to solve infinitely many proper sub-problems of problem X converges to zero, which violates the assumption that X is solvable. □

The difficulty of proving equilibrium of time complexity of a given problem may be made easier depending on its structure. Irregular problems seem to be very difficult to understand and unfortunately most problems fall into this category. It is, however, much easier to establish an equilibrium for a class of well behaved problems that fall into the category of reducible and regular problems. It turns out that once equilibrium is reached for the finest form of this problem, the equilibrium will certainly be attained for the actual problem. We make this discussion formal in the following results.

Theorem 11.11 (extension principle)**.** *Let T be a regular and a reducible problem with solution U. If T_k is a sub-problem of T with solution U_k such that there exist no $T_j \in \{T_i\}_{i\geq 1}$ with $T_j \not< T_k$ and that $C_r(T_k, U_k) = C_v(T_k, U_k)$, then $C_r(T, U) = C_v(T, U)$.*

Proof. Suppose that T is a regular problem with solution U and let $\{T_i\}_{i\geq 1}$ be the sequence of all sub-problems of T with the corresponding sequence of solutions $\{U_i\}_{i\geq 1}$, where each U_i solves T_i for each $i \geq 1$. Since T is reducible, it has a sub-problem with no proper sub-problem. Let T_k be this sub-problem of T, then by the regularity of T, we can arrange the sequence of all sub-problems of T in the following way:

$$T_k \leq T_{k-1} \leq T_{k-2} \leq \cdots \leq T_1$$

with

$$U_k \leq U_{k-1} \leq U_{k-2} \leq \cdots U_1$$

where each T_i is a sub-problem of T_{i-1} and U_i is a sub-solution of U_{i-1}. Under the equilibrium $C_r(T_k, U_k) = C_v(T_k, U_k)$ and since problem T_{k-1} is solved by providing a solution to all of its proper sub-problems, it follows that $C_r(T_{k-1}, U_{k-1}) = C_v(T_{k-1}, U_{k-1})$. Similarly, problem T_{k-2} is solved by providing a solution to all of its sub-problems and it follows that

$$\begin{aligned} C_r(T_{k-2}, U_{k-2}) &= C_r(T_k, U_k) + C_r(T_{k-1}, U_{k-1}) \\ &= C_v(T_k, U_k) + C_v(T_{k-1}, U_{k-1}) \\ &= C_v(T_{k-2}, U_{k-2}). \end{aligned}$$

We can iterate this process to reach the equilibrium $C_r(T, U) = C_v(T, U)$. □

Corollary 11.12. *Let T be a regular and a reducible problem with solution U. Let T_k be a sub-problem of T with solution U_k such that there exist no $T_j \in \{T_i\}_{i\geq 1}$ with $T_j \not< T_k$ and that $C_r(T_k, U_k) = C_v(T_k, U_k)$. If $C_v(T, U) \ll polynomial\ time$ then $C_r(T, U) \ll polynomial\ time$.*

Proof. It follows from Theorem 11.11 that $C_r(T, U) = C_v(T, U)$ so that under the hypothesis $C_v(T, U) \ll polynomial\ time$ then $C_r(T, U) \ll polynomial\ time$. □

Remark 11.13. Corollary 11.12 suggests that under a certain mild condition, if a certain class of well-behaved problems have a solution that are easy to verify for correctness then they must also be easy to solve at the same level.

11.1. **The time complexity of problem and solution spaces.** In this section, we study the notion of time complexity on problem and solutions spaces, as opposed to a specific problem and its solution.

Definition 11.14. Let $\mathcal{P}_Y(X)$ and $\mathcal{S}_Y(X)$ be the problem and solution spaces induced by providing solution X to problem Y. By the *resolution complexity* of the problem space $\mathcal{P}_Y(X)$, we mean the sum of each resolution complexity of each problem in the space. For each problem $T \in \mathcal{P}_Y(X)$, there exists a solution $L \in \mathcal{S}_Y(X)$ that solves T. We denote the resolution complexity of the space by

$$\mathcal{P}^r_Y(X) := \sum_{\substack{T\in\mathcal{P}_Y(X)\\ L\in\mathcal{S}_Y(X)}} C_r(T, L)$$

and the verification complexity by

$$\mathcal{S}^v_Y(X) := \sum_{\substack{L\in\mathcal{S}_Y(X)\\ T\in\mathcal{P}_Y(X)}} C_v(T, L).$$

Proposition 11.15. *Let $\mathcal{P}_Y(X)$ and $\mathcal{S}_Y(X)$ be the problem and solution spaces induced by providing a solution X to the problem Y. If for each $T \in \mathcal{P}_Y(X)$ and each $L \in \mathcal{S}_Y(X)$ that solves T, $C_r(T, L) = C_v(T, L)$, then $\mathcal{P}^r_Y(X) = \mathcal{S}^v_Y(X)$.*

Proof. This follows trivially from the proof of Proposition 11.9. □

12. Analysis on the topology of problem spaces

In this section, we introduce and develop the analysis of the theory of problem and their solution spaces. We adapt some classical concepts in functional analysis to study problems and their corresponding solution spaces. We introduce the notion of *compactness*, *density*, *convexity*, *boundedness*, *amenability* and the *interior*. We examine the overall interplay among these concepts in theory.

12.1. **Compact problems and solutions.** In this section, we study the notion of compactness of problems and their corresponding solutions.

Definition 12.1. Let $\mathcal{P}_X(Y)$ and $\mathcal{S}_X(Y)$ denote the problem and solutions spaces, respectively, induced by providing solution X to problem Y. We say that the problem space $\mathcal{P}_X(Y)$ is *compact* if and only if there exists a finite number of problem spaces $\mathcal{P}_{U_1}(V_1), \mathcal{P}_{U_2}(V_2), \ldots, \mathcal{P}_{U_k}(V_k)$ such that

$$\mathcal{P}_X(Y) \subset \mathcal{P}_{U_1}(V_1) \cup \mathcal{P}_{U_2}(V_2) \cup \cdots \cup \mathcal{P}_{U_k}(V_k).$$

Similarly, we say that the solution space $\mathcal{S}_X(Y)$ is *compact* if and only if there exists a finite number of solution spaces $\mathcal{S}_{U_1}(V_1), \mathcal{S}_{U_2}(V_2), \ldots, \mathcal{S}_{U_k}(V_k)$ such that

$$\mathcal{S}_X(Y) \subset \mathcal{S}_{U_1}(V_1) \cup \mathcal{S}_{U_2}(V_2) \cup \cdots \cup \mathcal{S}_{U_k}(V_k).$$

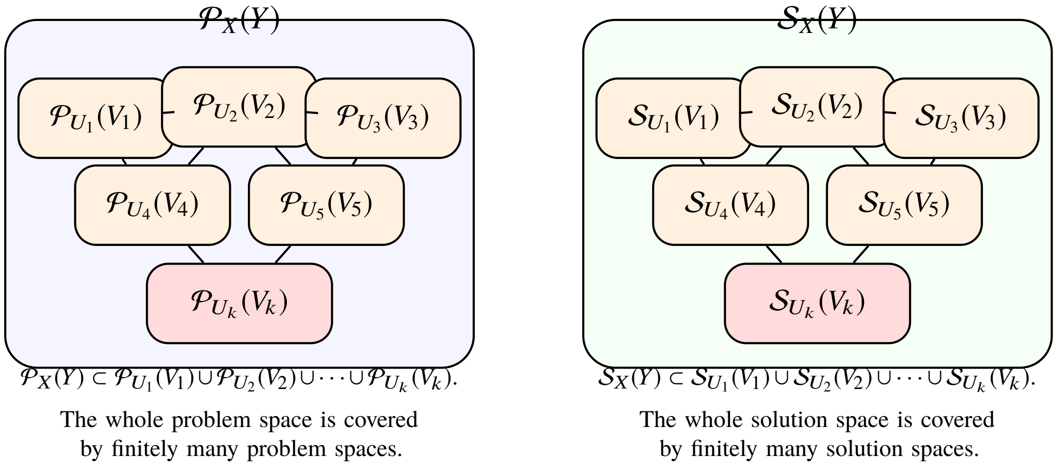


FIGURE 8. Compact problem and solution spaces in problem theory. Compactness means that the given problem space or solution space is contained in a finite union of problem spaces or solution spaces, respectively.

Proposition 12.2. *Let $\mathcal{P}_X(Y)$ be the problem space induced by providing solution a Y to the problem X. If $\mathcal{P}_X(Y)$ is compact, then the problem space $\mathcal{P}_{X_i}(Y_i)$ with $\mathcal{P}_{X_i}(Y_i)$ is also compact.*

Proof. Suppose that $\mathcal{P}_X(Y)$ is compact, then it follows that for a finite $k \in \mathbb{N}$ there exist problem spaces $\mathcal{P}_{U_1}(V_1), \mathcal{P}_{U_2}(V_2), \ldots, \mathcal{P}_{U_k}(V_k)$ such that

$$\mathcal{P}_X(Y) \subset \mathcal{P}_{U_1}(V_1) \cup \mathcal{P}_{U_2}(V_2) \cup \cdots \cup \mathcal{P}_{U_k}(V_k).$$

The compactness of $\mathcal{P}_{X_i}(Y_i)$ follows trivially since $\mathcal{P}_{X_i}(Y_i) \subset \mathcal{P}_X(Y)$. □

Proposition 12.3. *Let $\mathcal{P}_X(Y)$ be the problem space induced by providing solution Y to problem X and let $\mathcal{P}_{X_i}(Y_i) \subset \mathcal{P}_X(Y)$. If $\mathcal{P}_{X_i}(Y_i)$ is compact and principal, then $\mathcal{P}_X(Y)$ is compact.*

Proof. Let $\mathcal{P}_{X_i}(Y_i) \subset \mathcal{P}_X(Y)$ and suppose that $\mathcal{P}_{X_i}(Y_i)$, then there exists a sub-problem $X_j \leq X$ such that we can write $\mathcal{P}_X(Y) = \mathcal{P}_{X_i}(Y_i) \cup \{X_j\}$. Under the requirement that $\mathcal{P}_{X_i}(Y_i)$ is compact, it follows that for a finite $k \in \mathbb{N}$ there exist problems spaces $\mathcal{P}_{U_1}(V_1), \mathcal{P}_{U_2}(V_2), \ldots, \mathcal{P}_{U_k}(V_k)$ such that

$$\mathcal{P}_{X_i}(Y_i) \subset \mathcal{P}_{U_1}(V_1) \cup \mathcal{P}_{U_2}(V_2) \cup \cdots \cup \mathcal{P}_{U_k}(V_k)$$

and we have

$$\mathcal{P}_X(Y) \subset \{X_j\} \cup \mathcal{P}_{U_1}(V_1) \cup \mathcal{P}_{U_2}(V_2) \cup \cdots \cup \mathcal{P}_{U_k}(V_k).$$

This proves that the space $\mathcal{P}_X(Y)$ is also compact. □

Proposition 12.4. *Let $\mathcal{P}_X(Y)$ be the problem space induced by providing solution Y to problem X, where X is a regular problem. If $X_i < X$ is the maximal proper sub-problem of X and $\mathcal{P}_{X_i}(Y_i)$ is compact, then $\mathcal{P}_X(Y)$ is also compact.*

Proof. Suppose that X is regular problem and let X_i be the maximal proper sub-problem of X, then we can write $X > X_j > X_{j+1} > \cdots$ where $X_{j+n} > X_{j+n+1}$ indicates that X_{j+n+1} is the maximal proper sub-problem of X_{n+j} for $n = 1, 2, \ldots$, by virtue of the regularity of the problem X. The above sequence contains all the sub-problems of X so that we can put $\bigcup\limits_{n\geq 1} \mathcal{P}_{X_{j+n}}(Y_{j+n}) \subseteq \mathcal{P}_{X_j}(Y_j)$. Since a problem is solved by providing a solution to each sub-problem and X_j is the maximal problem sub-problem of X, we deduce that $\bigcup\limits_{n\geq 1} \mathcal{P}_{X_{j+n}}(Y_{j+n}) \cup \{X\} \subseteq \mathcal{P}_{X_j}(Y_j) \cup \{X\} = \mathcal{P}_X(Y)$ and it follows that

$$\mathcal{P}_X(Y) \subset \{X\} \cup \mathcal{P}_{U_1}(V_1) \cup \mathcal{P}_{U_2}(V_2) \cup \cdots \cup \mathcal{P}_{U_k}(V_k)$$

since $\mathcal{P}_{X_j}(Y_j)$ was assumed to be compact. This shows that the space $\mathcal{P}_X(Y)$ is compact. □

12.2. **Dense problems and solution spaces.** In this section, we study the concept of *density* of problems and their corresponding solution spaces.

Definition 12.5. Let $\mathcal{P}_X(Y)$ and $\mathcal{S}_X(Y)$ be the problem and solution spaces, respectively, induced by providing a solution Y to the problem X. Let $X_i \in \mathcal{P}_X(Y)$ with an induced sub-space $\mathcal{P}_{X_i}(Y_i) \subset \mathcal{P}_X(Y)$ and the corresponding solution space $\mathcal{S}_{X_i}(Y_i)$. We say that the subspace $\mathcal{P}_{X_i}(Y_i)$ is *dense* in the space $\mathcal{P}_X(Y)$ if and only if for any problem $Z \in \mathcal{P}_X(Y)$ with $Z \neq X$, there exists a proper subspace $\mathcal{P}_{X_j}(Y_j)$ with $Z \in \mathcal{P}_{X_j}(Y_j)$ such that $\mathcal{P}_{X_i}(Y_i) \cap \mathcal{P}_{X_j}(Y_j) \neq \emptyset$. Similarly, we say that the subspace $\mathcal{S}_{X_i}(Y_i)$ is *dense* in the space $\mathcal{S}_X(Y)$ if and only if for any solution $W \in \mathcal{S}_X(Y)$ with $W \neq Y$, there exists a proper subspace $\mathcal{S}_{X_j}(Y_j)$ with $W \in \mathcal{S}_{X_j}(Y_j)$ such that $\mathcal{S}_{X_i}(Y_i) \cap \mathcal{S}_{X_j}(Y_j) \neq \emptyset$.

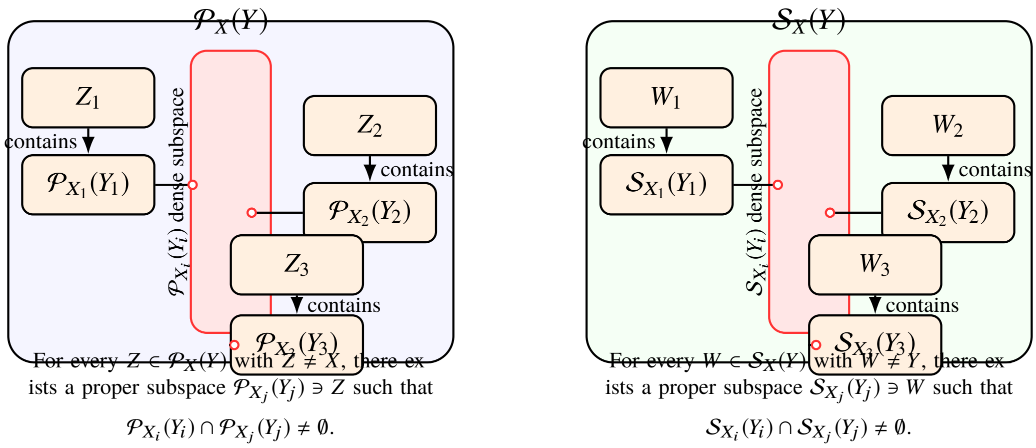


FIGURE 9. Density of a subproblem space in a problem space, and of a subsolution space in a solution space, in problem theory. The dense subspace is shown as a distinguished region that intersects the proper subspace associated with each ambient problem or ambient solution.

Theorem 12.6 (Characterization theorem)**.** *Let $\mathcal{P}_X(Y)$ be the problem space induced by providing a solution Y to the problem X. Then $\mathcal{P}_X(Y)$ is separable if and only if it contains no dense subspace.*

Proof. Suppose that the problem space $\mathcal{P}_X(Y)$ is separable, then there exists subspaces $\mathcal{P}_{X_i}(Y_i)$ and $\mathcal{P}_{X_j}(Y_j)$ such that $\mathcal{P}_X(Y) = \mathcal{P}_{X_i}(Y_i) \cup \mathcal{P}_{X_j}(Y_j)$ with $\mathcal{P}_{X_i}(Y_i) \cap \mathcal{P}_{X_j}(Y_j) = \emptyset$. Now, let $\mathcal{P}_{X_k}(Y_k) \subset \mathcal{P}_X(Y)$, then we must have one of these possibilities: $\mathcal{P}_{X_k}(Y_k) \subset \mathcal{P}_{X_i}(Y_i)$ or

$\mathcal{P}_{X_k}(Y_k) \subset \mathcal{P}_{X_i}(Y_i)$. Suppose that there exist problems $Z, U \in \mathcal{P}_{X_k}(Y_k)$ such that $Z \in \mathcal{P}_{X_i}(Y_i)$ and $U \in \mathcal{P}_{X_j}(Y_j)$, then we have for their corresponding problem spaces induced with, say, the solutions W and T the following properties $\mathcal{P}_Z(W) \subset \mathcal{P}_{X_i}(Y_i)$ and $\mathcal{P}_U(T) \subset \mathcal{P}_{X_j}(Y_j)$. We know that $\mathcal{P}_U(T) \subseteq \mathcal{P}_{X_k}(Y_k)$ and $\mathcal{P}_Z(W) \subseteq \mathcal{P}_{X_k}(Y_k)$ so that we must have $\mathcal{P}_{X_k}(Y_k) \subseteq \mathcal{P}_{X_i}(Y_i)$ and $\mathcal{P}_{X_k}(Y_k) \subseteq \mathcal{P}_{X_j}(Y_j)$. Suppose without loss of generality that $\mathcal{P}_{X_i}(Y_i) \subset \mathcal{P}_{X_k}(Y_k)$ then we will have

$$\mathcal{P}_{X_i}(Y_i) \cup \mathcal{P}_{X_j}(Y_j) = \mathcal{P}_X(Y) \subset \mathcal{P}_{X_k}(Y_k) \cup \mathcal{P}_{X_j}(Y_j) \subset \mathcal{P}_X(Y)$$

which is absurd. This implies that $\mathcal{P}_{X_i}(Y_i) \cap \mathcal{P}_{X_j}(Y_j) \neq \emptyset$, which violates the requirement that $\mathcal{P}_X(Y)$ is separable. Without loss of generality, we put $\mathcal{P}_{X_k}(Y_k) \subseteq \mathcal{P}_{X_i}(Y_i)$ and choose a problem $V \in \mathcal{P}_{X_j}(Y_j)$ then $\mathcal{P}_V(T) \subseteq \mathcal{P}_{X_j}(Y_j)$. It follows that $\mathcal{P}_{X_k}(Y_k) \cap \mathcal{P}_V(T) = \emptyset$ and since $V \notin \mathcal{P}_{X_l}(Y_l) \subseteq \mathcal{P}_{X_i}(Y_i)$ for subspace $\mathcal{P}_{X_l}(Y_l)$ of $\mathcal{P}_{X_i}(Y_i)$, the problem space $\mathcal{P}_{X_k}(Y_k)$ cannot be dense in $\mathcal{P}(X)(Y)$. Since $\mathcal{P}_{X_k}(Y_k)$ was an arbitrary problem subspace, it follows that the space $\mathcal{P}_X(Y)$ contains no dense sub-problem space. Conversely, suppose that the space $\mathcal{P}_X(Y)$ contains a dense problem sub-space but that the space is separable, then there exists proper sub-spaces $\mathcal{P}_{X_i}(Y_i)$ and $\mathcal{P}_{X_j}(Y_j)$ such that $\mathcal{P}_{X_i}(Y_i) \cup \mathcal{P}_{X_j}(X_j) = \mathcal{P}_X(Y)$ such that $\mathcal{P}_{X_i}(Y_i) \cap \mathcal{P}_{X_j}(Y_j) = \emptyset$. Let $\mathcal{P}_{X_k}(Y_k) \subset \mathcal{P}_X(Y)$ be dense in $\mathcal{P}_X(Y)$ then for $V \in \mathcal{P}_{X_i}(Y_i)$ and $U \in \mathcal{P}_{X_j}(Y_j)$. Since these subspaces are the largest subspaces in the space $\mathcal{P}_X(Y)$ containing the problems V and U, it follows by the density of the subspace $\mathcal{P}_{X_k}(Y_k)$ that $\mathcal{P}_{X_k}(Y_k) \cap \mathcal{P}_{X_i}(Y_i) \neq \emptyset$ and $\mathcal{P}_{X_k}(Y_k) \cap \mathcal{P}_{X_j}(Y_j) \neq \emptyset$. This contradicts the assumption that the space $\mathcal{P}_X(Y)$ is separable. □

12.3. **Bounded problem and solution spaces.** In this section, we study the notion of *bounded* problem and solution spaces.

Definition 12.7. Let $\mathcal{P}_X(Y)$ be a problem space induced by providing solution a Y to the problem X. We say that the space $\mathcal{P}_X(Y)$ is *bounded* if and only if it has finite complexity. If we denote the complexity of the space by $\mathbb{C}[\mathcal{P}_X(Y)]$, then we say that $\mathcal{P}_X(Y)$ is bounded if and only if $\mathbb{C}[\mathcal{P}_X(Y)] < \infty$. Similarly, we say that the corresponding solution space $\mathcal{S}_X(Y)$ is bounded if only if it has a finite index. If we denote the index of this space by $\mathbb{I}[\mathcal{S}_X(Y)]$, then $\mathcal{S}_X(Y)$ is bounded if and only if $\mathbb{I}[\mathcal{S}_X(Y)] < \infty$.

Proposition 12.8. *Let $\mathcal{P}_X(Y)$ be the problem space induced by providing solution a Y to the problem X. If $\mathbb{C}[\mathcal{P}_X(Y)] < \infty$, then $\mathcal{P}_X(Y)$ contains a reducible problem.*

Proof. Suppose that each problem $X_i \in \mathcal{P}_X(Y)$ is irreducible, then we can construct the infinite nested sequence of sub-problem spaces $\cdots \subset \mathcal{P}_{X_2}(Y_2) \subset \mathcal{P}_{X_1}(Y_1) \subset \mathcal{P}_X(Y)$ with $X_1 > X_2 > \cdots$, where $X_{j+1} < X_j$ indicates that X_{j+1} is a proper sub-problem of X_j. This implies that the space $\mathcal{P}_X(Y)$ contains infinitely many problems and thus $\mathbb{C}[\mathcal{P}_X(Y)] = \infty$. □

12.4. **The interior of problem and solution spaces.** In this section, we study the topological notion of *interior* of problem and solution spaces.

Definition 12.9. Let $\mathcal{P}_X(Y)$ and $\mathcal{S}_X(Y)$ be the problem and the solutions spaces induced by providing solution a Y to the problem X. We say that a problem $Z \in \mathcal{P}_X(Y)$ is an *interior* problem if there is no problem space $\mathcal{P}_S(T)$ with $\mathcal{P}_S(T) \nsubseteq \mathcal{P}_X(Y)$ such that $Z \in \mathcal{P}_S(T)$. We call the collection of all such problems in $\mathcal{P}_X(Y)$ the interior of $\mathcal{P}_X(Y)$ and denote for this collection $Int[\mathcal{P}_X(Y)]$. We say that the interior is non-empty if $Int[\mathcal{P}_X(Y)] \neq \emptyset$; otherwise, we say that the interior is empty. Similarly, we say that a solution $W \in \mathcal{S}_X(Y)$ is an *interior* solution if there is no solution space $\mathcal{S}_R(T)$ with $\mathcal{S}_R(T) \nsubseteq \mathcal{S}_X(Y)$ such that $W \in \mathcal{S}_R(T)$. We call the collection of all such solutions in $\mathcal{S}_X(Y)$ the interior of $\mathcal{S}_X(Y)$ and denote for this collection $Int[\mathcal{S}_X(Y)]$. We say that the interior is non-empty if $Int[\mathcal{S}_X(Y)] \neq \emptyset$; otherwise, we say that the interior is empty.

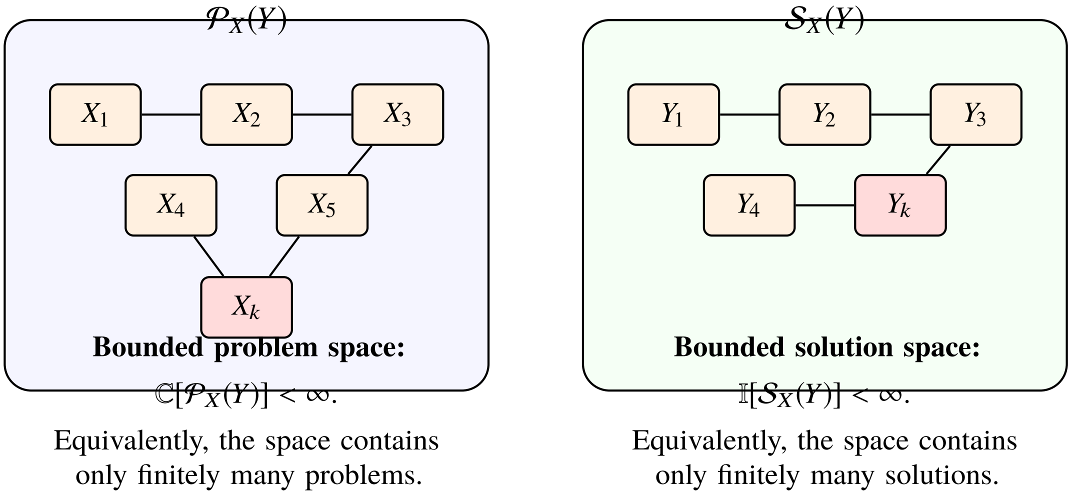


FIGURE 10. Bounded problem and solution spaces in problem theory. A problem space is bounded precisely when its complexity is finite, and a solution space is bounded precisely when its index is finite.

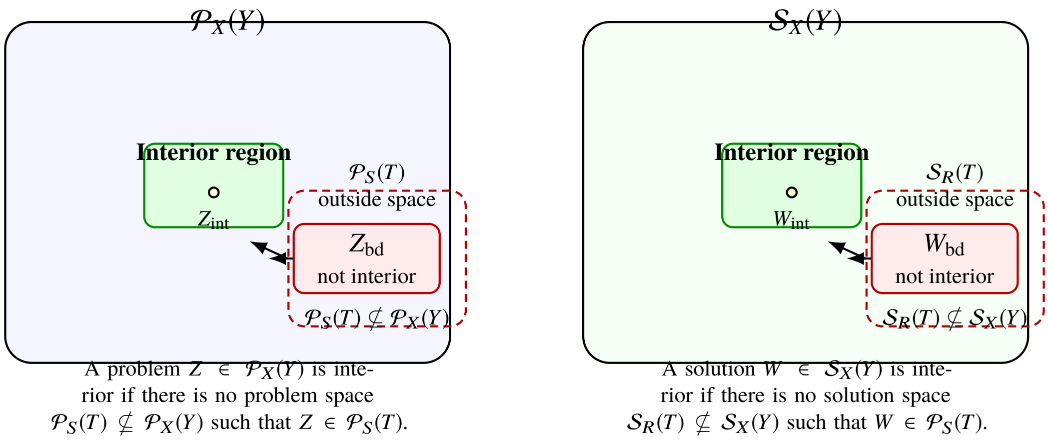


FIGURE 11. Interior of problem and solution spaces in problem theory. The interior consists of those problems or solutions that do not lie in any larger space outside the ambient problem or solution space.

Theorem 12.10. *Let $\mathcal{P}_X(Y)$ be the problem space induced by providing the solution Y to the problem X. If $Int[\mathcal{P}_X(Y)] = \emptyset$ and $\mathbb{C}[\mathcal{P}_X(Y)] < \infty$, then $\mathcal{P}_X(Y)$ is compact.*

Proof. Suppose that $\mathbb{C}[\mathcal{P}_X(Y)] < \infty$, then $\mathcal{P}_X(Y) = \{X, X_1, \ldots, X_k\}$ for a finite $k \in \mathbb{N}$. Since $Int[\mathcal{P}_X(Y)] = \emptyset$, it follows that there exist problem spaces $\mathcal{P}_{T_1}(R_1), \ldots, \mathcal{P}_{T_k}(R_k)$ with $\mathcal{P}_{T_i}(R_i) \nsubseteq \mathcal{P}_X(Y)$ for $i = 1, \ldots, k$ such that $X_i \in \mathcal{P}_{T_i}(R_i)$ for each i. It follows that we can put $\mathcal{P}_X(Y) \subset \bigcup\limits_{i=1}^{k} \mathcal{P}_{T_i}(R_i) \cup \{X\}$. This shows that the problem space $\mathcal{P}_X(Y)$ is compact. □

12.5. **Convex problem and solution spaces.** In this section, we introduce and study the notion of *convexity* of problems and solution spaces.

Definition 12.11. Let $\mathcal{P}_X(Y)$ be the problem space induced by providing the solution Y to the problem X. We say that the space $\mathcal{P}_X(Y)$ is *convex* if for any problem $X_i, X_j \in \mathcal{P}_X(Y)$ $(X_i, X_j \neq X)$, there exists a problem $X_k \in \mathcal{P}_X(Y)$ such that $\{X_i\} \cup \{X_j\} = \{X_k\}$. Similarly, we say that the solution space $\mathcal{S}_X(Y)$ is *convex* if for any solution $Y_i, Y_j \in \mathcal{S}_X(Y)$ $(Y_i, Y_j \neq Y)$, there exists a solution $Y_k \in \mathcal{S}_X(Y)$ such that $\{Y_i\} \cup \{Y_j\} = \{Y_k\}$.

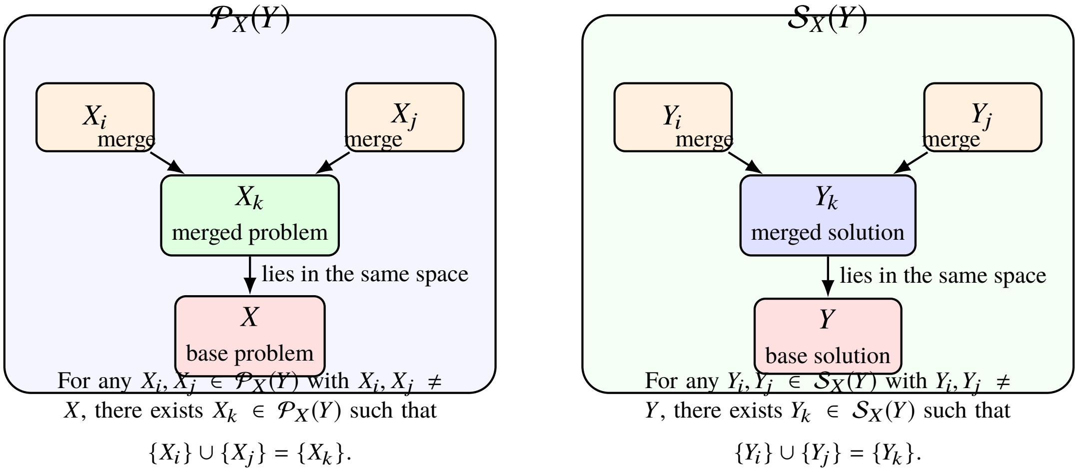


FIGURE 12. Convex problem and solution spaces in problem theory. The figure depicts the defining rule that any two non-base problems (or solutions) in the space can be merged to form another element of the same space.

The notion of *convexity* of a problem (resp. solution) spaces suggests that each problem in the *convex* problem space is a sub-problem of some problem in the space. It is worth noting that the convexity of problem and solutions do not unconditionally extend to *convexity* of sub-problem spaces.

Proposition 12.12. *Let $\mathcal{P}_X(Y)$ be the problem space induced by providing solution Y to problem X. If $\mathcal{P}_X(Y)$ is* convex *and bounded with $\mathbb{C}[\mathcal{P}_X(Y)] \geq 4$, then $\mathcal{P}_X(Y)$ has a principal subspace $\mathcal{P}_{X_k}(Y_k)$ with $\mathbb{C}[\mathcal{P}_{X_k}(Y_k)] \geq 3$.*

Proof. Suppose that $\mathcal{P}_X(Y)$ is bounded, then $\mathbb{C}[\mathcal{P}_X(Y)] < \infty$ so that $\mathcal{P}_X(Y)$ contains finitely many problems. Let $X_i, X_j \in \mathcal{P}_X(Y)$ so that under the requirement $\mathcal{P}_X(Y)$ is *convex*, then $\{X_i\} \cup \{X_j\} = \{X_k\}$, where $X_k \in \mathcal{P}_X(Y)$. That is, we can merge problems in the space to produce another problem in the space. It follows that $X_i \leq X_k$ and $X_j \leq X_k$. That is, X_i and X_j are sub-problems of X_k. By the minimality of the complexity of the space $\mathbb{C}[\mathcal{P}_X(Y)] \geq 4$, we can repeat this construction using the newly constructed problems X_k with some $X_s \in \mathcal{P}_X(Y)$ with $X_s \neq X_i, X_j$ to produce a sub-problem space which is principal and has complexity ≥ 3. □

The next result purports that each subspace of a problem space must be *dense* in their mother space.

Theorem 12.13. *Let $\mathcal{P}_X(Y)$ be the problem space induced by providing the solution a Y to the problem X. If $\mathcal{P}_X(Y)$ is* convex *then every subspace $\mathcal{P}_{X_i}(Y_i) \subset \mathcal{P}_X(Y)$ is* dense *in $\mathcal{P}_X(Y)$.*

Proof. Suppose that the problem space $\mathcal{P}_X(Y)$ is *convex* and put $\mathcal{P}_{X_i}(Y_i) \subset \mathcal{P}_X(Y)$. Next, we arbitrarily pick a problem $V \in \mathcal{P}_X(Y)$. Due to the *convexity* of the space, there exists a problem $W \in \mathcal{P}_X(Y)$ such that $\{X_i\} \cup \{V\} = \{W\}$. This implies $X_i < W$ and $V < W$; that is, X_i and V are proper sub-problems of W. Since $W \in \mathcal{P}_X(Y)$, it has a solution, so let $T \in \mathcal{S}_X(Y)$ be the solution to W. We obtain the induced problem space $\mathcal{P}_W(T) \subset \mathcal{P}_X(Y)$ with $V \in \mathcal{P}_W(T)$. Because $X_i < W$ and is the maximal sub-problem in the space $\mathcal{P}_{X_i}(Y_i)$, it follows that $\mathcal{P}_{X_i}(Y_i) \subset \mathcal{P}_W(T)$. We find $\mathcal{P}_{X_i}(Y_i) \cap \mathcal{P}_W(T) \neq \emptyset$ with $V \in \mathcal{P}_W(T)$. Since V was arbitrarily chosen in the space $\mathcal{P}_X(Y)$, it follows that $\mathcal{P}_{X_i}(Y_i)$ is *dense* in $\mathcal{P}_X(Y)$. Because the sub-problem space was arbitrarily chosen, it follows that each sub-problem space is a dense problem space $\mathcal{P}_X(Y)$. This completes the proof of the claim. □

12.6. **Amenable problem spaces.** In this section, we study the notion of *amenability* of problem spaces.

Definition 12.14. Let $\mathcal{P}_X(Y)$ be the problem space induced by providing a solution Y to the problem X. We say that the problem space $\mathcal{P}_X(Y)$ is *partially amenable* if there exist proper sub-problem $X_i, X_j \in \mathcal{P}_X(Y)$ such that X_i and X_j are equivalent problems ($X_i \equiv X_j$). We say that the space $\mathcal{P}_X(Y)$ is *totally amenable* if for any sub-problem $X_i, X_j \in \mathcal{P}_X(Y)$ then $X_i \equiv X_j$. We say that a problem is *amenable* if it is a problem in some totally *amenable* problem space.

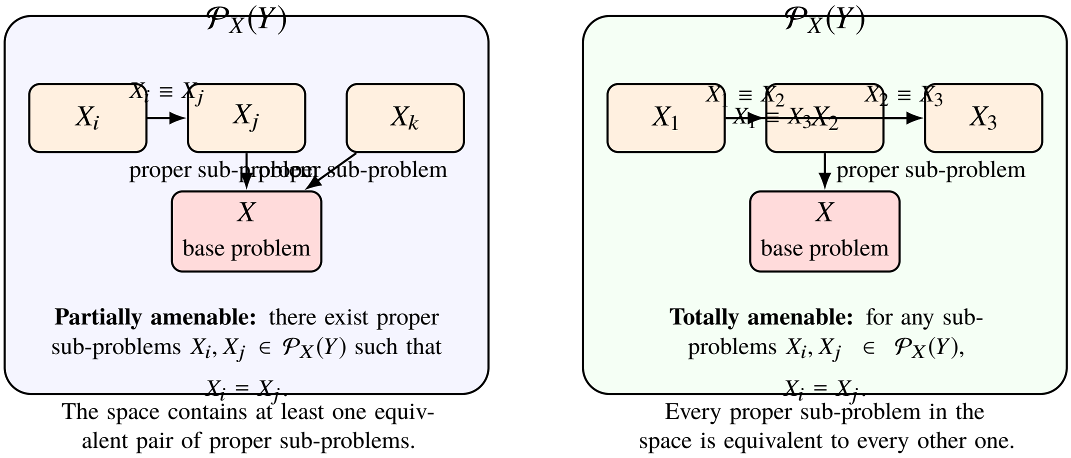


FIGURE 13. Amenable problem spaces in problem theory. The left panel shows partial amenability, where at least one pair of proper sub-problems is equivalent. The right panel shows total amenability, where every pair of proper sub-problems in the space is equivalent.

Amenable problems are naturally easily tractable. This notion holds much significance, because if we can identify some totally amenable space that contains a specific problem, then finding a solution will reduce to finding a solution to a much easier problem in the same space. Subsequent studies will be devoted to a detailed and much more specialized study of this important concept and its overall interplay with the theory. Next, we launch a result that basically purports the compactness of a space provided one can identify a compact sub-problem space.

Theorem 12.15. *Let $\mathcal{P}_X(Y)$ be a totally amenable problem space. If there exists a sub-problem space $\mathcal{P}_{X_i}(Y_i)$ such that $\mathcal{P}_{X_i}(Y_i)$ is compact, then $\mathcal{P}_X(Y)$ is compact.*

Proof. Put $\mathcal{P}_{X_i}(Y_i) \subset \mathcal{P}_X(Y)$ and suppose that $\mathcal{P}_X(Y)$ is an amenable space. This implies that for any problem $X_j \in \mathcal{P}_X(Y)$ then $X_j \equiv X_i$. The induced problem space $\mathcal{P}_{X_j}(Y_j)$ contains the problem X_j and is the maximal sub-problem of this space. Since $\mathcal{P}_{X_j}(Y_j) \subset \mathcal{P}_X(Y)$, it follows by amenability of the space that we can replace X_j with X_i and Y_j with Y_i, since problem and solution spaces remain invariant upon replacement with equivalent problems and alternative solutions, so that under the requirement that $\mathcal{P}_{X_i}(Y_i)$ is compact, we can put

$$\mathcal{P}_{X_j}(Y_j) = \mathcal{P}_{X_i}(Y_i) \subset \bigcup_{s=1}^{k} \mathcal{P}_{S_s}(T_s)$$

for a fixed $k \in \mathbb{N}$. It follows that

$$\bigcup_{i\geq 1} \mathcal{P}_{X_i}(Y_i) \cup \{X\} = \mathcal{P}_X(Y) \subset \bigcup_{s=1}^{k} \mathcal{P}_{S_s}(T_s) \cup \{X\}$$

for a fixed $k \in \mathbb{N}$. This shows that the problem space $\mathcal{P}_X(Y)$ is compact. □

13. Maps between problem and solution spaces

In this section, we study the analysis of the map between problem spaces and solution spaces. We examine how the notions of *boundedness* and *compactness* are preserved under the map.

Definition 13.1. Let $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ be a map between problem spaces. We say that f is *continuous* if and only if for any subspace $\mathcal{P}_R(U) \subseteq \mathcal{P}_S(T)$ with complexity $\mathbb{C}[\mathcal{P}_R(U)] \geq k$ there exists a subspace $\mathcal{P}_W(Z) \subseteq \mathcal{P}_X(Y)$ with complexity $C[\mathcal{P}_W(Z)] \geq k$ such that $f(\mathcal{P}_W(R)) \subseteq \mathcal{P}_R(U)$. Similarly, we say that the map $f : \mathcal{S}_X(Y) \longrightarrow \mathcal{S}_S(T)$ between the solution spaces is *continuous* if and only if for any subspace $\mathcal{S}_R(U) \subseteq \mathcal{S}_S(T)$ with index $\mathbb{I}[\mathcal{S}_R(U)] \geq k$ there exists a subspace $\mathcal{S}_W(Z) \subseteq \mathcal{S}_X(Y)$ with index $\mathbb{I}[\mathcal{S}_W(Z)] \geq k$ such that $f(\mathcal{S}_W(R)) \subseteq \mathcal{S}_R(U)$.

Definition 13.2. Let $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ be a map between problem spaces. We say that f is *bounded* if $f(\mathcal{P}_U(T))$ is a finite subset of problems in $\mathcal{P}_S(T)$ for each bounded $\mathcal{P}_U(T) \subset \mathcal{P}_X(Y)$.

Definition 13.3. Let $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ be a map between problem spaces. We say f is *compact* if and only if $f(\mathcal{P}_X(Y))$ is *compact*.

We expose the fact that *compactness* of a map between problem spaces can be inherited from the compactness of the space on which it acts.

Theorem 13.4 (Stability theorem)**.** *Let $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ be a map between problem spaces. If $\mathcal{P}_X(Y)$ is compact, then f is compact.*

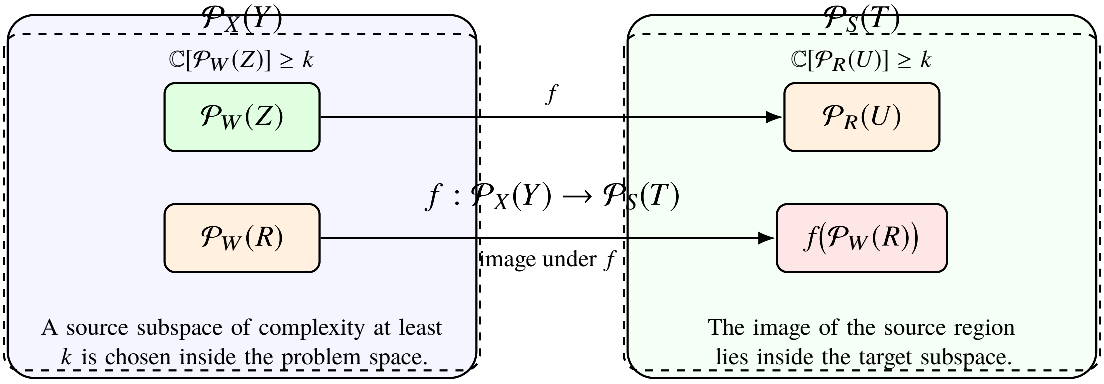


FIGURE 14. A continuous map between problem spaces. For every target subspace of complexity at least k, there exists a source subspace of complexity at least k whose image is contained in it.

Proof. Let $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ be a map between problem spaces and suppose that the space $\mathcal{P}_X(Y)$ is *compact*. Then there exists a finite number of problems spaces $\mathcal{P}_{K_1}(L_1), \cdots, \mathcal{P}_{K_n}(L_n)$ such that

$$\mathcal{P}_X(Y) \subset \mathcal{P}_{K_1}(L_1) \cup \cdots \cup \mathcal{P}_{K_n}(L_n).$$

We observe that $f(\mathcal{P}_X(Y) \cap \mathcal{P}_{K_1}(L_1)) \subseteq f(\mathcal{P}_{K_1}(L_1)))$. Using this relation, we can put

$$f(\mathcal{P}_X(Y)) \subseteq \bigcup_{j=1}^{n} f(\mathcal{P}_X(Y) \cap \mathcal{P}_{K_j}(L_j)) \subseteq \bigcup_{j=1}^{n} f(\mathcal{P}_{K_j}(L_j)).$$

This shows that the range $f(\mathcal{P}_X(Y))$ is *compact*, and therefore f is also compact. □

14. Isotope and Isotope problem and solution spaces

In this section, we introduce and study the notion of an *isotope* of problem and solution spaces.

Definition 14.1. Let V and U be any two problems. We say that V and U are *compatible* if there exists a problem space $\mathcal{P}_X(Y)$ such that $V, U \in \mathcal{P}_X(Y)$. We denote this compatibility by $V \diamond U$ or $U \diamond V$. Similarly, we say that two solutions R, S to some (possibly) distinct problems are compatible if there exists a solution space $\mathcal{S}_X(Y)$ such that $R, S \in \mathcal{S}_X(Y)$. We denote this compatibility by $R \diamond S$ or $S \diamond R$.

Definition 14.2. Let U and V be compatible problems. We say V and U admit a *merger* in the space $\mathcal{P}_X(Y)$ if there exists a problem $S \in \mathcal{P}_X(Y)$ such that $V < S$ and $U < S$ and V, U are the only maximal subproblem of S. In notation, we write $V \bowtie U = S \in \mathcal{P}_X(Y)$ or $U \bowtie V = S \in \mathcal{P}_X(Y)$. Similarly, let R and T be compatible solutions. We say that R and T admit a *merger* in the space $\mathcal{S}_X(Y)$ if there exists a solution $W \in \mathcal{S}_X(Y)$ such that $R < W$ and $T < W$ and R, T are the only maximal sub-solutions of W. In notation, we write $R \bowtie T = W \in \mathcal{S}_X(Y)$ or $R \bowtie T = W \in \mathcal{P}_X(Y)$

We now introduce the notion of an *isotope*.

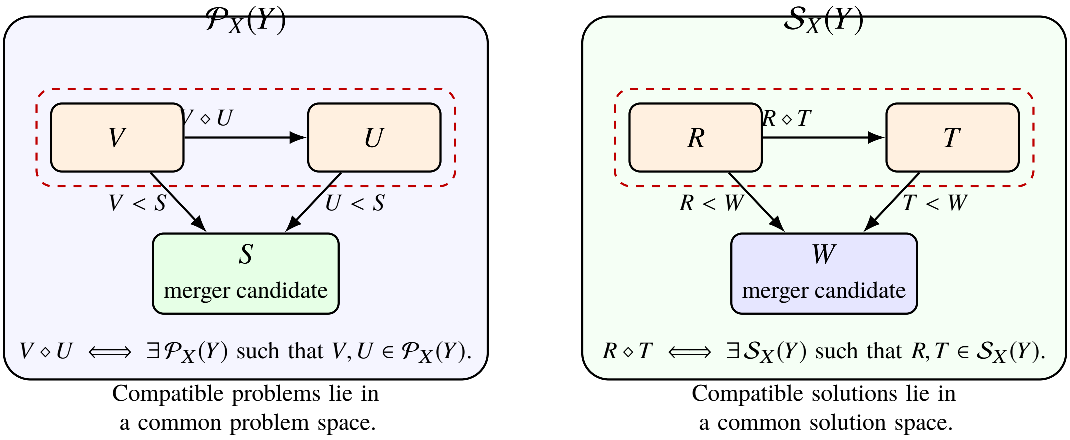


FIGURE 15. Compatible problems and compatible solutions in problem theory. Two problems are compatible precisely when they belong to a common problem space; two solutions are compatible precisely when they belong to a common solution space.

Definition 14.3. Let $\mathcal{P}_X(Y)$ and $\mathcal{S}_X(Y)$ be the problem space and the corresponding solution space, induced by assigning the solution Y to the problem X. We call an *isotope* on $\mathcal{P}_X(Y)$ as the map $\mathrm{Iso} : \mathcal{P}_X(Y) \longrightarrow \mathbb{R}$ such that

(i) $\mathrm{Iso}(V) \geq 0$ for each $V \in \mathcal{P}_X(Y)$ and
(ii) $\mathrm{Iso}(V \bowtie U) \leq \mathrm{Iso}(V) + \mathrm{Iso}(U)$ provided $U, V \in \mathcal{P}_X(Y)$ admits a merger.

A similar axiom also holds for solution spaces.

The notion of an *isotope* may not be viewed as an abstract notion. For example, if we consider a problem $V \in \mathcal{P}_X(Y)$ with a solution $U \in \mathcal{S}_X(Y)$ and the induced problem space $\mathcal{P}_V(U) \subset \mathcal{P}_X(Y)$, then we can associate a number with the problem V as

$$(\mathbb{C}[\mathcal{P}_V(U)])^{\frac{1}{\mathbb{C}[\mathcal{P}_V(U)]}-1}$$

where $\mathbb{C}[\mathcal{P}_V(U)]$ as usual denotes the complexity of the space. Similarly, for a solution U in the solution space $\mathcal{S}_X(Y)$, we can assign a number to the solution U to be

$$(\mathbb{I}[\mathcal{S}_V(U)])^{\frac{1}{\mathbb{I}[\mathcal{P}_V(U)]}-1}$$

where $\mathbb{I}[\mathcal{P}_V(U)]$ as usual denotes the index of the space. We could verify that these two maps satisfy the axioms of an *isotope*. In particular, an isotope may be viewed as a pseudo semi-norm.

Definition 14.4. Let $\mathcal{P}_X(Y)$ and $\mathcal{S}_X(Y)$ be a problem and a corresponding solution space whose topology admits an *isotope*. A problem (resp. solution) space equipped with an isotope is an isotope problem (resp. isotope solution) space. We denote these spaces by $(\mathcal{P}_X(Y), \mathrm{Iso}(\cdot))$ and $(\mathcal{S}_X(Y), \mathrm{Iso}(\cdot))$, respectively.

Definition 14.5. Let $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ be a map between isotope problem spaces. We put the isotope of f, denoted $\mathrm{Iso}(f)$, to be

$$\mathrm{Iso}(f) := \sup_{\substack{V \in \mathcal{P}_X(Y) \\ \mathrm{Iso}(V) \neq 0}} \frac{\mathrm{Iso}(f(V))}{\mathrm{Iso}(V)}.$$

We say that f is bounded if $\mathrm{Iso}(f) < \infty$. A similar characterization also holds for solution spaces.

Proposition 14.6. *Let* $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ *be a map between problem spaces. Then* $\mathrm{Iso}(f) < \infty$ *if and only if there exists an absolute constant* $c > 0$ *such that* $\mathrm{Iso}(f(V)) \leq c\,\mathrm{Iso}(V)$ *for all* $V \in \mathcal{P}_X(Y)$.

Proof. Suppose that $\mathrm{Iso}(f) < \infty$ then by definition 14.5 there exists an absolute constant $c > 0$ such that $\frac{\mathrm{Iso}(f(V))}{\mathrm{Iso}(V)} \leq c$ for all $V \in \mathcal{P}_X(Y)$. It immediately implies that $\mathrm{Iso}(f(V)) \leq c\,\mathrm{Iso}(V)$ for all $V \in \mathcal{P}_X(Y)$. Conversely, suppose that $\mathrm{Iso}(f(V)) \leq c\,\mathrm{Iso}(V)$ for all $V \in \mathcal{P}_X(Y)$ then

$$\mathrm{Iso}(f) := \sup_{\substack{V \in \mathcal{P}_X(Y) \\ \mathrm{Iso}(V) \neq 0}} \frac{\mathrm{Iso}(f(V))}{\mathrm{Iso}(V)} < \infty.$$

□

14.1. **Bounded isotope problem spaces.** In this section, we introduce and study the notion of a *bounded* isotope problem and solution spaces.

Definition 14.7. Let $\mathcal{P}_X(Y)$ be an isotope problem space induced by providing solution Y to problem X. We say that the space $\mathcal{P}_X(Y)$ is bounded if $\mathrm{Iso}(V) < \infty$ for all $V \in \mathcal{P}_X(Y)$.

Remark 14.8. We now show that a bounded map between problem spaces maps bounded subspaces to a bounded set of problems.

Proposition 14.9. *Let* $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ *be a map between isotope problem spaces. Suppose that* $\mathcal{P}_K(L) \subset \mathcal{P}_X(Y)$ *is a bounded sub-problem space. If* $\mathrm{Iso}(f) < \infty$*, then* $f(\mathcal{P}_K(L))$ *is bounded in* $\mathcal{P}_S(T)$.

Proof. Consider the map $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ such that $\mathrm{Iso}(f) < \infty$. Then there exists an absolute constant $c > 0$ such that Iso(f(V)) $\leq c\,\mathrm{Iso}(V)$ for all $V \in \mathcal{P}_X(Y)$. The requirement that $\mathcal{P}_K(L)$ is bounded implies that $\mathrm{Iso}(V) < \infty$ for all $V \in \mathcal{P}_K(L)$. This implies that $\mathrm{Iso}(f(V)) \leq d$ for all $V \in \mathcal{P}_K(L)$. This shows that $f(\mathcal{P}_K(L))$ is bounded in $\mathcal{P}_S(T)$. □

A similar characterization could be performed and proofs can be constructed by replacing the problem spaces $\mathcal{P}_K(L)$ with the corresponding induced solution spaces $\mathcal{S}_K(L)$.

14.2. **Continuous maps between isotope problem and solution spaces.** In this section, we introduce the notion of *continuity* of a map between isotope problem spaces.

Definition 14.10. Let $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ be a map between isotope problem spaces. We say that f is *continuous* if for any $\epsilon > 0$ there exists some $\delta > 0$ such that with $\mathrm{Iso}(V) < \delta$, then $\mathrm{Iso}(f(V)) < \epsilon$ for $V \in \mathcal{P}_X(Y)$.

We expose the relationship that exists between *continuity* and *boundedness* of maps between problem space. In fact, we show that these two seemingly disparate notions are equivalent in problem theory.

Theorem 14.11. *Let* $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ *be a map between isotope problem spaces. Then* $\mathrm{Iso}(f) < \infty$ *if and only if* f *is continuous.*

Proof. Let $f : \mathcal{P}_X(Y) \longrightarrow \mathcal{P}_S(T)$ be a map between isotope problem spaces. Suppose that $\mathrm{Iso}(f) < \infty$, then there exists an absolute constant $c > 0$ such that $\mathrm{Iso}(f(V)) \leq c\ \mathrm{Iso}(V)$ for all $V \in \mathcal{P}_X(Y)$. Let $\epsilon > 0$ and choose $\delta := \frac{\epsilon}{c}$ so that with $\mathrm{Iso}(V) < \delta$ then $\mathrm{Iso}(f(V)) \leq c\ \mathrm{Iso}(V) < c\delta = \epsilon$. This proves that f is continuous. Conversely, suppose that f is continuous and assume that f is not bounded. Then for each $n \geq 1$ there exists a sequence $\{V_n\} \subset \mathcal{P}_X(Y)$ such that $\mathrm{Iso}(f(V_n)) > n\ \mathrm{Iso}(V_n)$ for all $n \geq N_o > 0$. Putting $\frac{1}{n} < \mathrm{Iso}(V_n) < 1 - \frac{1}{n}$, then (by continuity) we get $1 < n\ \mathrm{Iso}(V_n) < \mathrm{Iso}(f(V_n)) < 1$, which is absurd. □

15. Application to the P vs NP problem

In this section, we provide a sketch solution to the P vs NP problem. We show that P = NP using ideas drawn from problem theory. We begin as follows:

15.1. **A sketch solution.** Let p be a problem in NP, then the solution q to the problem p is verifiable in polynomial time. In keeping with the notation of the theory, we let $\mathcal{P}_p(q)$ and $\mathcal{S}_p(q)$ denote the problem and the solution spaces induced equipped with an **isotope**. That is, we work in the isotope problem and solution spaces $(\mathcal{P}_p(q), \mathrm{Iso}(\cdot))$ and $(\mathcal{S}_p(q), \mathrm{Iso}(\cdot))$. We obtain for the corresponding resolution and verification complexity

$$\mathcal{P}_p^r(q) := \sum_{\substack{u \in \mathcal{P}_p(q) \\ k \in \mathcal{S}_p(q)}} C_r(u,k) = C_r(p,q)$$

and the verification complexity with

$$\mathcal{S}_p^v(q) := \sum_{\substack{k \in \mathcal{S}_p(q) \\ u \in \mathcal{P}_p(q)}} C_v(u,k) = C_v(p,q)$$

where $C_r(u,k)$ and $C_v(u,k)$ denotes the resolution and verification time complexity of each problem u in the problem space with solution k in the corresponding solution space. Because we have assumed that $p \in \mathrm{NP}$, it follows that for the verification time complexity $C_v(p,q) \ll$ **polynomial time**. Hence, for each $u \in \mathcal{P}_p(q)$ with solution $k \in \mathcal{S}_p(q)$, we must have $C_v(u,k) \ll$ **polynomial time**. It follows necessarily that for the index of the solution space $\mathcal{S}_p(q)$, we must have $\mathbb{I}[\mathcal{S}_p(q)] < \infty$. Consequently, we have, for the complexity of the problem space $\mathbb{C}[\mathcal{P}_p(q)] < \infty$.

Now, we consider a surjective map $f : \mathcal{P}_x(y) \longrightarrow \mathcal{P}_p(q)$ for $x \in \mathrm{P}$ and $p \in \mathrm{NP}$. Since $f(\mathcal{P}_x(y)) = \mathcal{P}_p(q)$ and $\mathbb{C}[\mathcal{P}_p(q)] < \infty$, it implies that f is **bounded**. It follows that $\mathrm{Iso}(f) < \infty$, where Iso denotes the *isotope* of the map f. Therefore, there exists an absolute constant $c_1 > 0$ such that $\mathrm{Iso}(f(l)) \leq c_1 \mathrm{Iso}(l)$ for all $l \in \mathcal{P}_x(y)$.

Now, we define $\mathrm{Iso} : \mathcal{P}_x(y) \longrightarrow \mathbb{R}$ by

$$\mathrm{Iso}(l) := \sum_{\substack{w \in \mathcal{S}_l(t) \\ z \in \mathcal{P}_l(t)}} C_r(z,w).$$

It is easy to check that this definition satisfies the axioms of an *Isotope*. Because $x \in \mathrm{P}$, it follows that $C_r(x, y) \ll$ **polynomial time** and hence $\mathrm{Iso}(x) \ll$ **polynomial time**. It follows that

$$\mathrm{Iso}(p) := C_r(p, q) := \sum_{\substack{k \in \mathcal{S}_p(q) \\ u \in \mathcal{P}_p(q)}} C_r(u, k) \leq c_1 \mathrm{Iso}(h) \ll \textbf{polynomial time}$$

for some $h \in \mathcal{P}_x(y)$ such that $f(h) = p \in \mathcal{P}_p(q)$. This proves $p \in \mathrm{P}$.

16. Conclusion and further remarks

This work represents a significant advancement in our understanding of problem and solution spaces, particularly with respect to their algebraic, topological, and computational properties. By introducing novel concepts such as isotopic maps, separability, amenability, and the isotope viewed as a pseudo semi-norm, we have opened new avenues for exploring the intricate relationships between problems, their solution spaces, and the transformations between them.

The Characterization Theorem introduced in this work provides a robust framework for categorizing problem spaces, a critical step toward understanding the fundamental structure of problems and the conditions under which they can be solved. The study of separability and amenability within these spaces has highlighted essential conditions that influence the solvability of problems, while the examination of isotopic maps and their properties has bridged the gap between theoretical exploration and practical application, particularly in the context of time complexity. The equivalence of boundedness and continuity of isotopic maps provides a key insight into how problem spaces can be transformed while preserving their complexity, offering valuable tools for further studies in computational complexity theory.

As we have shown, the introduction of the isotope pseudo semi-norm provides a new approach for assessing the complexity of problem spaces. This measure is a vital contribution to problem theory, facilitating a deeper understanding of solvability and the structure of solution spaces.

However, despite the progress made, several open questions remain, particularly in relation to the complexity of problem transformations and the limits of current theories. The following conjectures arise naturally from the findings of this work and serve as promising directions for future research.

Conjecture 16.1. *Let $\mathcal{P}$ be a problem space and* $\mathrm{Iso} : \mathcal{P} \longrightarrow \mathbb{R}$. *There exists a bounded isotope* $\mathrm{Iso}()$ *such that for any $p_1, p_2 \in \mathcal{P}$ with $p_1 \neq p_2$ and $p_1 \not\equiv p_2$, then* $\mathrm{Iso}(p_1) \neq \mathrm{Iso}(p_2)$.

Conjecture 16.2. *Let $\mathcal{P}$ be a problem space and $p_1, p_2 \in \mathcal{P}$. If $|\mathrm{Iso}(p_1) - \mathrm{Iso}(p_2)| < \epsilon$ for some small $\epsilon > 0$, then there exists absolute constants $C_1, C_2 > 0$ such that*

$$\frac{C_v(p_1)}{C_v(p_2)} \leq C_1$$

and

$$\frac{C_r(p_1)}{C_r(p_2)} \leq C_2.$$

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