Infinite Numbers in Mathematics

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Abstract

A method named 'motional construction' is introduced in this article. Questions involving continuum hypothesis and incompleteness theorems of formal systems are answered, but a major concern of this article is a nature of infinity: Infinity implies paradoxes.

Many conclusions contradicting orthodox mathematics are proved, such as: a set of real numbers does not exist, Lebesgue measure of any set is zero, ZF axioms are not logically consistent, etc. Unreliable results are common in sub-fields of mathematics where infinite sets are used intensively.

A valid mathematical conclusion describes finiteness in essence.

When infiniteness is clear, the third mathematical crisis shall be over.

Contents

1 Introduction

Given that many misunderstandings have been lasting for a long time and are so deceptive that even deceived many gifted minds, five paradoxes are prepared in the fist few sections including: real numbers can generate paradoxes without axiom of choice (AC), there is no smallest infinite set, continuum hypothesis is about invalid concepts, the set of real numbers is not a power set of the set of natural numbers, the set of natural numbers is uncountable.

Obstacles in dealing with infinite numbers are caused by methods of descriptions, e.g., a definition of a set $\{x\colon x \text{ meets conditions} \ldots\}$. A simple method for mathematical description is shown and named as 'motional construction' (MC) because of a motional view. AC and MC are the only tools available by now to handle infinity. Any axioms designed for infinity shall be consistent with AC and MC.

Despite of the inconsistency implied by infiniteness, a consistent mathematics could be built with paradoxes bypassed. An updated understanding of infinity leads to inevitable updates of set theory, logic, analysis etc. In this article, symbol $\mathbb{N}$ denotes the set of positive natural numbers, $\mathbb{Z}$ denotes the set of integer numbers, $\mathbb{Q}$ denotes the set of rational numbers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers, the power set of a set S is denoted by 2^S rather than $\mathcal{P}(S)$.

2 Extending rod paradox

A rod can be extended by finite steps of separations, rotations, translations and contractions. Remainder operation 'mod' for real numbers is used in this section, i.e., $\forall a, b \in \mathbb{R}, b \neq 0$, $a \bmod b = a - b\left\lfloor \frac{a}{b} \right\rfloor$. Every '$n$' in this paradox denotes an integer number, i.e., $n \in \mathbb{Z}$.

Let L be a line section of length $\sqrt{2}$:

$$L = \left\{ x \in \mathbb{R} \colon 0 \leq x \leq \sqrt{2} \right\}$$

Separate L into three parts L_0, L_1 and L_2:

$$L_0 = \{0\}$$
$$L_1 = \left\{ x \in \mathbb{R} \colon \frac{x}{\sqrt{2}} \notin \mathbb{Q},\, 0 < x \leq \sqrt{2} \right\}$$
$$L_2 = \left\{ x \in \mathbb{R} \colon \frac{x}{\sqrt{2}} \in \mathbb{Q},\, 0 < x \leq \sqrt{2} \right\}$$

Process L_1 as follows:

Step1: Rotate L_1 anticlockwise 45° around original point $(0, 0)$ to obtain L_1':

$$L_1' = \left\{ (x, y) \in \mathbb{R}^2 \colon y \notin \mathbb{Q},\, 0 < y \leq 1, x = y \right\}$$

Step2: Because $\forall y \notin \mathbb{Q}, \forall n_1, n_2 \in \mathbb{Z}, n_1 \neq n_2 \implies 2^{n_1} y \not\equiv 2^{n_2} y \pmod{1}$, so every element in every following set is uniquely specified by 'n'. Besides, it can be checked that any sets to be merged in following steps are disjoint. Suppose we have a draft plane rather than the plane containing L_1. On the draft plane, define A_1 as:

$$A_1 = \left\{ (x, y) \in \mathbb{R}^2 \colon y \notin \mathbb{Q}, 0 < y \leq 1, x = (2^n y) \bmod 1, n \neq 0 \right\}$$

Move L_1' onto the draft plane and keep its coordinate position to obtain $A_2 = A_1 \cup L_1'$:

$$A_2 = \left\{ (x, y) \in \mathbb{R}^2 \colon y \notin \mathbb{Q}, 0 < y \leq 1, x = (2^n y) \bmod 1 \right\} \tag{1}$$

Step3: Separate A_2 into two parts A_3 and A_4:

$$A_3 = \left\{ (x, y) \in \mathbb{R}^2 \colon y \notin \mathbb{Q}, 0 < y \leq 1, x = (2^n y) \bmod 2, x \leq 1 \right\}$$
$$A_4 = \left\{ (x, y) \in \mathbb{R}^2 \colon y \notin \mathbb{Q}, 0 < y \leq 1, x = (2^n y) \bmod 2 - 1, x > 0 \right\}$$

Then move A_4 along positive direction of x-axis by 1 to obtain A_5:

$$A_5 = \left\{ (x, y) \in \mathbb{R}^2 \colon y \notin \mathbb{Q}, 0 < y \leq 1, x = (2^n y) \bmod 2, x > 1 \right\}$$

So $A_6 = A_3 \cup A_5$ is obtained:

$$A_6 = \left\{ (x, y) \in \mathbb{R}^2 \colon y \notin \mathbb{Q}, 0 < y \leq 1, x = (2^n y) \bmod 2 \right\}$$

Step4: Separate A_6 into two parts A_7 and L_1'':

$$A_7 = \left\{(x, y) \in \mathbb{R}^2 \colon y \notin \mathbb{Q}, 0 < y \leq 1, x = (2^n y) \bmod 2, n \neq 1\right\}$$

$$L_1'' = \left\{(x, y) \in \mathbb{R}^2 \colon y \notin \mathbb{Q}, 0 < y \leq 1, x = 2y\right\}$$

Move L_1'' back to the plane which L_1 initially belongs to.

Step5: Compress A_7 along negative direction of x-axis by ratio $1/2$ to get A_8:

$$\begin{aligned} A_8 &= \left\{(x, y) \in \mathbb{R}^2 \colon y \notin \mathbb{Q}, 0 < y \leq 1, x = ((2^n y) \bmod 2)/2, n \neq 1\right\} \\ &= \left\{(x, y) \in \mathbb{R}^2 \colon y \notin \mathbb{Q}, 0 < y \leq 1, x = (2^{(n-1)} y) \bmod 1, n \neq 1\right\} \end{aligned}$$

Because $\mathbb{Z}$ contains infinitely many integers, so $\{n - 1 \colon n \in \mathbb{Z}, n \neq 1\}$ and $\{n \in \mathbb{Z} \colon n \neq 0\}$ are the same set. So $A_8 = A_1$.

Step6: Rotate L_1'' around original point $(0, 0)$ clockwise by angle $\arctan(1/2)$ to obtain L_3:

$$L_3 = \left\{x \in \mathbb{R} \colon \frac{x}{\sqrt{5}} \notin \mathbb{Q}, 0 < x \leq \sqrt{5}\right\}$$

After six steps, A_1 on the draft plane is unchanged, but L_1 becomes L_3. The only operation which is not a separation, rotation or translation is Step5, which reduces or keeps distances between any two points, so is a contraction.

Next, we alter L_2 first and then do similar operations. Separate L_2 into two parts L_4 and L_5:

$$L_4 = \left\{x \in \mathbb{R} \colon \frac{x}{\sqrt{2}} \in \mathbb{Q}, 0 < x \leq 1\right\}$$

$$L_5 = \left\{x \in \mathbb{R} \colon \frac{x}{\sqrt{2}} \in \mathbb{Q}, 1 < x \leq \sqrt{2}\right\}$$

Then move L_4 along positive direction of x-axis by $\sqrt{2} - 1$ to obtain L_4', and move L_5 along negative direction of x-axis by 1 to obtain L_5':

$$\begin{aligned} L_4' &= \left\{x \in \mathbb{R} \colon \frac{x - (\sqrt{2} - 1)}{\sqrt{2}} \in \mathbb{Q}, \sqrt{2} - 1 < x \leq \sqrt{2}\right\} \\ &= \left\{x \in \mathbb{R} \colon \frac{x + 1}{\sqrt{2}} \in \mathbb{Q}, \sqrt{2} - 1 < x \leq \sqrt{2}\right\} \end{aligned}$$

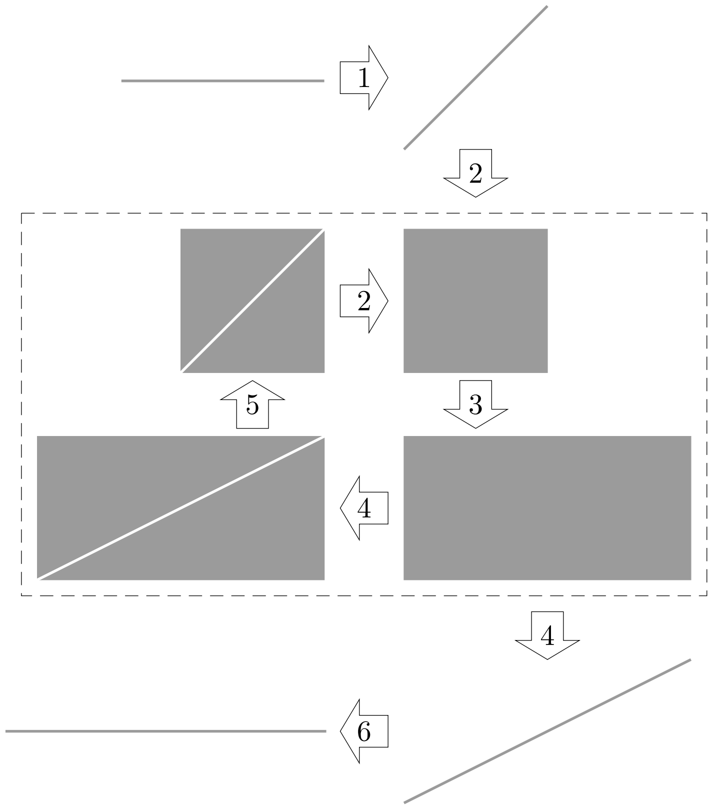

Figure 1: Steps of extending rod paradox

$$L'_5 = \left\{ x \in \mathbb{R} \colon \frac{x+1}{\sqrt{2}} \in \mathbb{Q}, 0 < x \leq \sqrt{2} - 1 \right\}$$

So $L_6 = L'_4 \cup L'_5$ is obtained:

$$L_6 = \left\{ x \in \mathbb{R} \colon \frac{x}{\sqrt{2}} + \frac{1}{\sqrt{2}} \in \mathbb{Q}, 0 < x \leq 1 \right\}$$

Because $x/\sqrt{2} + 1/\sqrt{2} \in \mathbb{Q} \implies x/\sqrt{2} \notin \mathbb{Q}$, so former six steps applied to L_1 can also be applied to L_6. Just replace all the '$\notin \mathbb{Q}$' with '$+1/\sqrt{2} \in \mathbb{Q}$', we will get six steps suitable for L_6. It can be checked that everything on the draft plane after these steps is also unchanged, and we can obtain L_7:

$$L_7 = \left\{ x \in \mathbb{R} \colon \frac{x}{\sqrt{5}} + \frac{1}{\sqrt{2}} \in \mathbb{Q}, 0 < x \leq \sqrt{5} \right\}$$

Then divide L_7 into two parts L_8 and L_9:

$$L_8 = \left\{ x \in \mathbb{R} \colon \frac{x}{\sqrt{5}} + \frac{1}{\sqrt{2}} \in \mathbb{Q}, 0 < x \leq \sqrt{5} - \frac{\sqrt{5}}{\sqrt{2}} \right\}$$

$$L_9 = \left\{ x \in \mathbb{R} \colon \frac{x}{\sqrt{5}} + \frac{1}{\sqrt{2}} \in \mathbb{Q}, \sqrt{5} - \frac{\sqrt{5}}{\sqrt{2}} < x \leq \sqrt{5} \right\}$$

Then move L_8 along positive direction of x-axis by $\sqrt{5}/\sqrt{2}$ to obtain L_8', and move L_9 along negative direction of x-axis by $\sqrt{5} - \sqrt{5}/\sqrt{2}$ to obtain L_9':

$$\begin{aligned} L_8' &= \left\{ x \in \mathbb{R} \colon \frac{x - \sqrt{5}/\sqrt{2}}{\sqrt{5}} + \frac{1}{\sqrt{2}} \in \mathbb{Q}, \frac{\sqrt{5}}{\sqrt{2}} < x \leq \sqrt{5} \right\} \\ &= \left\{ x \in \mathbb{R} \colon \frac{x}{\sqrt{5}} \in \mathbb{Q}, \frac{\sqrt{5}}{\sqrt{2}} < x \leq \sqrt{5} \right\} \\ L_9' &= \left\{ x \in \mathbb{R} \colon \frac{x + \sqrt{5} - \sqrt{5}/\sqrt{2}}{\sqrt{5}} + \frac{1}{\sqrt{2}} \in \mathbb{Q}, 0 < x \leq \frac{\sqrt{5}}{\sqrt{2}} \right\} \\ &= \left\{ x \in \mathbb{R} \colon \frac{x}{\sqrt{5}} \in \mathbb{Q}, 0 < x \leq \frac{\sqrt{5}}{\sqrt{2}} \right\} \end{aligned}$$

So $L_{10} = L_8' \cup L_9'$ is obtained:

$$L_{10} = \left\{ x \in \mathbb{R} \colon \frac{x}{\sqrt{5}} \in \mathbb{Q}, 0 < x \leq \sqrt{5} \right\}$$

Finally we get $\widehat{L} = L_0 \cup L_3 \cup L_{10}$:

$$\widehat{L} = \left\{ x \in \mathbb{R} \colon 0 \leq x \leq \sqrt{5} \right\}$$

After finite steps of separations, rotations, translations and contractions, a line segment is extended to a longer one. This process is replicable, consequently we get a paradox of extending line. If every point in the process is assigned with a line section vertical to the concerned plane, we would get a paradox of extending strap. If every point in the process is assigned with a plane face vertical to the point's plane in a 4-dimensional space, we would get a paradox of extending rod.

The process can be carried out in a reverse manner, i.e., by finite steps of separations, rotations, translations and expansions, a line, a strap or a rod can be compressed to a shorter and shorter one.

The step3 alone is a persuasive paradox. A_2 proportionally turns into a longer A_6 by displacement and could turn into 2^n times long by n displacements. This is not an expansion by increasing distances among points, since A_2 is not a set of disjoint points. The dimension of set A_2 along the extending direction is at least 1 (consider the diagonal). In a reverse way, the length of A_2 could be reduced to 2^{-n} by n displacements.

Every paradox similar to the Banach-Tarski paradox[1] involves three crucial features:

- Irrational numbers in 1st mathematical crisis.
- Infinite series in 2nd mathematical crisis.
- Set concepts in 3rd mathematical crisis.

3 Countable cardinality paradox

This paradox shows that there is no smallest infinite cardinal number.

Partition of a set is used in this paradox, so its definition is recalled: Let P be a set formed by nonempty subsets of set S. If any two elements in P do not intersect, and a union of all elements in P equals S, then P is called a partition of S.

Definition 3.1. (Larger Than) A set S_1 is larger than a set S_2 if and only if there is no bijection between S_1 and any subset of S_2.

Lemma 3.1. *An infinite set exists that its power set is no larger than the set of natural numbers.*

Proof. This is a constructive proof. Let a set sequence $D_1, D_2, D_3, \cdots$ be

$$D_i = \left\{x \in \mathbb{N} \colon (x-1) \bmod 2^i < 2^{i-1}\right\}$$

Let E be a set consisting of a finite number of D_i. Let 2^E be the power set of E.

Define a mapping $f \colon 2^E \mapsto f(2^E)$ as

$$f(\alpha) = \left(\bigcap_{\beta\in\alpha} \beta\right) \cap \left(\bigcap_{\beta\in E, \beta\notin\alpha} \beta^c\right), \alpha \in 2^E$$

where β^c represents the complementary set of β.

It follows the above definition that $f(2^E)$ is a partition of $\mathbb{N}$. A detailed proof is included here. But this proof can be skipped, just consider that a binary notation of every natural number exists and is unique.

$\forall\alpha \in 2^E$, let $I_1 = \{i \colon D_i \in \alpha\}$ denote all the indexes of β in α, let $I_2 = \{i \colon D_i \in E, D_i \notin \alpha\}$ denote all the indexes of β in E but not in α. Choose a natural number $x^* = 1 + \sum_{i \in I_2} 2^{(i-1)}$. $\forall \beta \in 2^E$, suppose $\beta = D_j$. If $\beta \in \alpha$, then $j \in I_1 \implies (x^* - 1) \bmod 2^j = \sum_{i \in I_2, i \leq j} 2^{(i-1)} = \sum_{i \in I_2, i \leq j-1} 2^{(i-1)} \leq \sum_{i \leq j-1} 2^{(i-1)} = 2^{(j-1)} - 1 < 2^{(j-1)} \implies x^* \in D_j \implies x^* \in \bigcap_{\beta \in \alpha} \beta$. If $\beta \notin \alpha$, then $j \in I_2 \implies (x^* - 1) \bmod 2^j = \sum_{i \in I_2, i \leq j} 2^{(i-1)} \geq 2^{(j-1)} \implies x^* \notin D_j \implies x^* \in \beta^c \implies x^* \in \bigcap_{\beta \notin \alpha} \beta^c$. So $x^* \in f(\alpha) \implies f(\alpha) \neq \varnothing$. Therefore, $\forall f(\alpha) \in f(2^E), f(\alpha) \neq \varnothing$.

$\forall \alpha_1, \alpha_2 \in 2^E, \alpha_1 \neq \alpha_2, \exists \beta^* \in E$, satisfying $\beta^* \in \alpha_1, \beta^* \notin \alpha_2$, or $\beta^* \notin \alpha_1, \beta^* \in \alpha_2$, therefore

$$
\begin{aligned}
f(\alpha_1) \cap f(\alpha_2) &= \Big(\bigcap_{\beta \in \alpha_1} \beta \Big) \cap \Big(\bigcap_{\substack{\beta \in E \\ \beta \notin \alpha_1}} \beta^c \Big) \cap \Big(\bigcap_{\beta \in \alpha_2} \beta \Big) \cap \Big(\bigcap_{\substack{\beta \in E \\ \beta \notin \alpha_2}} \beta^c \Big) \\
&= \beta^* \cap \beta^{*c} \cap \Big(\bigcap_{\substack{\beta \in \alpha_1 \cup \alpha_2 \\ \beta \neq \beta^*}} \beta \Big) \cap \Big(\bigcap_{\substack{\beta \in E \\ \beta \notin \alpha_1 \cap \alpha_2 \\ \beta \neq \beta^*}} \beta^c \Big) = \varnothing
\end{aligned}
$$

$\forall x \in \mathbb{N}$, consider $\alpha^* = \{\beta \in E \colon x \in \beta\}$, so $\forall \beta \in E, \beta \notin \alpha^* \implies x \notin \beta$, so $x \in \big(\bigcap_{\beta \in \alpha^*} \beta \big) \cap \big(\bigcap_{\beta \in E, \beta \notin \alpha^*} \beta^c \big) \implies x \in f(\alpha^*)$, therefore $\bigcup_{\gamma \in f(2^E)} \gamma = \mathbb{N}$. Now we have proved that $f(2^E)$ is a partition of $\mathbb{N}$.

In the proof, $\forall \alpha \in 2^E, f(\alpha) \neq \varnothing$ and $\forall \alpha_1, \alpha_2 \in 2^E, \alpha_1 \neq \alpha_2, f(\alpha_1) \cap f(\alpha_2) = \varnothing$ implies f is a bijection.

Then we proceed to define a set E which cannot be defined in an ordinary form like $\{x \colon x \text{ meets} \cdots\}$. Let's begin with an empty set $E = \varnothing$, and process as follows: Check $D_1, D_2, D_3, \cdots$ one by one in sequence. Let $E' = E \cup \{D_i\}$. if E' meets all the following conditions, then Add D_i into set E (i.e., make E' the new E). Otherwise, stop the process. The conditions are:

(I) f is a bijection.
(II) $f\big(2^{E'}\big)$ is a partition of $\mathbb{N}$.
(III) Other conditions needed to make this paradox more convincing ...

Then we ask a question: Does this process of adding elements to E have to be stopped at some finite step?

Assume this process fails when trying to add D_n. Then $n-1$ elements $D_1, D_2, \ldots, D_{n-1}$ have been successfully added to E, so it can be checked that all the conditions are also satisfied by $E \cup \{D_n\}$, which contradicts our step-n-stop assumption. So, we have to accept that this process can be done infinitely, to get at least one set E consisting of infinitely many elements.

Then we ask a second question: Does this infinite set E meet all the conditions?

Assume E does not meet one of the conditions. Then we should have stopped before obtaining E, since our process is to check all the conditions

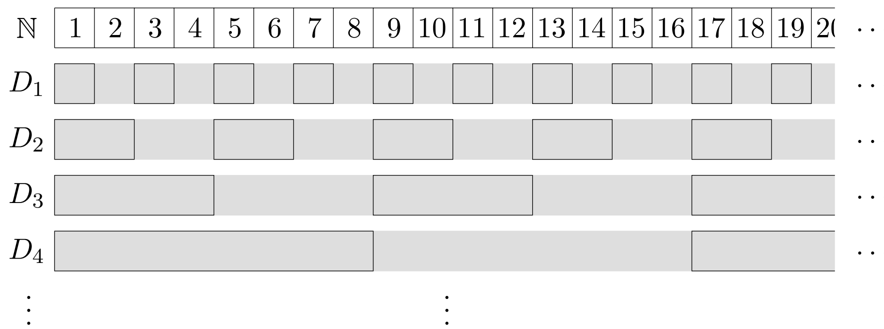

Figure 2: Partition of the set of natural numbers

prior to altering E. So, we did not get E, which contradicts the set-E-got assumption. So, we have to accept that the infinite set E meets all the conditions.

Since $f(2^E)$ is a partition of $\mathbb{N}$, we can define a mapping[1]

$$p\colon f(2^E) \mapsto \mathbb{N}$$

that $\forall S \in f(2^E)$, $p(S) =$ the smallest number in S. Because f is a bijection and $f(2^E)$ is a partition of $\mathbb{N}$, so $\forall \alpha_1, \alpha_2 \in 2^E, \alpha_1 \neq \alpha_2 \implies f(\alpha_1) \neq f(\alpha_2) \implies f(\alpha_1) \cap f(\alpha_2) = \varnothing \implies p \circ f(\alpha_1) \neq p \circ f(\alpha_2)$, i.e., $p \circ f$ is a bijection from (2^E) to a subset of $\mathbb{N}$. Therefore, by Definition 3.1, E is an infinite set whose power set is no larger than the set of natural numbers. □

Let $card()$ denote a cardinality. Since $p \circ f$ is a bijection from 2^E to a subset of $\mathbb{N}$, we have $card(2^E) \leq card(\mathbb{N})$. According to Cantor's theorem about cardinality of power set: the cardinality of a power set of any set is larger than the cardinality of the set itself, so

$$card(E) < card(2^E) \leq card(\mathbb{N})$$

Hence the cardinal number of E is smaller than $\mathbb{N}$. Name this E as

$$\log_2 \mathbb{N}$$

The cardinality of $\log_2 \mathbb{N}$ is not the smallest either, because we can repeat this process to obtain:

$$\cdots \log_2 \log_2 \log_2 \mathbb{N}$$

Besides, $\forall m \in \mathbb{N}$, we can define $\log_m \mathbb{N}$ through a similar process.

[1]Alternatively the existence of mapping p could be added to the conditions in the process defining E.

4 Continuum hypothesis paradox

This is a proof of continuum hypothesis[2].

The former section shows that there is no smallest infinite cardinal number, but it is not answered whether there exists a third cardinal number between cardinality of an infinite set and cardinality of its power set. Continuum hypothesis guesses that there is no other cardinal number between the cardinality of the set of natural numbers and cardinality of its power set.

If we assume that any cardinal number no greater than the cardinality of a set can be represented by a subset of it, then it is sufficient to check cardinality of every subset of the power set of set of natural numbers, namely, check all the subsets of $2^{\mathbb{N}}$.

An idea of this proof is to find a bijection from a subset of $2^{\mathbb{N}}$ to a second subset of $2^{\mathbb{N}}$, and make sure the second subset meets: If a set S comprising some natural numbers is in it, then the power set of S is a subset of it. The bijection to be obtained cannot be defined by an ordinary expression like $f(x) = expression(x)$.

For convenience of statement, five kinds of sets are used in this proof: primitive set composed of some natural numbers, subscript set composed of some natural numbers, set of some subscript sets, parting set composed of some primitive sets, set composed of some parting sets.

Definition 4.1. (Counted-Binary Order of Finite Sets of Natural Numbers) Let S_1 and S_2 be two different sets comprising finitely many natural numbers. A counted-binary order between them is decided by: If one set contains less natural numbers than the other, then it is prior to the other. If they contain the same number of natural numbers, then compute $\sum_{n\in S_1} 2^n$ and $\sum_{n\in S_2} 2^n$, the set with a smaller result is prior to the other.

Lemma 4.1. $\forall J \subseteq 2^{\mathbb{N}}$, $\exists J^*$ *which is a set of sets, there exists a bijection between* J *and* J^*, *and* $\forall \eta \in J^*, 2^\eta \subseteq J^*$.

Proof. (Proofs of required lemmas are nested inside this proof.)

$\forall J \subseteq 2^{\mathbb{N}}$, try to separate J into many sets $\{J_\xi\}$, where every index ξ is a subscript set consisting of some natural numbers. Name J_ξ as parting set. Name ξ as subscript set. Begin with $\{J_\xi\} = \{J\}$, i.e., initially $\{J_\xi\}$ contains only J, of which the subscript set ξ is an empty set. Process $\{J_\xi\}$ step by step. In the ith step ($i = 1, 2, 3, \cdots$), process $\{J_\xi\}$ as follows:

(1) (Process 1) Obtain two parting sets from every parting set in $\{J_\xi\}$ according to number 'i':

Let J_η be a parting set currently dealt with. One set obtained is J_{η_0}. Specify its subscript set as

$$\eta_0 = \eta$$

From J_η select every primitive set which does not contain number 'i' to form this parting set J_{η_0}, that is

$$J_{\eta_0} = \{\zeta \colon \zeta \in J_\eta, i \notin \zeta\}$$

The other set obtained is J_{η_1}. Specify its subscript set as

$$\eta_1 = \eta \cup \{i\}$$

From J_η select every primitive set which contains number 'i' to form this parting set J_{η_1}, that is

$$J_{\eta_1} = \{\zeta \colon \zeta \in J_\eta, i \in \zeta\}$$

It can be verified that: If $J_\eta = \varnothing$, then $J_{\eta_0} = J_{\eta_1} = \varnothing$. If $J_\eta \neq \varnothing$, then at most one of J_{η_0} and J_{η_1} might be empty set.
After every parting set in $\{J_\xi\}$ was processed, a new set formed by all the new parting sets is obtained and denoted by $\{J_{\xi'}\}$.

(2) (Process 2) Process all the parting sets in $\{J_{\xi'}\}$ one by one in counted-binary order of their subscript sets in following way:
Let J_η be a parting set currently dealt with.
If $J_\eta = \varnothing$, then do nothing.
If $J_\eta \neq \varnothing$, then from $\{J_{\xi'}\}$ select all the parting sets whose subscript set is a subset of η, to form a set of parting sets. Let K denote this set of parting sets temporarily, i.e.,

$$K = \{J_\sigma \colon J_\sigma \in \{J_{\xi'}\}, \sigma \subset \eta\}$$

Then search in this temporary K in counted-binary order of subscript sets to find one empty parting set. If no empty parting set could be found in K, then do nothing. If a first empty parting set is found, then stop searching, note this empty parting set as $J_{\eta'}$, and move all the contents of J_η into $J_{\eta'}$ (i.e., J_η becomes empty, and $J_{\eta'}$ becomes a parting set consisting of all the primitive sets originally in J_η). The Figure 3 illustrates an example of this Process 2.

(3) Check whether $\{J_{\xi'}\}$ meets all the following conditions:
(a) (Condition 3a) All the nonempty parting sets in $\{J_{\xi'}\}$ form a partition of J.

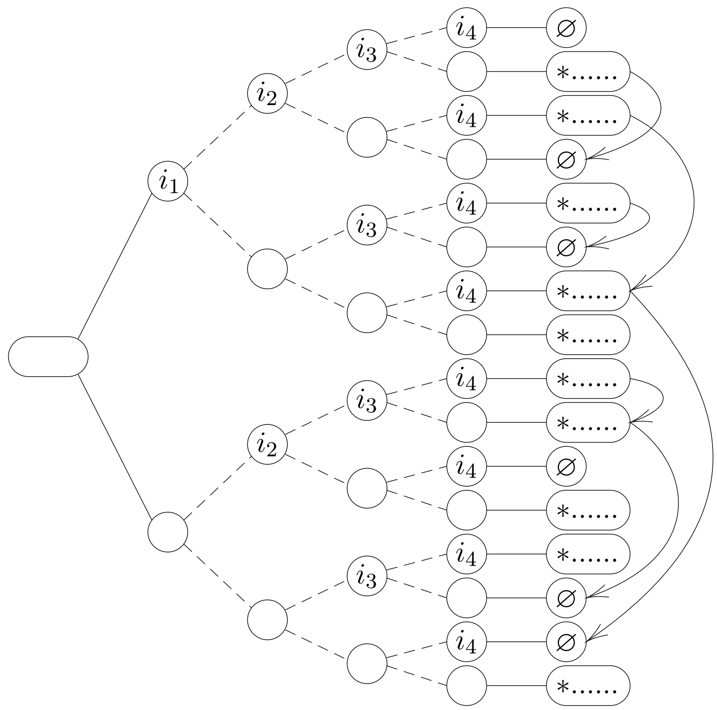

Figure 3: Example of Process 2

(b) (Condition 3b) $\forall J_\eta \in \{J_{\xi'}\}, \forall \sigma \subset \eta \implies \sigma$ is subscript set of a parting set in $\{J_{\xi'}\}$, i.e., $J_\sigma \in \{J_{\xi'}\}$.

(c) (Condition 3c) $\forall J_\eta \in \{J_{\xi'}\}$, $J_\eta \neq \varnothing$, $\forall \sigma \subset \eta \implies J_\sigma \neq \varnothing$.

(d) Other conditions needed to make this proof more convincing ...

If all the conditions are satisfied, then replace $\{J_\xi\}$ with $\{J_{\xi'}\}$ and proceed to the next step. If any condition is violated, then stop the process.

Now we prove that every condition is guaranteed in any finite step:

Proof. Proof of Condition 3a

During every step, the only possible maneuver operated on any primitive set is moving it from one parting set to another. So all the nonempty parting sets in $\{J_{\xi'}\}$ always form a partition of J. □

Proof. Proof of Condition 3b

According to process 1 up to the ith step, let $I = \{m \in \mathbb{N} \colon m \leq i\}$, then all the subscript sets form a power set of I, i.e., every subset of I is a subscript set. Therefore, $\forall J_\eta \in \{J_{\xi'}\} \implies \eta \subseteq I \implies \forall \sigma \subset \eta, \sigma \subset I \implies J_\sigma \in \{J_{\xi'}\}$, i.e., any subset of a subscript set is also a subscript set. □

Proof. Proof of Condition 3c

This condition is guaranteed by Process 2. Let J_η be a nonempty parting set currently dealt with in process 2.

(i) If no empty set is found in the temporary K.

(A) According to Condition 3b, $\forall \sigma \subset \eta, J_\eta \in \{J_{\xi'}\} \implies J_\sigma \in \{J_{\xi'}\}$.
According to Definition 4.1, $\sigma \subset \eta \implies \sigma$ is prior to η.
According to Process 2, $J_\sigma \in \{J_{\xi'}\}, \sigma$ is prior to $\eta \implies J_\sigma \neq \varnothing$.

(B) Because $\forall \sigma \subset \eta \implies \sigma$ is prior to η, so J_σ has been processed before this stage and would not be changed to empty set afterwards. And J_η itself would not be changed afterwards. Therefore, $J_\eta \neq \varnothing$ and $\forall \sigma \subset \eta, J_\sigma \neq \varnothing$ holds afterwards.

(ii) If a first empty set $J_{\eta'}$ is found in the temporary K. Then J_η would become empty, and $J_{\eta'}$ would be filled with all the primitive sets originally in J_η.

(A) (Reason iiA) Replace every 'η' with 'η'' in the first part of above proof, then a similar conclusion could be derived: $\forall \sigma \subset \eta' \implies J_\sigma \neq \varnothing$.

(B) (Reason iiB) Because $\eta' \subset \eta$, so $\forall \sigma \subset \eta' \implies \sigma \subset \eta \implies \sigma$ is prior to η, so J_σ has been processed before this stage and would not be changed to empty set afterwards. And $J_{\eta'}$ itself would not be changed afterwards. Therefore, $J_{\eta'} \neq \varnothing$ and $\forall \sigma \subset \eta', J_\sigma \neq \varnothing$ holds afterwards.

(C) J_η becomes empty set. Suppose it will be changed later, according to above Reason iiA and Reason iiB, it is supposed to be filled with contents of some other parting set to meet $J_\eta \neq \varnothing, \forall \sigma \subset \eta, J_\sigma \neq \varnothing$. And after that, J_η and $\forall \sigma \subset \eta, J_\sigma$ are supposed to keep unchanged. Therefore, at this point, a definite conclusion could already be got that either $J_\eta = \varnothing$ or $J_\eta \neq \varnothing, \forall \sigma \subset \eta, J_\sigma \neq \varnothing$ holds afterwards.

Since every parting set would be processed to meet either $J_\eta = \varnothing$ or $J_\eta \neq \varnothing, \forall \sigma \subset \eta, J_\sigma \neq \varnothing$, and this status would be maintained afterwards, we conclude that Condition 3c is guaranteed. □

Now that every condition is guaranteed at every finite step, we ask a question: Can this process be executed for all finite steps? Assume this process cannot be executed for all finite steps, then there should be some finite step at which this process has to stop, suppose it is the nth step. Then the process has been successfully executed for $n-1$ steps, but fails at the nth step. But after checking all the conditions, no condition could be violated in the nth step, which contradicts the step-n-stop assumption. So, we have to accept that this process can be executed for all finite steps to obtain a $\{J_\xi\}$.

Then we ask a second question: Does the $\{J_\xi\}$ obtained meet all the conditions? Assume $\{J_\xi\}$ violates some condition. Then we should have stopped before obtaining $\{J_\xi\}$, since our process is to assure all the conditions satisfied before updating $\{J_\xi\}$. So, we do not get $\{J_\xi\}$, which contradicts our set-$\{J_\xi\}$-got assumption. So, we have to accept that $\{J_\xi\}$ meets all the conditions.

Moreover, since $\{J_\xi\}$ is obtained after processes of all finite steps, following four conclusions can be proved:

(I) For any primitive set in J, the subscript set of the parting set it belongs to is definite.

Proof. For some primitive set in J, let the subscript set of the parting set containing it be denoted by η. Then it can be checked that the process might only add 'i' into η at the ith step. Other than that, an only operation the process might perform is to remove natural numbers from η. So that $\forall i \in \mathbb{N}$, the relationship between 'i' and η falls into three possible cases:

- 'i' is never added to η.
- 'i' is added to η at the ith step and remains in η.
- 'i' is added to η at the ith step and removed from η at some later step.

No matter which possibility, 'i' in η or not in η is definite after finite steps. So η is a definite set. □

(II) Every nonempty parting set in $\{J_\xi\}$ contains only one primitive set.

Proof. For any two different primitive sets in J, suppose the smallest natural number in one of them but not in another is 'm', then the two primitive sets would be moved into distinct parting sets during the mth step. Therefore, any two different primitive sets are in distinct parting sets, i.e., every nonempty parting set contains only one primitive set. □

(III) According to Condition 3a, all the nonempty parting sets in $\{J_\xi\}$ form a partition of J, therefore for any primitive set $\zeta \in J$, there exists a unique $J_\eta \in \{J_\xi\}$, $\zeta \in J_\eta$. Therefore, a mapping from primitive sets to subscript sets can be defined:

Definition 4.2. A mapping $g\colon J \mapsto g(J)$ is defined by an equivalent relation:

$$g(\zeta) = \eta \iff \zeta \in J_\eta,\ J_\eta \in \{J_\xi\}$$

It has been proved that every nonempty parting set in $\{J_\xi\}$ contains only one primitive set, so $\forall \zeta_1, \zeta_2 \in J$, $\zeta_1 \neq \zeta_2$, $g(\zeta_1) = \eta_1$, $g(\zeta_2) = \eta_2 \implies \zeta_1 \in J_{\eta_1}, \zeta_2 \in J_{\eta_2} \implies \zeta_2 \notin J_{\eta_1} \implies J_{\eta_1} \neq J_{\eta_2} \implies \eta_1 \neq \eta_2 \implies g(\zeta_1) \neq g(\zeta_2)$, therefore the mapping g is a bijection. In addition, $J_\eta \in \{J_\xi\}$, $J_\eta \neq \varnothing \iff \exists \zeta \in J$, $\zeta \in J_\eta$, $J_\eta \in \{J_\xi\} \iff \exists \zeta \in J$, $g(\zeta) = \eta \iff \eta \in g(J)$. In short, we have

$$J_\eta \in \{J_\xi\}, J_\eta \neq \varnothing \iff \eta \in g(J)$$

(IV) $\forall \eta \in g(J), 2^\eta \subseteq g(J)$.

Proof. Use Condition 3b, Definition 4.2, Condition 3c, and Definition 4.2 in sequence, $\forall \eta \in g(J)$, $\forall \sigma \subset \eta \implies \eta \in g(J)$, $\sigma \subset \eta$, $J_\sigma \in \{J_\xi\} \implies J_\eta, J_\sigma \in \{J_\xi\}$, $J_\eta \neq \varnothing, \sigma \subset \eta \implies J_\sigma \in \{J_\xi\}$, $J_\sigma \neq \varnothing \implies \sigma \in g(J)$. In short, we have

$$\forall \eta \in g(J), \forall \sigma \subset \eta \implies \sigma \in g(J)$$

which is equivalent to $\forall \eta \in g(J)$, $2^\eta \subseteq g(J)$. □

This $g(J)$ is a J^* satisfying Lemma 4.1. □

Then we ask a question: Is there a subscript set $\eta \in J^*$ containing infinitely many natural numbers?

Suppose there exists such a subscript set, then $card(\eta) = card(\mathbb{N})$. So $\eta \in J^* \implies 2^\eta \subseteq J^* \implies card(J^*) \geq card(2^\eta) = card(2^{\mathbb{N}}) \implies card(J) \geq card(2^{\mathbb{N}})$. On the other hand, $J \subseteq 2^{\mathbb{N}} \implies card(J) \leq card(2^{\mathbb{N}})$, therefore

$$card(J) = card(2^{\mathbb{N}}) = \aleph_1$$

Suppose there is no such a subscript set, in other words, $\forall \eta \in J^*$, η contains at most finitely many natural numbers, then count J^* according to a sequence:

$$\begin{gathered}\text{First count subscript sets in } \left\{\eta \in J^* \colon 0 = \textstyle\sum_{m\in\eta} m\right\},\\ \text{then count subscript sets in } \left\{\eta \in J^* \colon 1 = \textstyle\sum_{m\in\eta} m\right\},\\ \text{then count subscript sets in } \left\{\eta \in J^* \colon 2 = \textstyle\sum_{m\in\eta} m\right\},\\ \vdots\end{gathered}$$

So J^* is countable. Therefore, J is also countable, which means

$$card(J) = card(\mathbb{N}) = \aleph_0$$

Hence the cardinality of any subset of the power set of the set of natural numbers is either $\aleph_0$ or $\aleph_1$, the continuum hypothesis is proved.

5 Real number cardinality paradox

This paradox states that the set of real numbers could be larger than the power set of set of natural numbers. A lemma is proved first, in which a mathematical object in a sequence is called a 'bit'.

Lemma 5.1. *Let S be a set consisting of infinitely long sequences. Let $\bar{s}$ be an infinitely long sequence. If $\forall n \in \mathbb{N}, \exists s \in S$, the first n bits of s and $\bar{s}$ are the same, then $\bar{s} \in S$.*

Proof. Let s^* be a sequence. Initially, let s^* equal to an arbitrary sequence in S. Then process s^* step by step:

At the ith step $(i = 1, 2, 3, \dots)$, check in S whether there is a sequence s that at least the first i bits are the same with $\bar{s}$. If such a sequence s exists, update s^* with s. If there is no such a sequence, then stop the process.

Then we ask a question whether this process can be carried out for all finite steps? Assume this process cannot be carried out for all finite steps, it has to stop at some finite step. Let the step stopped be the nth step, then $n-1$ steps have been executed, but the nth step fails. A cause of failure must be: In S there is no sequence with the first n bits the same as $\bar{s}$. But this contradicts the premise. So, we have to accept that this step do not need to stop at any finite step.

Now s^* is a string obtained after all finite steps. Is this s^* in S? We have to accept $s^* \in S$, because at every step, s^* equals to a sequence in S.

Because s^* is obtained after all the finite steps, so $\forall i \in \mathbb{N}$ the first i bits of s^* and $\bar{s}$ are the same, namely $s^* = \bar{s}$. Therefore, $\bar{s} \in S$. □

The above Lemma 5.1 contradicts mainstream mathematical intuition, which judges infinite series by convergence. But it is a true result. S includes $\bar{s}$ is an inevitable consequence of that S contains sequences with arbitrary many first bits the same as $\bar{s}$. If we accept Lemma 5.1, then it will be shown that $\mathbb{R}$ could be larger than $2^{\mathbb{N}}$.

Henceforth in this proof, symbols 'i' and 'j' denote natural numbers $1, 2, 3, \dots$ Let H be a set formed by 2-parameter sequences (or tables) comprising infinitely many '0', '1':

$$H = \left\{\psi \colon \psi = \{h_{ij}\}, h_{ij} \in \{0, 1\}, j \leq 2^i\right\}$$

Define a mapping $f_1 \colon H \mapsto f_1(H)$ such that $\forall \psi \in H$,

$$f_1(\psi) = \sum_{i=1}^{\infty} \sum_{j=1}^{2^i} h_{ij} 3^{-2^i - j}$$

in which $h_{ij} \in \psi$. As $h_{ij} \in \{0,1\}$, so

$$f_1(\psi) \le \sum_{i=1}^{\infty}\sum_{j=1}^{2^i} 3^{-2^i-j} = \sum_{i=1}^{\infty} \frac{1}{2}\left(3^{-2^i} - 3^{-2^{i+1}}\right) = \frac{1}{2}3^{-2^1} = \frac{1}{18}$$

Therefore, $\forall \psi \in H$, $f_1(\psi)$ is defined by a bounded monotonic series, so $f_1(\psi) \in \mathbb{R}$, i.e., $f_1(H) \subseteq \mathbb{R}$.

$\forall b_{i_1 j_1}, b_{i_2 j_2} \in \psi$, $i_1 < i_2 \implies 2^{i_1} + j_1 - 2^{i_2} - j_2 < 2^{i_1} + 2^{i_1} - 2^{i_2} - 0 = 2^{i_1+1} - 2^{i_2} \le 0$. And $i_1 = i_2, j_1 \ne j_2 \implies 2^{i_1} + j_1 - 2^{i_2} - j_2 = j_1 - j_2 \ne 0$. So

$$i_1 \ne i_2 \vee j_1 \ne j_2 \implies 2^{i_1} + j_1 \ne 2^{i_2} + j_2$$

Therefore, in definition of $f_1(\psi)$ the exponents of '3' never repeat.

$\forall \psi_1, \psi_2 \in H, \psi_1 \ne \psi_2$, note $\psi_1 = \{h_{ij}\}$, $\psi_2 = \{\hbar_{ij}\}$. Compare bits of ψ_1 and ψ_2 in order of $\left(2^i + j\right)$, assume index of the first different bit is i^*, j^*, so

$$f_1(\psi_1) - f_1(\psi_2) = \sum_{i=1}^{\infty}\sum_{j=1}^{2^i} \left(h_{ij} - \hbar_{ij}\right) 3^{-2^i-j} = \left(h_{i^*j^*} - \hbar_{i^*j^*}\right) 3^{-2^{i^*}-j^*} + rst.$$

in which

$$\left(h_{i^*j^*} - \hbar_{i^*j^*}\right) 3^{-2^{i^*}-j^*} = \pm 3^{-2^{i^*}-j^*}$$

$$|rst.| \le \sum_{n > 2^{i^*}+j^*}^{\infty} 3^{-n} = \frac{1}{2} \cdot 3^{-2^{i^*}-j^*}$$

so $f_1(\psi_1) - f_1(\psi_2) \ne 0$. Therefore, f_1 is a bijection.

$\forall W \subseteq 2^{\mathbb{N}}$, assume there exists a bijection $f_2 \colon f_1(H) \mapsto W$. Then $f_2 \circ f_1$ is a bijection from H to W, note $f_3 = f_2 \circ f_1$.

Definition 5.1. $\forall \zeta \in W$, if $\forall h_{ij} \in f_3^{-1}(\zeta), j = 1 + \sum_{m\in\zeta}^{m\le i} 2^{m-1} \implies h_{ij} = 1$, then call ζ as coded. Otherwise, call ζ as non-coded.

Lemma 5.2. $\forall i$, $\forall \zeta \subseteq \mathbb{N}$, $0 \le 1 + \sum_{m\in\zeta}^{m\le i} 2^{m-1} \le 2^i$.

Lemma 5.2 ascertains the h_{ij} in Definition 5.1 always exists. Therefore, ζ is non-coded $\iff \exists h_{ij} \in f_3^{-1}(\zeta), j = 1 + \sum_{m\in\zeta}^{m\le i} 2^{m-1}, h_{ij} \ne 1 \vee \nexists h_{ij} \iff$

$$\exists h_{ij} \in f_3^{-1}(\zeta), j = 1 + \textstyle\sum_{m\in\zeta}^{m\le i} 2^{m-1}, h_{ij} = 0$$

Let F be a set consisting of all the non-coded ζ in W:

$$F = \left\{\zeta \colon \zeta \in W, \exists h_{ij} \in f_3^{-1}(\zeta), j = 1 + \textstyle\sum_{m\in\zeta}^{m\le i} 2^{m-1}, h_{ij} = 0\right\}$$

Because of Lemma 5.2, we can construct a $\psi^* \in H$ such that

$$\psi^* = \left\{h_{ij} \colon h_{ij} = 1 \iff \exists \zeta \in F, j = 1 + \sum_{m \in \zeta}^{m \le i} 2^{m-1}, \text{ otherwise } h_{ij} = 0\right\}$$

It is evident that $f_3(\psi^*) \in W$, because f_3 is a bijection. Then we ask a question about whether $f_3(\psi^*)$ is in F?

Assume $f_3(\psi^*) \in F$, then according to the definition of F,

$$\exists h_{ij} \in f_3^{-1}(f_3(\psi^*)), j = 1 + \sum_{m \in f_3(\psi^*)}^{m \le i} 2^{m-1}, h_{ij} = 0$$

but according to the definition of ψ^*, because $f_3(\psi^*) \in F$, so

$$\forall h_{ij} \in \psi^*, j = 1 + \sum_{m \in f_3(\psi^*)}^{m \le i} 2^{m-1} \implies h_{ij} = 1$$

This is a contradiction.

Assume $f_3(\psi^*) \notin F$, then $f_3(\psi^*)$ is coded, according to Definition 5.1,

$$\forall h_{ij} \in f_3^{-1}(f_3(\psi^*)), j = 1 + \sum_{m \in f_3(\psi^*)}^{m \le i} 2^{m-1} \implies h_{ij} = 1$$

And by definition of ψ^*,

$$\forall h_{ij} \in \psi^*, h_{ij} = 1 \implies \exists \zeta \in F, j = 1 + \sum_{m \in \zeta}^{m \le i} 2^{m-1}$$

therefore

$$\forall h_{ij} \in \psi^*, j = 1 + \sum_{m \in f_3(\psi^*)}^{m \le i} 2^{m-1} \implies \exists \zeta \in F, j = 1 + \sum_{m \in \zeta}^{m \le i} 2^{m-1}$$

By definition of H, the above relation is equivalent to

$$\forall i, \forall j \le 2^i, j = 1 + \sum_{m \in f_3(\psi^*)}^{m \le i} 2^{m-1} \implies \exists \zeta \in F, j = 1 + \sum_{m \in \zeta}^{m \le i} 2^{m-1}$$

By Lemma 5.2, the above relation is equivalent to

$$\forall i, j = 1 + \sum_{m \in f_3(\psi^*)}^{m \le i} 2^{m-1} \implies \exists \zeta \in F, j = 1 + \sum_{m \in \zeta}^{m \le i} 2^{m-1}$$

which is equivalent to

$$\forall i \implies \exists \zeta \in F, 1 + \sum_{m \in f_3(\psi^*)}^{m \le i} 2^{m-1} = 1 + \sum_{m \in \zeta}^{m \le i} 2^{m-1}$$

which is equivalent to

$$\forall i, \exists \zeta \in F, \sum_{m \in \zeta}^{m \le i} 2^{m-1} = \sum_{m \in f_3(\psi^*)}^{m \le i} 2^{m-1}$$

which is equivalent to

$$\forall i, \exists \zeta \in F, \{m \colon m \in \zeta, m \leq i\} = \{m \colon m \in f_3(\psi^*), m \leq i\}$$

which means $\forall n \in \mathbb{N}$, $\exists \zeta \in F$, the first n bits of ζ and $f_3(\psi^*)$ are the same, then according to Lemma 5.1, $f_3(\psi^*) \in F$. This contradicts the assumption.

Therefore, $\forall W \subseteq 2^{\mathbb{N}}$ there is no bijection between $f_1(H)$ and W, so there is no bijection between $\mathbb{R}$ and W. According to Definition 3.1, $\mathbb{R}$ is larger than $2^{\mathbb{N}}$.

The reason why Lemma 5.1 is needed in this paradox is that a set like $2^{2^{\mathbb{N}}}$ is required which is difficult to be constructed otherwise.

6 Natural numbers uncountable paradox

It will be shown in this section that natural numbers are larger than any set and therefore uncountable.

Lemma 6.1. *A set containing all the finite natural numbers contains an infinite number.*

Proof. Let $\bar{G}$ be the set concerned. Construct a set G from finite natural numbers step by step. Let G be an empty set initially. At the ith step ($i = 1, 2, 3, \cdots$), check $G \cup \{i\}$ by the following conditions:

(I) The maximum number in $G \cup \{i\}$ equals to the number of elements of $G \cup \{i\}$.

(II) Other conditions needed to make this paradox more convincing ...

If all the conditions are satisfied, then add 'i' into G, proceed to the next step. If any condition is violated, stop the construction.

After all the allowed steps, is G a finite set or not? Assume G is a finite set, then let m be the number of its elements. Because its maximum number equals to the number of its elements, so that the max number in G is m. Because $G \cup \{m+1\}$ meets all the conditions, so $m+1$ could be added to G to make G a set with $m+1$ elements. This contradicts m-number-elements assumption. So that G cannot be a finite set. G must be an infinite set with infinite number of elements, so that the maximum number in G is infinite.

Because at every step, the finite number added to G belongs to $\bar{G}$, so that $G \subseteq \bar{G}$. Therefore, the infinite number in G is also in $\bar{G}$. □

Theorem 6.2. *The set of natural numbers is larger than any infinite set.*

Proof. Because $\mathbb{N}$ contains all the finite natural numbers, by Lemma 6.1, there exists an infinite number in $\mathbb{N}$. Let N^* be a set[2] comprising all the infinite numbers in $\mathbb{N}$.

Assume $\mathbb{N}$ is not larger than a set S. Then by Definition 3.1 there is a bijection f^* to $\mathbb{N}$ from a subset of S, this mapping is also a bijection between N^* and another subset (denoted by S^*) of S. Then

$$f^* : S^* \mapsto N^*$$

Because $\mathbb{N}-N^*$ contains all the finite natural numbers, it follows Lemma 6.1 that $\mathbb{N} - N^*$ contains an infinite number n^*. And

$$n^* \in \mathbb{N} - N^* \implies n^* \notin N^* \implies n^* \notin f^*(S^*)$$

which contradicts f^* is a bijection between S^* and all the infinite numbers of $\mathbb{N}$. The assumption that $\mathbb{N}$ is not larger than a set leads to a contradiction, so that $\mathbb{N}$ is larger than any set. □

7 Cause of paradoxes

The previous five paradoxes account for a small portion of paradoxes hidden in mathematics. Before explaining the cause, four definitions are given.

Definition 7.1. (Motional Construction) If a process on a mathematical object satisfies:
(I) The process is carried out step by step.
(II) At each step, only finite number of attributes of the mathematical object is processed.
(III) There are finite number of conditions. At each step the process satisfies these conditions.
then this process is a motional construction.

Definition 7.2. (Set) A collection is a set if and only if: For any element, conclusions about the element belonging to it or not are the same through all reasoning paths.

Definition 7.3. (Class) A collection is a class if and only if: For any element, a conclusion about the element belonging to it or not is definite through any reasoning path.

[2]This set N^* can also be constructed from $\mathbb{N}$ by removing all finite numbers one by one.

Definition 7.4. (Infinite) In mathematics, if an attribute is larger than any finite number, then it is infinite.

In an ordinary definition of a set '$\{x : x \text{ meets conditions} \ldots\}$' or '$S = \bigcup_i S_i$', every property of every element is defined 'locally' such that every element is described with predefined elements, while elements defined afterwards are not mentioned. When discussing a finite set, such kind of 'locality' does not cause problems, because there must be a final element dominating the whole scope. But while dealing with an infinite set, there is no such a final element at hand. Thus, we are trapped in a dilemma that on one hand whole-scope properties are questioned, on the other hand all the tools available are local.

In order to overcome this difficulty, some assumptions were conjectured as axioms, such as counting infinite set by one-to-one correspondence of finite elements (Hilbert's hotel) and Cantor's infinite cardinality[3]. But as previous sections showed, such assumptions about infiniteness produces confusions rather than methods.

Consider to weigh a large carpet with a small carat scale. Even if weighing it region by region, data got is still meaningless. A task of weighing a carpet requires a scale large enough. In the motional construction (Definition 7.1), all the properties are assured globally at each step, so the difficulty caused by the locality of the ordinary set definition does not exist.

Infinity sometimes appears as magnitude, sometimes as fineness. An ordinary form defining a mapping or function like '$f(x) = formula(x)$' is also powerless for infinite problems. To describe all the relationships between infinitely many elements with expressions with finite symbols in finite length is like trying to measure circuits on a CPU chip with an ordinary ruler. A structure of nanoscale fineness cannot be measured with a ruler of millimeter resolution. Tools fine enough are required, such as the AC or MC, to identify relations between infinitely many elements.

Mathematical expressions are inventions for convenience, but not constraints. As long as every step is well-defined, a mathematical sequence is built successfully.

A theorem about infinity explains all the five paradoxes:

Theorem 7.1. *Any mathematical collection with infinitely many elements is not a set.*

Proof. Let X be a mathematical collection with infinitely many elements. Then by Motional Construction, an injection f from all the finite natural numbers to X can be established. Let X^* be all the elements in X mapped from finite natural numbers. Then $f^{-1}(X^*)$ consists of all the finite numbers. According to Lemma 6.1, there exists an infinite number n^* in $f^{-1}(X^*)$, therefore $f(n^*)$ exists and $f(n^*) \in X^*$.

(I) $f(n^*) \in X^* \implies f(n^*) \in X$.

(II) n^* is infinite and $\forall x \in X^*, f^{-1}(x)$ is finite $\implies n^* \notin f^{-1}(X) \implies f(n^*) \notin X^*$. Because $f(n^*) \in X^* \implies f(n^*) \notin X - X^*$, so $f(n^*) \notin X$.

Since the conclusions about $f(n^*)$ in X or not in X are different through different reasoning paths, X is not a set. □

Corollary 7.2. *A set is finite.*

Corollary 7.3. *The collection of real numbers is a class but not a set, so is the collection of rational numbers, integer numbers or natural numbers.*

Conclusions involving infinity are path-dependent. Reasoning by contradiction could be used, but it is only responsible for itself. Some equivalent statements or expressions in set math are not equivalent in class math, e.g., following statements about a class are not equivalent:

- x meets conditions Ω.
- $x \in \{x\colon x$ meets conditions $\Omega\}$.
- $x \notin \{x\colon x$ does not meet conditions $\Omega\}$.

Set implies finiteness. Not surprisingly a lot of infinite classes treated as sets in orthodox math would cause problems. A strictness of mathematics assumed for a long time has never been established. In order to show respect to work of predecessors, false ideas are interpreted herein as enlightening jokes intentionally devised for thorough understandings.

Definition 7.5. (Joking) A mathematical concept or result is called 'joking' in this article if it is defined or derived by strict reasoning, but with an invalid premise.

8 Set theory

There are many joking concepts and conclusions in set theory. Take the 'countable' as an example. In fact, every proof claiming that real numbers are uncountable presupposes the class of real numbers to be a power class (typically $3^{\mathbb{N}}$) of the class of natural numbers. A relation between sizes of any two infinite classes entirely depends on how the comparison is specified. If we represent real numbers with binary-decimals with digits indexed by $\log_2 \log_2 \mathbb{N}$, then the class of natural numbers is a power class of the class of real numbers. 'Countable' is a joking concept.

By Theorem 6.2, the class of natural numbers could be not only larger than the class of real numbers but also larger than itself. It is meaningless to discuss the size of an infinite class, because it could be larger than, smaller than or of the same size of any other infinite class at the same time.

The proof of continuum hypothesis in §4 is a paradox, because cardinality of infinity itself is a joking concept.

'Ordinal number' introduced by Gödel, is also a joking concept not only because of its infinite nature. An order between infinite classes inevitably implies a multidimensional structure which cannot be dealt in Gödel's way.

Forcing method[4] is an instance of the motional construction. The forcing method is logically valid and sound, but because of the class nature of objects in its implementations, results derived could only be joking.

Efforts executed in finding some hierarchy of infinite numbers could only yield joking results. In set theory, any two infinities are not distinguishable. In terms of size, '∞' is unique.

8.1 ZF axioms

ZF-axioms system[5, 6] is not logically consistent, since it contains an axiom of infinity. By Theorem 7.1 we claim:

Corollary 8.1. *Any logical system involving infinite elements is not consistent.*

ZF-axioms system is a local definition of sets, and even more local than a naive set theory. Due to this weakness when facing infinity, ZF-axioms system avoids some paradoxes by resisting mentioning them.

Axiomatic set theories attempt to eliminate paradoxes by imposing restrictions on math, but paradoxes are not caused by the approach towards infinity but by infinity itself. Hence these axiom systems are generally joking.

8.2 Axiom of Choice

Many paradoxes are attributed to the axiom of choice (AC), but AC is not needed for those paradoxes. Take the Banach-Tarski paradox as an example: AC is used to select a point from every rotation-equivalent set. Here shows a definite selection by motional construction without AC:

Let (r, x, y) be some coordinates of $\mathbb{R}^3$, for example the radical-spherical coordinate. For a set of rotational equivalence $\mathcal{S}$ on a sphere with radius r, at ith step, divide $\mathcal{S}$ into 4 parts:

$$\begin{cases} \mathcal{S}_1' = \{(r,x,y) \in \mathcal{S} \colon x \in [x_1, (x_1+x_2)/2], y_1, y \in [y_1, (y_1+y_2)/2]\} \\ \mathcal{S}_2' = \{(r,x,y) \in \mathcal{S} \colon x \in [x_1, (x_1+x_2)/2], y_1, y \in [(y_1+y_2)/2, y_2]\} \\ \mathcal{S}_3' = \{(r,x,y) \in \mathcal{S} \colon x \in [(x_1+x_2)/2, x_2], y_1, y \in [y_1, (y_1+y_2)/2]\} \\ \mathcal{S}_4' = \{(r,x,y) \in \mathcal{S} \colon x \in [(x_1+x_2)/2, x_2], y_1, y \in [(y_1+y_2)/2, y_2]\} \end{cases}$$

From these parts, select the one larger than the others as new $\mathcal{S}$ and update x_1, x_2, y_1, y_2. If no one is larger than the others, select the one with the smallest superscript as $\mathcal{S}$ and update x_1, x_2, y_1, y_2. This process must be able to be carried out for all finite steps, and after all the finite steps, the $\mathcal{S}^*$ obtained is not empty and contains only one point, otherwise contradictions would be deduced. So, we select a point without AC. This process could be applied for every paradox where elements are defined by parameters.

Such a selection was rejected by mathematicians because of a simple reason, e.g., let $\mathcal{S}_i = \left\{x \colon x \in \mathbb{R}, x \notin \mathbb{Q}, 1 - 2^{-i} \le x \le 1 + 2^{-i}\right\}$, then $\forall i, \mathcal{S}_i \neq \varnothing$, but $\bigcap_{i=1}^{\infty} \mathcal{S}_i = \varnothing$. But this is also a misconception of infinity.

A well-known form of AC is stated with mapping, but AC is not about mapping. It poses a question:

> *A set can be constructed by adding elements in any order. Is it allowed to pick an element at will from the set formed?*

Just take a glimpse is enough to know AC is innocent.

AC provides a random but fine probe into infiniteness. For a 'set', the only possible reason why AC derives a paradox is that: this 'set' is not a set at all. For over a century, AC has been addressing that many mathematical collections deemed as 'sets' are not sets, but its voice was ignored.

9 Logic

In mathematical logic, there are 'theorems' about completeness and consistency of 'formal systems', such as Gödel's 1st and 2nd incompleteness theorem[7]. So a natural question is whether these conclusions revealed the nature of infiniteness? The answer is no, because these theorems are not as they alleged.

In Gödel's proof, a set or relation S is strongly representable in a formal system F if there exists a formula $A(\cdot)$ in F, $\forall n \in \mathbb{N}$

$$n \in S \iff F \vdash A(n)$$

A job of mathematics is to categorize propositions into equivalent collections, but this definition of 'representable' messes up all the equivalent propositions with each other by not distinguishing them. There is a third logic system hiding in the ' $\iff$ ' which involves extra work to link an arithmetic function and a statement in the formal system. For instance, let

$$S_1 = \{n \in \mathbb{N} \colon \exists a, b, c \in \mathbb{N}, a^n + b^n = c^n\}$$

Let $A_1(n)$ be $n = 1 \vee n = 2$, then S_1 is represented by formula $A_1(n)$, if Fermat's last theorem is right. Another example, let

$$S_2 = \left\{n \in \mathbb{N} \colon n^6 - 2n^5 - 70n^4 + 14n^3 + 451n^2 - 20n - 630 = 0\right\}$$

(the formula of n is an expansion of $(n^2 - 2)(n^2 - 5)(n + 7)(n - 9)$). Let $A_2(n)$ be $n = 9$, then S_2 is represented by formula $A_2(n)$, if it is verified that $n = 9$ is the only natural number solution of $n^6 - 2n^5 - 70n^4 + 14n^3 + 451n^2 - 20n - 630 = 0$.

The minimization operator 'μ' together with the primitive recursive function packs a searching process in a short statement. In the proof of lemma of 'recursive functions are all representable', a time consuming statement involving remainder operations is constructed in order to represent a recursive function. The computational complexity increases rapidly with respect to 'n' in that statement. If statements with abnormal length are not acceptable, why such statements with abnormal time complexity is adopted?

Due to the packing ability of 'μ' operator, Gödel constructed a short statement for every recursive function. Then by a 'Gödel-numbering', a $K(n)$ that cannot be represented in F is constructed. But actually, this $K(n)$ is representable with an expression of infinite length. Let $g_1, g_2, g_3, \ldots$ be a sequence of all the Gödel numbers satisfying $K(n)$ in natural order, then $K(n)$ can be represented as:

$$n = g_1 \vee n = g_2 \vee n = g_3 \vee \ldots$$

The PA system does not reject infinite deductions. $\forall n \in \mathbb{N}, K(n)$ is derivable or not can be confirmed within finite steps of computations, so this statement is computable and really represents $K(n)$.

Gödel's theorems shall be revised as:

(I) Gödel's 1st incompleteness theorem: If F is consistent, then even if taking a very loose definition of 'representable', there exist statements unable to be represented with statements of finite length in F.

(II) Gödel's 2st incompleteness theorem: If F is consistent, then even if taking a very loose definition of 'representable', the consistency of F is unable to be represented with statements of finite lengths in F.

These theorems are about a particular kind of representability, but irrelevant to properties of F. Whereas, even these conclusions about a loose definition are not really obtained, because of the prerequisite 'consistent'.

A formal system is a synonym of ordinary mathematical language. It is exactly the math it symbolizes. Just like arithmetic notations and chess moves in a chess game, which are the same thing in different appearances. If a method adopted is included in the formal system, then this method should be carried out by symbol calculation of this formal system. If any mathematical method F' adopted is beyond the formal system F, then $F \cup F'$ is studied indeed. With properties about F in question (e.g., the consistency of F), how could a result derived by $F \cup F'$ be trusted? The only method allowed for studying a formal system is symbol calculation (methods of deductions defined by the formal system).

Theorem 9.1. *Consistency of any logical system could only be proved by enumerating all its deduction paths.*

Proof. Suppose there is a non-enumerating proof P_0 of consistency of a logical system F_0. Then P_0 is either in itself or in another larger logical system F_1. Suppose P_0 is in F_0 itself, then if F_0 is inconsistent then the non-enumerating proof P_0 does not prevent a non-consistency proof through other path.

Suppose P_0 is in another larger logical system F_1, then whether P_0 prevents a counter-proof depends upon the consistency of F_1. Suppose there is a non-enumerating proof P_1 of consistency of F_1, then the proof P_1 is in F_1 or in some larger logical system F_2. Maybe a logical systems sequence $F_1, F_2, F_3, F_4, \dots$ are concerned.

If there is no largest logical system, then this consistency question cannot be answered, because there is no final answer to consistency of any system. Even if there is a largest system F^*, suppose F^* is inconsistent then a non-enumerating proof P^* does not prevent a non-consistency proof through other path.

In logical systems sequence $F_1, F_2, F_3, F_4, \dots$, only an enumerating proof of consistency of some F_n could answer the consistency of F_0. Because $F_0 \subset F_n$, so an enumerating proof of consistency of F_n is also an enumerating proof of consistency of F_0. $\square$

Consistency of any logical system rejects a proof like the one Gödel provided. With the consistency of a logical system unknown, every theorem does not prevent its counter-theorem. Therefore, in some sense, any theorem in present mathematics might be a part of a paradox. And we cannot really know which one is or not, unless every deduction path has been enumerated. Besides since the proof of Theorem 9.1 is not enumerating, the Theorem 9.1 itself could also be a part of paradox.

Even if we ignore Corollary 8.1, in an infinite mathematics, the consistency of theorems without notion of infinity is likely unanswerable, because deduction paths are practically not enumerable.

Definition 9.1. (Origins of Paradoxes) Origins of paradoxes are independent paradoxes that cause all other paradoxes in a logical system.

Infinity implies paradoxes. But it does not mean everything of a non-set class is in disorder. '$P \wedge \neg P \vdash$ *all propositons*' is not a fact. In a non-consistent logic system, whether a statement is derivable or not shall also be verified by enumerating deduction paths. There could be many consistent propositions in a non-consistent logical system, as long as the deduction path bypasses all the origins of paradoxes. For example, the only paradox in the class of natural numbers or integer numbers is that it contains numbers which are both finite and infinite.

Undoubtedly, the proof of Gödel's 1st and 2nd incompleteness theorems are right. In general math, a convincing fact showed by these theorems are that a particular kind of representability is incapable of representing every proposition with finitely-long statement. But as far as mathematical logic is concerned, these theorems are joking.

In fact, if we look over their proofs, the definition of Gödel number is too causal, and the main lemma — Chinese remainder theorem is too weak. It is unrealistic to expect to solve a sophisticate problem, like the consistency or completeness of Peano arithmetic system, with causal and weak tools.

Representability shall be revised. Statement A in a formal system represents statement B in ordinary mathematics, only if symbols in A are one-to-one mapped with expressions in B and some criterions are satisfied:

(I) A shall be of equivalent statement length with B.
(II) A shall be of equivalent computation complexity with B.
(III) Other criterion required.

With a revised representability concept, it can be proved that many functions are not representable in PA system.

Lemma 9.2. *Exponential function is not representable in PA.*

Proof. Suppose A is a representation of an exponential function 2^n. Because A keeps equivalent computation complexity and there is no quantifier in 2^n, so there shall be no quantifier in A. Because A keeps equivalent statement length and 2^n comprises three symbols, so A shall comprise finite symbols. The largest number obtained in PA with m symbols is

$$\underbrace{n \times n \times \cdots \times n}_{(m+1)/2}$$

For sufficient large n, $2^n \gg n^{(m+1)/2}$, therefore 2^n is not representable in PA. □

Hyper operation is an extension of elementary operations, a natural way to define a binary hyper operator '$[i]$' by induction is cited here:

$$a[i]b = \underbrace{a[i-1]a[i-1]\cdots[i-1]a}_{b}$$

In hyper operations, the augments operated are defined on $\mathbb{N}$, and the binary operations are right-associative by default, e.g., $a[i]b[i]c = a[i](b[i]c)$. If let [1] represents the plus operation '+', then [2] represents the product operation'×', [3] represents the exponential operation, [4] represents the Knuth's up-arrow notation '↑'. All the hyper operations other than [1], [2] are not representable in PA system. A definition is given here to manifest how complicate the arithmetic is.

Definition 9.2. (Tower of arithmetic Operations)

(I) First level: An operator '$[i]$' is defined by

$$a[i]b = \underbrace{a[i-1]a[i-1]\cdots[i-1]a}_{b}$$

(II) Second level: An operator $[\kappa]$, that $\kappa = \{i_n, i_{n-1}, \cdots, i_2, i_1\}$, $i_j, n \in \mathbb{N}$, $m = 0, 1, 2, \cdots$ is defined by:

$$a[\kappa', i]b = \underbrace{a[\kappa', i-1]a[\kappa', i-1]\cdots[\kappa', i-1]a}_{b}$$

$$a[\kappa', i, \underbrace{1, \cdots, 1}_{m}]b = a[\kappa', i-1, \underbrace{b, \cdots, b}_{m}]b$$

(III) Third level: An operator $[\mu]$, that $\mu = \{\kappa_n; \kappa_{n-1}; \cdots; \kappa_2; \kappa_1\}$, $\kappa_j = \{i_{j,n_j}, i_{j,n_j-1}, \cdots, i_{j,2}, i_{j,1}\}$, $i_{j,p_j}, p_j, j, n \in \mathbb{N}$, $m = 0, 1, 2, \cdots$ is defined by:

$$a[\mu'; \kappa_1', i]b = \underbrace{a[\mu'; \kappa_1', i-1]a[\mu'; \kappa_1', i-1]\cdots[\mu'; \kappa_1', i-1]a}_{b}$$

$$a[\mu';\kappa'_1,i,\underbrace{1,\cdots,1}_{m}]b = a[\mu';\kappa'_1,i-1,\underbrace{b,\cdots,b}_{m}]b$$

$$a[\mu';i;\underbrace{1;\cdots;1}_{m};1]b = a[\mu';i-1;\underbrace{b;\cdots;b}_{m};\underbrace{b,\cdots,b}_{b}]b$$

(IV) Fourth level: ...

⋮

Another layer of levels ...

⋮

Theorem 9.3. *Arithmetic exhausts expressions.*

Proof. No matter what form is used to express an arithmetic operation sequence, an arithmetic operation of higher grade could be defined with respect to some traits of the formalization. Therefore, a new form (such as name, symbol, format, etc.) is needed. □

Given a rear occurrence like x^{y^z} in present math, the math world we have explored is at a preliminary stage of '[3]'. Rational numbers and algebraic numbers are roots of arithmetic equations at stage '[1]' and '[2]' respectively.

The tower of arithmetic operations is so endless that a question about whether it can be formalized is essentially a question about relative size of two infinite classes. Whether formalizable or not is not answerable, not to mention proving them with statements of finite length in PA.

The question about consistency of math cannot be answered, therefore faith in math is based on intuitions. But it does not undermine the importance of formalization, because mind resource is precious and formalization helps to lighten mathematicians' intuition burden. Logic is a final ruler which cannot be calibrated anymore, just like the light-speed meter. A real responsibility of mathematical logic is to formalize concepts and methods to make them computable and be learned in a deeper level. 'Computablization' of math techniques is important, because we need to enumerate more reasoning paths to understand the consistency of the whole math.

Definition 9.3. (Finite Proposition and Reasoning) A proposition or reasoning is finite if and only if it does not involve infiniteness.[3]

[3] This definition shall be explained. If a form like $\forall x \cdots$ can be represented as 'for any definite x which could be any number, there is a conclusion ...', then this is still a finite proposition, because it is a collection of finite propositions. If a reasoning involves steps that increases with respect to some index, but this index is definite for fixed premises, then this is still a finite reasoning, because it is a collection of finite reasonings.

Definition 9.4. (Finitely Consistent and Finitely Complete) A logical system (mathematics) is finitely consistent if and only if all the conclusions of finite reasonings are consistent. A logical system (mathematics) is finitely complete if and only if every finite proposition can be proved true or false by a finite reasoning.

Conjecture 9.1. *(Consistency and completeness of Mathematics) We can have a finitely consistent and finitely complete math.*

10 Analysis

Analysis is built on a concept called 'limit'. A limit is defined in $\epsilon - \delta$ language, which provides a way to extract a set from the class of real numbers. Thus, the paradox nature of real numbers is reined. 'Limit' is an example of MC.

There is a viewpoint that 'limit' is not a moving concept. But I have to point out that this is a misconception of not only 'limit' but also time. Time is essentially the sequence of changes. In a description of 'limit', a finite number of points are extracted from a class, all the other points are ignored. This is the way 'limit' reins real numbers. If we reject an obvious moving feeling of the 'limit' and take all the points into consideration, we have to face a class with paradox nature, and the most valuable essence of 'limit' is discarded. So, it is the only proper way to consider 'limit' in a motional view.

There are more properties about numbers in analysis than in set theory, so we distinguish two concepts: 'by motional construction (by MC)' and 'by specification (by SP)'. A concept by MC refers to that its meaning is logically all-path, so that MC can be used in its interpretation. A concept by SP refers to that its meaning shall be literally adhering to its description, therefore reasonings permitted are not all-path. It is worth noting that paradoxes revealed by MC are not caused by MC, but indeed implied by a concept itself.

A class definition by MC is generally a closure of one by SP (possibly with some paradox elements added).

$$X_{MC} = \overline{X}_{SP} \tag{2}$$

In orthodox math, however a class S is used, e.g., $x \in S, \ldots$ or $S \cap \ldots$, it would be interpreted back to the specification of S eventually, therefore a notation in orthodox math are always by SP. In this article, default meanings of statements and expressions are still by SP, unless otherwise specified.

In this section, several present concepts like 'metric space', 'Cauchy sequence', 'limit point', 'completeness', 'Archimedean property', 'closed', 'open', 'inner point', 'compact' and 'measure' are discussed, some are redefined. Several new concepts are introduced.

10.1 Completeness of real numbers

Definition 10.1. (Metric Space[4]) For a class X, there exists a mapping $\|\cdot,\cdot\| : X \times X \mapsto \|X, X\|$ and an operation '+' on $\|X, X\|$ satisfying:
(i) $\forall d_1, d_2 \in \|X, X\|, d_1 + d_2 \in \|X, X\|$
(ii) $\exists 0 \in \|X, X\|, 0 + 0 = 0$
(iii) $\forall d_1 \in \|X, X\|, d_1 \geq 0$
(iv) $\forall d_1, d_2, d_3 \in \|X, X\|, d_1 < d_2 \implies d_1 + d_3 < d_2 + d_3,\ d_3 + d_1 < d_3 + d_2$
(v) $\forall d_1, d_2 \in \|X, X\|, d_1 < d_2 \implies \exists d_3 \in \|X, X\|, d_3 > 0,\ d_1 + d_3 = d_2$
(vi) $\forall x_1, x_2 \in X, x_1 = x_2 \iff \|x_1, x_2\| = 0$
(vii) $\forall x_1, x_2 \in X, \|x_1, x_2\| = \|x_2, x_1\|$
(viii) $\forall x_1, x_2, x_3 \in X, \|x_1, x_2\| \leq \|x_1, x_3\| + \|x_2, x_3\|$
Then X is a metric space, and $\|\cdot,\cdot\|$ is a distance on X.

Several definitions identical with orthodox math[8] are cited here, then a complete theorem is proved (X is a metric space. $\epsilon \in \|X, X\|$. i, i_1, i_2 denote natural numbers.):
(I) (Cauchy Sequence) Sequence $\{x_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence in X, if and only if $\{x_i\}_{i\in\mathbb{N}} \subseteq X$ and $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall i_1, i_2 \geq N, \|x_{i_1}, x_{i_2}\| < \epsilon$.
(II) (Limit Point) A point x^* is a limit point of X, if and only if $\exists \{x_i\}_{i\in\mathbb{N}} \subseteq X, \forall \epsilon > 0, \exists N \in \mathbb{N}, \forall i \geq N, \|x_i, x^*\| < \epsilon$.
(III) (Complete) X is complete if and only if any Cauchy sequence in X has a unique limit point in X.

Theorem 10.1. *A metric space is complete.*

Proof. Let $\{x_i\}_{i\in\mathbb{N}}$ be any Cauchy sequence in a metric space X. Apply the motional construction with k and y. Begin the process with $k = 1$ and $y = x_1$. At the ith step, let $i' = i$, check whether $x_{i'}$ meets:
(I) i' and $x_{i'}$ exist.
(II) $x_{i'} \in \{x_i\}_{i\in\mathbb{N}}$
(III) $i' \geq i$
if $x_{i'}$ meets these conditions then replace k with i', replace y with $x_{i'}$. Otherwise stop the process. This process can be executed for all the steps that $i \in \mathbb{N}$, otherwise there would be contradictions. After all the steps, the k and

[4] This definition is summarized from the proof of Theorem 10.1.

y obtained exist and meet $y = x_k \in \{x_i\}_{i\in\mathbb{N}}$ and $\forall i \in \mathbb{N}, k \geq i$. $\{x_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence $\implies \forall\epsilon > 0, \exists N \in \mathbb{N}, \forall i_1 \geq N, \|x_{i_1}, x_k\| < \epsilon$. Therefore, y is a limit point of $\{x_i\}_{i\in\mathbb{N}}$.

Let x_a^*, x_b^* be two limit points of $\{x_i\}_{i\in\mathbb{N}}$, $\forall\epsilon > 0$,

If $\exists\epsilon_1, \epsilon_2 \in \|X, X\|$, $\epsilon_1 > 0$, $\epsilon_2 > 0$, $\epsilon = \epsilon_1 + \epsilon_2$, then $\exists N_1 \in \mathbb{N}$, $\forall i \geq N_1$, $\|x_i, x_a^*\| < \epsilon_1$ and $\exists N_2 \in \mathbb{N}$, $\forall i \geq N_2$, $\|x_i, x_b^*\| < \epsilon_2$, then $\exists N = \max(N_1, N_2), \in \mathbb{N}$, $\forall i \in \mathbb{N}$, $i \geq N$, $\|x_a^*, x_b^*\| \leq \|x_i, x_a^*\| + \|x_i, x_b^*\| < \epsilon_1 + \epsilon_2 = \epsilon$.

If $\nexists\epsilon_1, \epsilon_2 \in \|X, X\|$, $\epsilon_1 > 0$, $\epsilon_2 > 0$, $\epsilon = \epsilon_1 + \epsilon_2$, then by Property v of Definition 10.1, $\nexists\epsilon' \in \|X, X\|$, $0 < \epsilon' < \epsilon$, then $\exists N_1 \in \mathbb{N}$, $\forall i \geq N_1$, $\|x_i, x_a^*\| < \epsilon \implies \|x_i, x_a^*\| = 0$ and $\exists N_2 \in \mathbb{N}$, $\forall i \geq N_2$, $\|x_i, x_b^*\| < \epsilon \implies \|x_i, x_b^*\| = 0$, then $\exists N = \max(N_1, N_2) \in \mathbb{N}$, $\forall i \geq N$, $\|x_a^*, x_b^*\| \leq \|x_i, x_a^*\| + \|x_i, x_b^*\| = 0 + 0 = 0 \implies \|x_a^*, x_b^*\| < \epsilon$.

Therefore, in each case $\forall\epsilon \in \|X, X\|, \epsilon > 0, \|x_a^*, x_b^*\| < \epsilon \implies \|x_a^*, x_b^*\| = 0 \implies x_a^* = x_b^*$. Therefore, $\{x_i\}_{i\in\mathbb{N}}$ has a unique limit point in X. □

Containing the unique limit point is necessary for a metric space to contain a Cauchy sequence. But the limit point might not satisfy the specification of the metric space. In contrast with results in functional analysis, Theorem 10.1 gives a definite conclusion that a metric space is not completable, but always complete. The completeness in orthodox math actually refers to that every limit point also satisfies the specification of the metric space, which is a completeness by SP indeed.

Therefore, no matter what a class of real numbers is, it is complete as long as it is a metric space. But the completeness of ordinary real numbers $\mathbb{R}$ involves an extra property:

Axiom 1. *(Archimedean property)* $\forall x \in \mathbb{R}$, $\exists n \in \mathbb{N}$, $n > x$. *(It is equivalent to Euclidean statement* $\forall\epsilon \in \mathbb{R}$, $\epsilon > 0$, $\forall m \in \mathbb{N}$, $\exists n \in \mathbb{N}$, $\epsilon n > m$.*)*

Accepting Axiom 1 or not would lead to different real numbers. In this article, real numbers are defined by rational numbers. Rational numbers are obtained from natural numbers by elementary operations '+,−,×,/'. An order on rational numbers is derived from the order of natural numbers. A distance on rational numbers is the absolute value of difference which is also a rational number.

Definition 10.2. (Class of Ordinary Real Numbers) Class $\mathbb{R}$ is a class of ordinary real numbers if and only if

(I) MC is included in the specification of $\mathbb{R}$, i.e., $\mathbb{R}_{SP} = \mathbb{R}_{MC}$.

(II) $\mathbb{R} \supseteq \mathbb{Q}_{SP}$.

(III) $\mathbb{R}$ satisfies Axiom 1.

Definition 10.3. y keeps an order '$<$' on a collection X if and only if:
(I) $\forall x_1, x_2 \in X, y < x_1 \wedge x_1 < x_2 \implies y < x_2$
(II) $\forall x_1, x_2 \in X, y > x_1 \wedge x_1 > x_2 \implies y > x_2$

By Theorem 10.1, $\mathbb{R}$ contains limit points of all the Cauchy sequences of rational numbers. Moreover, two lemmas can be proved that

Lemma 10.2. *A limit point of a Cauchy sequence containing limit points of a Cauchy sequence of rational numbers is also a limit point of a Cauchy sequence of rational numbers.*

Lemma 10.3. *A Cauchy sequence of rational numbers keeps the order on rational numbers, and keeps the order inferred by elementary operations '+,−,×,/'.*

Therefore, all that $\mathbb{R}$ contains are limit points of Cauchy sequences of rational numbers[5]. But whether $\mathbb{R}$ contains all the numbers keeping rational number order is a question of completeness of ordinary real numbers which cannot be answered without Axiom 1.

Theorem 10.4. *(Completeness of Class of Ordinary Real Numbers) The class of ordinary real numbers by Definition 10.2 is complete, that is: $\forall y$, if y keeps the order of rational numbers, then $\exists y_x \in \mathbb{R}$, $|y - y_x|$ is defined and $|y - y_x| = 0$.*

Proof. Suppose there does not exist a rational number x meeting $x < y$, then $y = -\infty$. Suppose there does not exist a rational number x meeting $x > y$, then $y = \infty$. Otherwise, there exist rational number a, b meeting $a < y < b$. Suppose there exists a rational number x meeting $|x - y| = 0$, then x is the y_x required. If there does not exists a rational number x meeting $|x - y| = 0$, then begin with $x_a = a$, $x_b = b$, process x_a and x_b step by step. At the ith step, if $(x_a + x_b)/2 > y$, then $x'_a = x_a$ and $x'_b = (x_a + x_b)/2$. If $(x_a + x_b)/2 < y$, then $x'_a = (x_a + x_b)/2$ and $x'_b = x_b$. Check whether x'_a, x'_b meet conditions:
(I) $x'_a \in \mathbb{R}$.
(II) $|y - x'_a|$ is defined.
(III) $|y - x'_a| < (b - a)/2^i$.

[5] Owing to MC, all the properties of rational numbers are naturally extended to real numbers, because these properties could be simply added in the conditions of MC process of Definition 10.2.

If all these conditions are satisfied, then replace x_a with x'_a, replace x_b with x'_b, otherwise stop the process. These steps can be carried out for all steps $i \in \mathbb{N}$, and after all the steps, the x_a and x_b obtained shall satisfy all the conditions, otherwise there would be contradictions. So that x_a is in $\mathbb{R}$, $|y - x_a|$ is defined and $\forall i \in \mathbb{N}, |y - x_a| < (b-a)/2^i$. Because $\mathbb{R}$ meets Axiom 1, so $\forall \epsilon \in \mathbb{R}, \epsilon > 0, \exists i > (b-a)/(2\epsilon) \implies |y - x_a| < (2\epsilon i)/2^i < \epsilon \implies |y - x_a| = 0$. Then x_a is the y_x required. □

Therefore, y is really in $\mathbb{R}$ in terms of distance. The completeness of class of ordinary real numbers means an interval between numbers could be small enough but not too small that cannot be measured by rational numbers.

It seems that for ordinary real numbers, there are only two origins of paradoxes: One is that there exist elements being both finite and infinite. Another is that any limit point of a class is included in it no matter the limit point satisfies the class specification or not. So, paradoxes are about magnitude abnormally large or small, which is a merit of ordinary real numbers.

From Definition 10.2 it is clear that a class of real numbers is defined by Cauchy sequence in fact. Because any sub-sequence of a Cauchy sequence converges to the same limit, so a definition by monotonic Cauchy sequences or monotonic Cauchy sequences with monotonic increments up to any degrees is equivalent. Furthermore, if replace $\mathbb{Q}_{SP}$ in Definition 10.2 with any $\mathcal{Q}_{SP}$ meeting $\mathcal{Q}_{SP} \subseteq \mathbb{Q}_{MC}$, $\mathbb{Q}_{SP} \subseteq \mathcal{Q}_{MC}$, the same real numbers would be obtained.

If we reject Axiom 1, then we can get $\mathbb{R}^*$ which contradicts Theorem 10.4. $\exists \epsilon \in \mathbb{R}^*, \epsilon > 0, m \in \mathbb{N}, \nexists n \in \mathbb{N}, \epsilon n > m \implies \forall n \in \mathbb{N}, \epsilon n \leq m \implies \forall q \in \mathbb{Q}, q > 0, \epsilon < q \implies \forall q, p \in \mathbb{Q}, q > 0, p > 0, \epsilon p < q$, so this ϵ is in a dimension other than rational numbers. The 'ϵ' in limit definition represents an ordinary number, but once an ϵ is defined explicitly, it is not an ordinary number any more, because Axiom 1 is violated. This class of different real numbers is the one adopted in nonstandard analysis, in which there are more paradoxes. The extra dimensions could be a family $\{\epsilon_\alpha\}$. A question for non-standard analysis is that why only one ϵ is added but not a family $\{\epsilon_\alpha\}$.

There are other definitions for ordinary real numbers in history (e.g. Cantor's and Dedekind's) and several axioms about the completeness of ordinary real numbers. Among them, some can be revised with the motional construction, others are joking.

10.2 Point topology

Rules of operations for real numbers can be derived from Equation 2.

Theorem 10.5. *(Class Operations on Metric Space)* $\forall X_1, X_2 \subseteq \mathbb{R}, \forall x \in \mathbb{R}$, *within MC meaning:*

(I) x *meets conditions* $\Omega \implies x \in \{x\colon x \text{ meets conditions } \Omega\}$

(II) $x \in X_1 \wedge x \notin X_2 \implies x \in X_1 - X_2$

(III) $(X_1 \cup X_2)^c = X_1^c \cap X_2^c$

(IV) $(X_1 \cap X_2)^c = X_1^c \cup X_2^c$

(V) Operations with no complementary operation involved are the same as set operations.

The only difference from familiar set operations (or operations by specifications) is the complementary operation. $\forall X \neq \varnothing, X \neq \mathbb{R} \implies X \cap X^c \neq \varnothing$, because the boarder of X belongs to both X and X^c.

Now, a concise edition of the extending rod paradox can be stated:

Paradox 10.1. *(Extending Rod 2nd Edition) Separate a rod*

$$L = \{(x, y, z) \in \mathbb{R}^3\colon x^2 + y^2 \leq 1,\, 0 \leq z \leq 10\}$$

into two parts: $L_1 = \{(x, y, z) \in L\colon z \in \mathbb{Q}\}$ *and* $L_2 = \{(x, y, z) \in L\colon z \notin \mathbb{Q}\}$. *Move* L_2 *along* $z+$ *direction by 10 to obtain* L_3. *Because every metric space is closed, so* $L_1 \cup L_3 = \{L = (x, y, z) \in \mathbb{R}^3\colon x^2 + y^2 \leq 1,\, 0 \leq z \leq 20\}$ *is obtained, which is one time longer than* L.

In fact in the extending rod paradox of §2, $L_1 = L_2 = L$ and $A_3 = A_4 = A_2 = \left\{(x, y) \in \mathbb{R}^2\colon 0 \leq x \leq 1, 0 \leq y \leq 1\right\}$. In the Banach-Tarski paradox, every part of a sphere equals to the whole sphere. The cause of these paradoxes is that a 'set' is defined by some specification, but because of an 'infinite' property of the specification, the 'set' defined means the entire class indeed.

By Theorem 10.1, every metric space is closed. In orthodox math, a concept of 'closed' is about specifications, 'closed' refers to that a class by MC equals to the class by SP:

Definition 10.4. (Closed Specification) A specification Ω is closed if and only if a class $S = \{s\colon s \text{ satisfies } \Omega\}_{MC}$ meets $\forall x \in S, x$ satisfies Ω.

A statement like '...is a rational number' is not a closed specification, but class $\mathbb{Q}$ by MC is closed. Every class is closed by MC.

Definition 10.5. (Inner Point) In a metric space, a point x is an inner point of a class C if and only if $\exists \epsilon > 0, \forall x^*, \|x^*, x\| \leq \epsilon \implies x^* \in C$.

Open cover is not about what a class be covered by, but about how a class be covered. An updated edition of open cover is given:

Definition 10.6. (Open Cover) In a metric space, a class collection M is an open cover of a class X, if and only if $\forall x \in X, \exists C \in M$, x is an inner point of C.

Hence the compactness could be defined in the same way as in orthodox math:

Definition 10.7. (Compact) A class X is compact if and only if there is a finite open cover of X in any open cover of X.

Compactness is finiteness of dimensions and values for real numbers. The paradox about amount 'there is some elements both finite and infinite' does not exist in a compact class. Paradoxes about dimensions in an infinite-dimensional space do not exist in a compact class either. For instance, $Z = \{z\colon z = \{b_i\}, b_i = 0 \text{ or } 1, i \in \mathbb{N}\}$ is a infinite-dimensional space, a distance on Z is defined as $\|z_1 - z_2\| = \max\{|b_{1,i} - b_{2,i}| : b_{1,i} \in z_1, b_{2,i} \in z_2\}$. By motional construction, a sequence

$$\{1,0,0,0,\dots\}, \{0,1,0,0,\dots\}, \{0,0,1,0,\dots\}\dots$$

must have a limit point $\{b_i\colon b_i = 0, i \in \mathbb{N}\}$. This more unacceptable paradox does not exist in a compact class.

'Open' is not opposite to 'closed'. 'Open' means that a dimension over a class is the same as the dimension of the universal space, only in this way can a class has an 'inner'. It can be proved that the following definitions of openness are equivalent:

Definition 10.8. (Open) A class X is open if and only if any following criterion is satisfied *within MC meaning:*
(I) $\forall x \in X, x$ *is a limit point of inner points of* X*.*
(II) $X^{cc} = X$
(III) Let $Y(x,\epsilon) = \{y\colon \|y,x\| \leq \epsilon\}$*, then* $X = \bigcup_{Y \subseteq X} Y$*.*
(IV) $\forall x \in X, \forall \epsilon > 0$*,* $\{y \in X\colon \|y,x\| \leq \epsilon\}^{cc} \neq \varnothing$*.*

Due to the paradox nature of the real numbers, every proof above does not prevent a negative proof. Real numbers are complicate in nature. But for realistic applications such as calculus, a discrete real space is enough.

10.3 Measure theory

The A_2 (Equation 1) on the draft plane in §2 is a challenge for a measure theory.

Theorem 10.6. *Any measure, which is countable additive and displacement invariant, measures everything as zero or infinity.*

Proof. Let the measure be $\mathcal{M}$. Let the class measured be X. By Theorem 6.2 and MC, we can construct a bijection from $\mathbb{N}$ to X. Let x_i denote an element in X mapped from i. $\mathcal{M}(X) = \sum_{i=1}^{\infty} \mathcal{M}(x_i) = \sum_{i=1}^{\infty} \mathcal{M}(x_0)$. If $\mathcal{M}(x_0) > 0$ then $\mathcal{M}(X) = \infty$. If $\mathcal{M}(x_0) = 0$ then $\mathcal{M}(X) = 0$. □

'Countable' is a joking concept, so a measure theory based on 'countable' is also joking. In order to reduce paradoxes, a measure should be defined by counting open classes.

Definition 10.9. (Utmost Connectivity) A class X is utmost-connected if and only if $\forall S \subseteq X$, S can be continuously contracted in X to a point.

Theorem 10.7. *For any class X in a metric space, the following propositions are equivalent:*
(I) X is utmost-connected.
(II) X can be continuously contracted in itself to a point.
(III) X is bounded, and there exists a continuous mapping $f\colon X \times [0,1] \mapsto X$ satisfying $\forall x \in X, f(x,0) = x$ and $\forall x_1, x_2 \in X, \|f(x_1,1), f(x_2,1)\| < \|x_1, x_2\|$.

Definition 10.10. (Open Partition) An open partition of $\mathbb{R}^n$ is a class collection $T = \{t\}$ satisfying:
(I) $\forall t \in T$, t is open and utmost-connected.
(II) $\bigcup_{t \in T} t = \mathbb{R}^n$.
(III) $\forall t_1, t_2 \in T, t_1 \neq t_2 \implies t_1 \cap t_2$ has no inner point and $t_1 \cap t_2$ is utmost-connected.
Call every t as a box of T. If $\exists \lambda > 0$, the max distance of every box is no greater than λ, then call that the resolution of T is no greater than λ.

Definition 10.11. (Box Cover) $X \subseteq \mathbb{R}^n$. T is an open partition of $\mathbb{R}^n$. A box cover B of X in T is a collection of boxes: $B(X) = \{t \in T\colon t \cap X \neq \varnothing\}$.

Definition 10.12. (Box Neighborhood) In an open partition, a neighborhood of a box t is a set formed by all the boxes intersecting t.

Definition 10.13. (Inner Box and Border Box) Let $B(X)$ be a box cover of X, If all the boxes in neighborhood of t belong to $B(X)$ then t is an inner box. If a box t belongs to $B(X)$ but at least one box in neighborhood of t does not belong to $B(X)$, then t is a border box. Note all the inner boxes as B°, note all the border boxes as ∂B.

Definition 10.14. (Discrete Real Space) A discrete real space of $\mathbb{R}^n$ is an open partition $\mathbb{R}^n_\Lambda = \left\{t_{(k_1,\cdots,k_n)}\right\}$, in which $\Lambda = (\lambda_1, \ldots, \lambda_n), \lambda_i > 0$ and

$$t_{(k_1,\cdots,k_n)} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \colon k_i\lambda_i \leq x_i \leq k_i\lambda_i + \lambda_i, k_i \in \mathbb{Z}\}$$

Call Λ as a resolution of $\mathbb{R}^n_\Lambda$. Call analysis in a discrete real space as 'discrete analysis'.

A proposition can be represented in discrete analysis if and only if a concerned error tends to zero as the resolution Λ tends to zero. In discrete analysis, the box cover of a class is manipulated instead of the class itself. A box cover of a bounded class is a finite set which is closed, compact and most importantly open. After a problem is solved, the resolution Λ could be increased to a required level, so there would be no precision lost. Some propositions cannot be represented in discrete analysis, such as the extending rod paradox and Banach-Tarski paradox. It is a necessary condition for a proposition to be a paradox that it cannot be represented in discrete analysis. This gives a clue to find other paradoxes.

Definition 10.15. (Counting Measure 1) Let $\mathbb{R}^n_\lambda$ be a discrete real space of $\mathbb{R}^n$ with a resolution $(\lambda, \ldots, \lambda)$, in which $\lambda > 0$. A counting measure $\mathcal{M}_\lambda$ of X is: $\mathcal{M}_\lambda(X) = \sum_{t \in B(X)} \lambda^n$.

This counting measure can measure classes with different dimensions at the same time, because dimensional information is kept.

Definition 10.16. (Max Dimension) A max dimension $\mathcal{D}$ of a class X is

$$\mathcal{D}(X) = n - \lim_{\lambda \to 0} \frac{\ln(\mathcal{M}_\lambda(X))}{\ln \lambda}$$

if the limit exists.

Definition 10.17. (Zero Measure) A class Z is a zero-measure class with respect to a class X if and only if $\lim_{\lambda \to 0} \frac{\mathcal{M}_\lambda(Z)}{\mathcal{M}_\lambda(X)} = 0$.

Corollary 10.8. $\mathcal{D}(Y) < \mathcal{D}(X) \implies Y$ *is a zero measure class with respect to* X.

The counting measure 1 is not rotation-invariant, so a counting measure 2 is given by increasing the resolution:

Definition 10.18. (Counting Measure 2) $X \subseteq \mathbb{R}^n$. $\mathbb{R}^n_{\lambda_1}$ and $\mathbb{R}^n_{\lambda_2}$ are two discrete real spaces of $\mathbb{R}^n$ with resolution $(\lambda_1, \ldots, \lambda_1)$ and $(\lambda_2, \ldots, \lambda_2)$ respectively, in which $\lambda_1, \lambda_2 > 0$. Note the box cover of X in $\mathbb{R}^n_{\lambda_1}$ and $\mathbb{R}^n_{\lambda_2}$ as B_1 and B_2 respectively. For every box $t \in B_1(X)$, calculate the measure M_t of $X \cap t$ as follows:

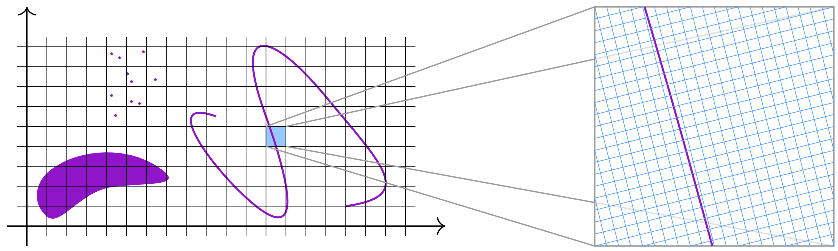

Figure 4: Counting Measure 2

(I) If every box in $B_2(X)$ intersecting t is an inner box, then $M_t = \lambda_1^n$.

(II) If among boxes in $B_2(X)$ intersecting t, there is a border box, then note a set of all the border boxes in $B_2(X)$ intersecting t as V_t.
Calculate the n-dimensional inertia matrix of V_t relative to the mass center of V_t. Then calculate an orthogonal matrix P for diagonalization of the inertia matrix. Define a new open partition by rotation and displacement of a copy of $\mathbb{R}^n_{\lambda_2}$: displace the origin of this copy at the center of box t, then rotate this copy to align its coordinates with vectors of P.
Note a box cover of $X \cap t$ in this open partition copy as B_2^t, then $M_t = \sum_{\tau \in B_2^t} \lambda_2^n$.

A counting measure $\mathcal{M}_{\lambda_1 - \lambda_2}$ of X is: $\mathcal{M}_{\lambda_1-\lambda_2}(X) = \sum_{t \in B_1(X)} M_t$.

The Figure 4 illustrates the counting measure 2. In practice, the resolutions shall meet $\lambda_2 \ll \lambda_1$. The counting measure 2 is selected from several candidates because of computation efficiency. Moreover, it keeps a dimensional relation, e.g., $\mathcal{M}_{\lambda_1-\lambda_2}(a\ square) = (\mathcal{M}_{\lambda_1-\lambda_2}(edge\ of\ the\ square))^2$.

Immeasurable classes manifest the inconsistency of a countable-additive measure theory. A consistent measure allows only finite additivity with errors. Even if a class is of infinite magnitude, its measure should be deemed as an addition of finite but arbitrarily many parts.

Only a class by MC should have a measure. Every class is measurable. But a measure of a class is also a class not a value. When the deviations are small, we deem the class of measure values as a single value. Since every class is closed by MC, errors could only be significant for non-open classes.

10.4 Calculus

Integral and differential are implementations of limit, so are all finite concepts. We only have Riemann integral[9] by now, because other integrals are more paradoxical.

In orthodox math, $\int_1^\infty x^{-2}dx = 1$, but this is not a definite result. If we interpret this integral as an area of $D = \{(x, y)\colon x \geq 1,\, 0 \leq y \leq x^{-2}\}$, then $D_1 = \{(x, 0)\colon x \in [1, +\infty]\}$ is a subset of D. Because of Theorem 6.2, D_1 contains more points than the whole x-y plane, so its area should be greater than the whole x-y plane. Then $\int_1^\infty x^{-2}dx$ is also infinite. In fact, we adopt

$$\int_1^\infty x^{-2}dx = \lim_{a\to\infty}\int_1^a x^{-2}dx$$

We have to specify the finite structure before altering a property to infinity. This is the only way to present a definite proposition relating to infinity.

Every limit exists by MC. Every differential, integral, limit of series exists. For example, this limit equals a class: $\lim_{x\to+\infty} x\sin(x) = [-\infty, +\infty]$. Another example is: $\lim_{x\to 0}\sin(1/x) = [0, 1]$.

The concept of limit existence in orthodox math actually refers to that the limit is a single value not a class of multiple values. Take the following limit as an example:

$$\lim_{\substack{x\to a\\ y\to b}} f(x, y)$$

A sufficient condition of interchangeability of limit operations usually used in orthodox analysis demands that the limit value is irrelevant to the way (x, y) approaching (a, b). Properties of limits depend on the finite structure they act on.

Consider a function similar to the Dirichlet function:

$$f_{12}(x) = \begin{cases} 1, & \text{if } x \text{ is irrational} \\ 2, & \text{if } x \text{ is rational} \end{cases}$$

It seems that $1 \leq \int_0^1 f_{12}(x)dx \leq 2$, because $1 \leq f_{12}(x) \leq 2$. Lebesgue integral says $\int_0^1 f_{12}(x)dx = 1$. But indeed $\int_0^1 f_{12}(x)dx$ could be any value between 1 and 3 depending on the definition of integral, because this f_{12} maps a point to a set: $f_{12}(x) = \{1, 2\}$.

Theorem 10.9. *Within MC meaning, a mapping is bijective if and only if it is continuous.*

Theorem 10.10. *Within MC meaning, a function on a finite domain of definition is bounded if and only if its function value is point-wise finite.*

A perfect definition of integral does not exist.

10.5 Mechanics

Owning to the Archimedean property of ordinary real numbers, mass points are all spinless in classical mechanics, material mechanics and fluid mechanics. Rotation is represented by relative motions of mass points. Even if we assign a spinning velocity to every point, there would be no difference in the principles of these mechanics.

Take the symmetry of continuous stress tensor as an example. When the size of a considered mass cube tends to zero, its rotation inertia reduces faster than the torch of stresses. Because neither an ϵ stress nor an ϵ^{-1} angular velocity can be expressed by ordinary real numbers, so the stress tensor must be symmetric. The symmetry of continuous stress tensor is an immediate consequence of the Archimedean property. A continuous but non-symmetric stress tensor could only exist in some non-standard analysis.

The linearity of continuous stress (Cauchy's theorem) is due to the same reason.

11 Topology

A new perspective of topology is introduced that topology is also a measure based on open partition.

A non-Euclidean geometry is a geometry in a subspace of Euclidean space. As long as a space is described by real numbers, it is a Euclidean space[10] in fact. In Euclidean space, the number of i-dimensional objects contained by an n-dimensional cube is $C_n^{n-i}2^{(n-i)}$. Suppose the counting measure of an i-dimensional object in a discrete real space is k^i, then the sum of measure of all the objects contained by this n-dimensional cube is $\sum_{i=0}^{n} C_n^{n-i}2^{(n-i)}k^i = (2+k)^n$.

Definition 11.1. (Topology Measure) Suppose the number of i-dimensional objects contained by an n-dimensional class A is m_i, then a topology measure of A is $\chi_k(A) = \sum_{i=0}^{n} m_i k^i$.

The 'k' in topology measure could be any complex number. Euler[11] characteristic $\chi_{-1}(A)$ is a topology measure with '-1' resolution. $\chi_{-2}(A)$ represents the number of points in a decomposition into basic i-dimensional cubes. Next we attempt to define another measure called 'corner measure'.

Definition 11.2. (Box Reshaping) B and b are two box collections in an open partition. A box reshaping is one of the two adjustments of B:

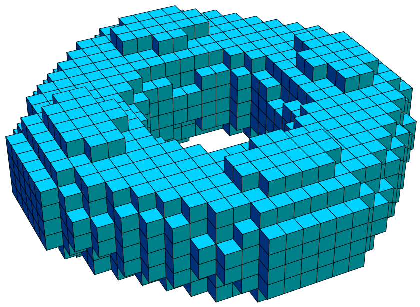

Figure 5: Box cover of torus body

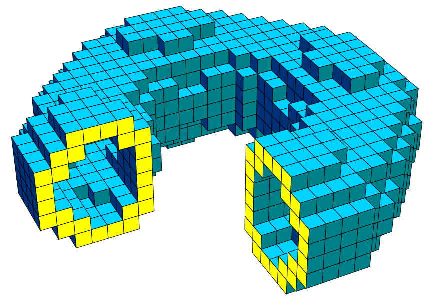

Figure 6: Box cover of torus surface (cut-away view)

(I) Adding: If b is utmost-connected, $t \cap B = \varnothing$ and $\bigcup_{t \in b} t \cap \bigcup_{t \in B} t$ is utmost-connected, then this b could be added to B.

(II) Removing: If b is utmost-connected, $t \subseteq B$ and $\bigcup_{t \in b} t \cap \bigcup_{t \in B-b} t$ is utmost-connected, then this b could be removed from B.

Definition 11.3. (Box Homeomorphism) X_1, X_2 are box-homeomorphic if and only if: $\forall \lambda > 0$, there exists an open partition T with a resolution no greater than λ, the box cover of X_1 and X_2 in T are $B(X_1)$ and $B(X_2)$ respectively, $B(X_1)$ and $B(X_2)$ can be box-reshaped to each other.

The box homeomorphism keeps connectivity, i.e., two box homeomorphic classes are either both n-connected or both not n-connected. Box homeomorphism does not distinguish dimensions. A point, a triangle and a ball are box-homeomorphic. If a topological class intersects itself, like the Klein bottle in $\mathbb{R}^3$, then there should be multiple copies of a box at the intersecting position . There are basic questions to be answered: For a class X, does there exist an open partition that $B(X)$ is box-homeomorphic with X? Is box homeomorphism an equivalence relation? Is box homeomorphism equivalent to homeomorphism for open classes... If all the questions are confirmed, then a class can be represented by a box cover that is box-homeomorphic with itself. Topology properties shall be studied through this box cover afterwards. The difference between simplex in orthodox topology and box is that a box is open and utmost-connected.

Definition 11.4. (Vertex) A vertex is a point belonging to at least $n + 1$ different boxes in an open partition of $\mathbb{R}^n$.

Definition 11.5. (Corner) A corner in an open cover is a set of all the boxes intersecting at one vertex.

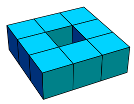

Figure 7: Box-homeomorphic of torus body

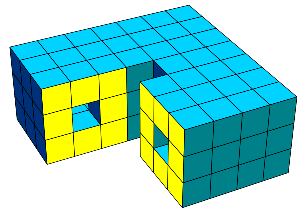

Figure 8: Box-homeomorphic of torus surface (cut-away view)

Definition 11.6. (Arithmetic Operation of Corners) For a box cover of a class, let C_k and $\widetilde{C}_k$ denote a corner at the same vertex before and after a box-reshaping. Corner '+' operation meets: $\sum C_k = \sum \widetilde{C}_k$. Commutative law: $\forall C_1, C_2, C_1+C_2 = C_2+C_1$. Cancellation law: $\forall C^*, C_i, C_j, C^*+\sum C_i = C^* + \sum C_j \iff \sum C_i = \sum C_j$. Multiplication law: $\forall C, \sum_m C = mC$.

Definition 11.7. (Trivial Corner) If a corner equals to $\varnothing$ by the arithmetic computation according to Definition 11.6, then it is a trivial corner.

'Corner' is a representation of 'vertex' in terms of open partition. Because '=' is defined by box-homeomorphic reshaping, so a corner $C = \varnothing$ if and only if $C = \sum C_k$ and C_k are all the corners at some vertices. In a box cover of a compact class, there exists a corner with the minimal volume angle, so that it cannot equal to $\varnothing$ and is non-trivial. Hence, the arithmetic operation of corners is not nonsense and provides some information. Definition 11.6 is an exemplary corner measure.

A Euclidean space is described by n independent real numbers, so its topology could be described in a discrete real space. In 2-d or 3-d discrete real space, all the corners can be enumerated.

If $B(X)$ is box-homeomorphic with X, then check the corners of $B(X)$ is enough. For example, Figure 5 shows a box cover of a slightly twisted torus body. Figure 6 shows a box cover (in cut-away view) of a torus surface. By box-reshaping, they are box-homeomorphic with Figure 7 and Figure 8 respectively. It can be verified that by Definition 11.6, they are both $\varnothing$ (i.e., adding or removing a torus does not change the sum of corners).

An n-dimensional X can be deemed as an n-dimensional knot in $\mathbb{R}^{n+2}$. The common knots only exist in $\mathbb{R}^3$, so can be represented by a sequence of turning boxes with 3 coordinates. In a box cover of a knot, a turning box is a box with non-trivial corners. For example, Figure 9 shows a box cover of

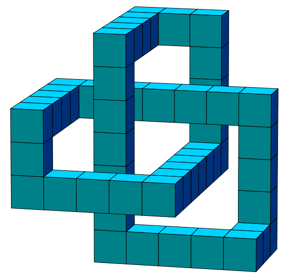

Figure 9: Box cover of left-hand trefoil

a left-handed trefoil $\mathcal{K}$. It could be represented by a $3 \times n$ matrix:

$$\mathcal{K} = \begin{bmatrix} 2 & 2 & 4 & 4 & 4 & 1 & 1 & 1 & 3 & 3 & 3 & 2 \\ 4 & 1 & 1 & 1 & 3 & 3 & 3 & 2 & 2 & 2 & 4 & 4 \\ 3 & 3 & 3 & 2 & 2 & 2 & 4 & 4 & 4 & 1 & 1 & 1 \end{bmatrix} \tag{3}$$

In matrix $\mathcal{K}$, a column denotes the three coordinates of a turning box, a row represents relative positions of turning boxes along a coordinate axis. Thus knot transformations could be computed by manipulating an integer matrix. A simpler form is

$$\mathcal{K} = \begin{bmatrix} & & 4 & & & 1 & & & 3 & & & 2 \\ & 1 & & & 3 & & & 2 & & & 4 & \\ 3 & & & 2 & & & 4 & & & 1 & & \end{bmatrix} \tag{4}$$

or

$$\mathcal{K} = \begin{bmatrix} 3_3 & 1_2 & 4_1 & 2_3 & 3_2 & 1_1 & 4_3 & 2_2 & 3_1 & 1_3 & 4_2 & 2_1 \end{bmatrix} \tag{5}$$

Every math counts something. Real analysis counts areas, complex analysis counts angles, topology counts corners. Quantitative theorems could be explained from a new viewpoint, e.g., Gauss-Bonnet theorem[12, 13] states that if corners are averaged by area and accumulated, then the result equals to counting corners directly.

12 Algebra

'Extract a set from a class, then study the set' is a basic method, e.g., the mathematical induction. In algebra a proof usually presumes a finite number that could be arbitrary large. For example, in proofs of Diophantine approximation problems, it is necessary to take algebraic equations with fixed degrees as a premise, otherwise because all the algebraic numbers generate the class of real numbers, a proof is unimaginable.

Theorem 12.1. *Let X be an algebraic closed field. If a real number belongs to X, then $\mathbb{R} \subseteq X_{MC}$. If a complex number belongs to X, then $\mathbb{C} \subseteq X_{MC}$, (i.e., 'algebraic closed field' refers to the whole complex plane within the meaning of MC).*

Proof. If X contains a real (or complex) number, then a sequence of algebraic equations could be constructed to make roots dense in real (or complex) numbers. □

In algebra geometry, finite separations of an infinite field is used. This is another way to extract a set from a class: represent a class with finite number of sub-classes, then study this set of sub-classes.

Next, we attempt to delve into a deeper infinite world. Several definitions and an axiom are introduced to give a 'proof' of one Erdős conjecture[14].

Definition 12.1. (Limit of Motional Construction) Let x be a mathematical object processed by an MC process p, then

$$\operatorname*{limc}_{p\to\infty} x$$

represents an x obtained after all the finite steps.

Definition 12.2. (Distantly Exclusive in Logic) Let x_1, x_2 be two values of a mathematical object x, if there exists $\epsilon \geq 0$ that for any MC process p_1, p_2, $\operatorname{limc}_{p_1\to\infty} x = \lim_{i\to\infty} x_{1i} = x_1$, $\operatorname{limc}_{p_2\to\infty} x = \lim_{j\to\infty} x_{2j} = x_2$, $\exists N \in \mathbb{N}, \forall i, j \geq N, \|x_{1i}, x_{2j}\| \geq \epsilon$, then x_1 and x_2 are distantly exclusive.

Definition 12.3. (Finitely imply) Proposition P_2 is a corollary of proposition P_1 in a logical system. If P_2 is consistent with all conclusions of finite reasonings, then represent this fact as $P_1 \underset{FD}{\Longrightarrow} P_2$.

Axiom 2. *In a finitely consistent logical system (finitely consistent mathematics), $\operatorname*{limc}_{p\to\infty} x = x_1$, x_1 and x_2 are distantly exclusive $\underset{FD}{\Longrightarrow} \operatorname*{limc}_{p\to\infty} x \neq x_2$.*

Conjecture 12.1. *(Erdős) If a subclass S of natural numbers satisfies $\sum_{n\in S} \frac{1}{n} = \infty$, then S contains arbitrarily long arithmetic progressions.*

Proof. Construct a subclass W of natural numbers step by step. Begin with $W = S$ and $d = 1$. At the ith step ($i \in \mathbb{N}$), check whether $i|d$. If $i|d$, then do nothing and proceed to the next step. If $i \nmid d$, then let $k = i/\gcd(i, d)$, divide W into k parts:

$$W_1 = W \cap \{kj + 1 \colon j = 0, 1, 2, \cdots\}$$
$$W_2 = W \cap \{kj + 2 \colon j = 0, 1, 2, \cdots\}$$
$$\vdots$$
$$W_k = W \cap \{kj + k \colon j = 0, 1, 2, \cdots\}$$

Because $\sum_{n\in W} \frac{1}{n} = \infty$, so among these classes there must be at least one class W^*, $\sum_{n\in W^*} \frac{1}{n} = \infty$. Let $d^* = kd$. Check whether W^* and d^* meet all the following conditions:

(I) $\sum_{n\in W^*} \frac{1}{n} = \infty$.

(II) For any progression interval a of W^*, $a \in \mathbb{N}$ and $d^*|a$.

(III) $i|d^*$.

(IV) $W^* \subseteq S$.

If W^* and d^* meet all the above conditions, then replace W with W^*, replace d with d^*, otherwise stop the process.

Assume the process ceases at a finite step, then contradictions would be derived. Therefore, this process shall be executed for all finite steps. Note a class obtained after all the finite steps as $\overline{W} = \mathrm{limc}_{p\to\infty} W$. Because for any progression differences a_1, a_2 of $\overline{W}$, we have $d|a_1$, $d|a_2$ and $\forall i \in \mathbb{N}$, $i|d$, so that $\forall i \in \mathbb{N}, i|a_1, i|a_2$. On the other hand, $a_1, a_2 \in \mathbb{N}$, so that $a_1|a_2, a_2|a_1$, so that $a_1 = a_2$. Then $\overline{W}$ is an arithmetic progression. $\sum_{n\in\overline{W}} \frac{1}{n} = \infty$, ∞ and 0 are distantly exclusive $\underset{FD}{\Longrightarrow}$ $\mathrm{limc}_{p\to\infty} W \neq 0$. This means $\sum_{n\in\overline{W}} \frac{1}{n} > 0$. Because the common difference of $\overline{W}$ can be arbitrarily large, so that there are arbitrarily long arithmetic progressions in $\overline{W}$. Because $\overline{W} \subseteq S$, so that S contains arbitrarily long arithmetic progressions. □

Many paradoxes can be obtained through reasoning paths involving infinity. A distantly exclusive reasoning seems to be more trustworthy than the others.

A job of math is to find a finite structure governing a concerned proposition. But it is not easy to find a finite structure governing Erdős conjecture. Because $\sum_{n\in S} \frac{1}{n} = \infty \iff \exists S^* \subset S,\ \sum_{n\in S^*} \frac{1}{n} = \infty$ and the difference values of progression S^* can be arbitrary large, so a subclass S^* could always be untouched, beating our effort of deducing a conclusion. In order to obtain a finite proof of Erdős conjecture, we have to overcome this obstacle.

13 Other Topics

All the three mathematical crises are due to infinity. This article aims to demonstrate a fact: Infiniteness implies inconsistency, which is a nature of infinity and cannot be avoided by adjusting a strategy of getting the infinity. Infinite math is paradox math.

Paradox 13.1. *Operations of abstract mathematical ideas are carried out by manipulating primitive intuitions in mathematicians' minds.*

Finiteness is intuitive in nature, but it is risky to presume an intuitive vision of infinity. A reliable way to study infinity is to construct a finite structure then expand it to infinity. If paradoxes derived are acceptable then we adopt the structure, otherwise we look for something else. AC and MC are the tools we have to explore infiniteness at present. Most proofs in this article are prototypes of exploiting MC. Practically, finiteness is all we have in math, so count it and make it count.

This is a brief introduction to a math with an updated understanding of infiniteness. Many topics are not covered, such as in probabilistic theory a lot of conclusions shall be reexamined because of the problem of measure theory, in group theory all set groups are finite groups, etc.

Definitions and Notations in this article are exemplary and temporary, e.g., it is a choice whether to use a new name 'class' or to update the concept of 'set'. This is the reason why 'set' is used in the first few sections while 'class' is used in the subsequent sections.

14 Author's Opinion

Existence of infinity is a problem in philosophy as well as in reality. If the world we live in is invariant both in rotations and in translations, then real numbers might exist. Otherwise, infinity is an approximation of a world with resolution.

Mathematics is a language to interpret sophisticate problems with simply trusted intuitions, but not to design a world. Any theory that is neither intuitively inspiring nor practically applicable is highly probably a joking one, because being applicable in mind or reality is an ultimate approval of a theory, i.e., an approval by our world's creator — God.

Logic is an abstraction of everyday life, but not a law of the world. There are many omnipotence paradoxes. For example, 'Could God create a rock he cannot lift up? No matter the answer is yes or no, God are not omnipotent.'

I disagree with this notion, because if God reply to your challenge, you might get a class rock. Omnipotence of the Almighty is beyond the thinking ability of humankind. For me, God is not a belief but real. The way to believe in God is to fight for truth and justice. Here is a fact I see:

God bless everyone fighting for truth and justice.

Math has been clumsy for a long time. Maybe someday when all the misunderstandings are corrected, math could become neat. We need a neat math because it is the only way for math to be prosperous, and a prosperous math is an indispensable move towards human civilization's new eve.

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