Thermal Relativity Revisited : Relativity of Entropy, Negative Mass, Entropy and Temperature

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Abstract

We review the basics of Thermal Relativity Theory and describe many of the novel physical consequences, including the *corrections* to the Schwarzschild black hole entropy. One of the most salient results is the existence of a minimal area $\frac{L_P^2}{4\pi}$, where L_P is the Planck length. The brief review paves the way to the study of the thermal relativistic analog of Lorentz transformations. A careful examination of these transformations imply that entropy is observer *dependent* and that one must also include negative masses, entropy and temperature in the formalism. A review of the literature on negative masses, entropy and temperature follows. In special relativity one has the equivalence of mass and energy. While in thermal relativity one finds an equivalence of proper thermal mass $\mathcal{M}$ (not to be confused with ordinary mass) and proper entropy $\mathbf{s}$. We conclude with a brief description of Born's Reciprocal Relativity Theory (BRRT), based on a maximal proper-force, a maximal speed of light, inertial and non-inertial observers, and explain how to extend the formulation of thermal relativity described in this work to cotangent spaces.

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1 A Brief Introduction to Thermal Relativity

The idea of describing classical thermodynamics using geometric approaches has a long history [1]. Among various treatments, Weinhold [3] used the Hessian of internal energy to define a metric for thermodynamic fluctuations. Ruppeiner

[4] used the Hessian of entropy for the same purpose. Quevedo [5] introduced a formalism called Geometrothermodynamics (GTD) which also introduces metric structures on the configuration space $\mathcal{E}$ of the thermodynamic equilibrium states spanned by all the extensive variables. The Quevedo metric is obtained via the pullback of the metric from the $2n+1$-dimensional thermodynamic phase space $\mathcal{T}$ (comprised of n extensive variables, n intensive variables, and the thermodynamic potential) to the n-dim configuration space $\mathcal{E}$. Geometrothermodynamics differs from earlier approaches in that it implements an invariance under Legendre transformations at the fundamental level. Unfortunately, one of the essence of Riemannian geometry, i.e. invariance under continuous coordinates transformations was not discussed in this picture.

For this reason Zhao [1] outlined the essential principles of Thermal Relativity; i.e. invariance under the group $\mathcal{G}$ of general coordinate transformations on the thermodynamic configuration space, and *introduced* a metric with a *Lorentzian* signature on the space. The line element was identified as the square of the *proper entropy*. Thus the first and second law of thermodynamics admitted an invariant formulation under general coordinate transformations, which justified the foundations for the principle of Thermal Relativity.

Following the proposal by [1] it allowed us [2] to derive the *exact* Thermal Relativistic *corrections* to the Schwarzschild, Reissner-Nordstrom, Kerr-Newman black hole entropies. In this work we shall display the thermal analog of Lorentz transformations and examine the physical implications. One finds that it is necessary to extend the positive domain of absolute temperatures, masses and entropy, to *negative* values. Furthermore, one arrives at the conclusion that thermodynamic entropy must be observer-dependent. Namely, an accelerated observer will assign a different entropy to a physical system than an inertial observer devoid of acceleration.

We review briefly the results in [2] and derive the exact Thermal Relativistic corrections to the Schwarzschild black hole entropy. It is important to emphasize that the thermodynamic configuration space (thermodynamic manifold) associated with the Schwarzschild black hole solution is $flat$ despite that the Schwarzschild metric itself is not flat.

In what follows we shall adopt the units $\hbar = c = k_B = 1$. Then one may implement Zhao's formulation [1] of Thermal Relativity to the flat analog of Minkowski space by starting with the infinitesimal thermal interval

$$(d\mathbf{s})^2 \ = \ (T_P dS)^2 \ - \ (dM)^2 \ \leftrightarrow \ (d\tau)^2 \ = \ (cdt)^2 \ - \ (dx)^2 \qquad (1.1)$$

The maximal Planck temperature T_P plays the role of the speed of light, and $\mathbf{s}$ is the so-called *proper entropy* which is invariant under the thermodynamical version of Lorentz transformations [1]. Note the $\mathbf{s} \leftrightarrow \tau$ correspondence. Thus the flow of the proper entropy $\mathbf{s}$ is consistent with the arrow of time.

The left hand side of (1.1) yields, after recurring to the first law of Thermodynamics $TdS = dM$, and factoring out $(T_P dS)^2$, the following

$$(d\mathbf{s})^2 \;=\; (T_P dS)^2 \left(1 \;-\; \frac{T^2}{T_P^2} \right) \;\Rightarrow\; (d\mathbf{s}) \;=\; (T_P\ dS) \sqrt{\left(1 \;-\; \frac{T^2}{T_P^2} \right)} \;=$$

$$T_P\ (\frac{dM}{T}) \sqrt{\left(1 \;-\; \frac{T^2}{T_P^2} \right)} \;\Rightarrow\; dM \;=\; \frac{T}{T_P} \frac{1}{\sqrt{1 - \frac{T^2}{T_P^2}}}\ d\mathbf{s} \qquad (1.2)$$

Eq-(1.2) is the one we shall use in order to *derive* the thermal relativistic corrections to the Black Hole Entropy.

The numerical factor $(1 - \frac{T^2}{T_P^2})^{-\frac{1}{2}}$ is the *thermal dilation* factor γ_T analog of the Lorentz dilation factor $(1 - \frac{v^2}{c^2})^{-\frac{1}{2}}$, and such that

$$1 \;\leq\; \gamma_T \;\equiv\; (1 - \frac{T^2}{T_P^2})^{-\frac{1}{2}} \;\leq\; \infty, \;\; 0 \;\leq\; T \;\leq\; T_P \qquad (1.3)$$

Given the thermal dilation factor one can always define an “effective” temperature as

$$T_{eff} \;=\; \frac{T}{\sqrt{1 - \frac{T^2}{T_P^2}}} \qquad (1.4)$$

such that the last term in eq-(1.2) $dM \;=\; \gamma(T)T(d\mathbf{s}/T_P)$ becomes then the thermal relativistic analog of the Energy-Momentum relations $E = m_o c^2(1 - \frac{v^2}{c^2})^{-\frac{1}{2}}$, $\vec{p} = m_o\vec{v}(1 - \frac{v^2}{c^2})^{-\frac{1}{2}}$ in Special Relativity, in terms of the rest mass m_o, velocity v, and maximal speed of light c. This line of reasoning behind eq-(1.2) is what leads to the notion of a “Thermal Relativity Theory”, in agreement with [1], and which must not be confused with other notions of “thermal Relativity”, ”thermal gravitation” in the past.

In the case of black holes, the thermal-dilation factor γ_T is given in terms of the Hawking temperature T_H

$$\gamma_T \;\equiv\; \frac{1}{\sqrt{1 - \frac{T_H^2}{T_P^2}}} \qquad (1.5)$$

For a Schwarzschild black hole, one has

$$T_H \;=\; \frac{1}{2\pi} a \;=\; \frac{1}{4\pi} |(\frac{\partial g_{tt}}{\partial r})|_{r = r_h = 2GM} \;=\; \frac{1}{8\pi GM} \qquad (1.6)$$

Thus, after renaming $\tilde{S} \equiv (\mathbf{s}/T_P)$, in terms of the *proper* entropy $\mathbf{s}$, the first law of black hole thermal-relativistic dynamics $dM \;=\; \gamma(T_H)T_H d\tilde{S}$ yields the *corrected* entropy

$$\int_{\tilde{S}_o}^{\tilde{S}} d\tilde{S} \;=\; \tilde{S} - \tilde{S}_o \;=\; \int_{M_o}^{M} \frac{dM}{\gamma(T_H)T_H} \;=\; \int_{M_o}^{M} dM\ \frac{\sqrt{1 - (T_H^2/T_P^2)}}{T_H} \qquad (1.7)$$

By inserting $T_H(M) = (8\pi GM)^{-1}$ into eq-(1.7) leads, after setting $(T_P)^{-2} = (M_P)^{-2} = L_P^2 = G$, to the following integral

$$\tilde{S} - \tilde{S}_o \ = \ \int_{M_o}^{M} dM \ (8\pi GM) \sqrt{1 - \frac{G}{(8\pi GM)^2}} \ = \ \int_{M_o}^{M} dM \ \sqrt{(8\pi GM)^2 \ - \ G} \tag{1.8}$$

The indefinite integral

$$\int dx \ \sqrt{a^2x^2 \ - \ b} \ = \ \frac{ax \ \sqrt{a^2x^2 \ - \ b}}{2a} \ - \ \frac{b}{2a} \ ln \Big(\ a \ [\sqrt{a^2x^2 \ - \ b} \ + \ ax] \ \Big) \tag{1.9}$$

permits to evaluate the definite integral in the right hand side of (1.8) between the *upper* limit M, and a *lower* limit M_o defined by $(8\pi GM_o)^2 - G = 0$, giving

$$\tilde{S} - \tilde{S}_o \ = \ \frac{A}{4G} \sqrt{1 - \frac{1}{16\pi}(\frac{A}{4G})^{-1}} - \frac{1}{16\pi} \ ln \left(\ 4\sqrt{\pi} \ (\frac{A}{4G})^{\frac{1}{2}} \ [\ 1 \ + \ \sqrt{1 - \frac{1}{16\pi}(\frac{A}{4G})^{-1}} \ ] \ \right) \tag{1.10}$$

after using the relations for the ordinary entropy in the Schwarzschild black hole

$$S = \frac{A}{4G} \ = \ 4\pi GM^2 \ \Rightarrow \ M \ = \ (\frac{A}{16\pi G^2})^{\frac{1}{2}} \tag{1.11}$$

and $(8\pi GM_o)^2 \ = \ G \ \Rightarrow \ 8\pi GM_o \ = \ \sqrt{G}$. The lower limit M_o of integration is required in eq-(1.8) to ensure the terms inside the square root are positive definite and the integral is real-valued.

Due to the relations

$$A_o \ = \ 4\pi(2GM_o)^2 \ = \ 16\pi G^2 M_o^2, \ \frac{A_o}{4G} \ = \ \frac{1}{16\pi}, \ M_o^2 \ = \ \frac{1}{64\pi^2 G} \Rightarrow M_o = \frac{M_P}{8\pi} \tag{1.12}$$

the *minimum* value of $M_o = (M_P/8\pi)$ corresponds to the *minimum* value of the horizon area A_o given by $\frac{A_o}{4G} = \frac{1}{16\pi} \Rightarrow A_o = \frac{G}{4\pi} = \frac{L_P^2}{4\pi}$. The constant $\tilde{S}_o$ is the corrected entropy associated with the minimum area (mass) and is found to be zero

$$\tilde{S}_o \ \equiv \ \tilde{S}(A = A_o) \ = \ ln[\ 4\sqrt{\pi} \ (\frac{A_o}{4G})^{\frac{1}{2}} \ ] \ = \ ln[1] = 0 \tag{1.13}$$

Because the corrected entropy corresponding to a *zero* area leads to an imaginary expression in the right hand side of eq-(1.10)

$$\tilde{S}(A = 0) \ - \ \tilde{S}_o \ = \ - \ \frac{ln(i)}{16\pi} \ = \ - \ \frac{ln(e^{i\pi/2})}{16\pi} \ = \ - \ \frac{i}{32} \tag{1.14}$$

this is the reason why one must have a *minimal* non-zero horizon area $A_o = \frac{L_P^2}{4\pi}$ and whose corresponding corrected entropy is $\tilde{S}_o = 0$.

Some of the relevant and novel physical consequences of Thermal Relativity can be found in [2], like

(i) The Thermal Relativistic dilation factor $\gamma_T = (1 - (\frac{T_H}{T_P})^2)^{-1/2}$ yields an infinite value when $T_H \rightarrow T_P$ so that $M \rightarrow M_o$ reaches its *minimum* nonzero value of $M_o = (M_P/8\pi)$. At that stage the Hawking evaporation *stops* leaving a black hole *remnant* of the order of the Planck mass.

(ii) There are modifications to the black hole emission rate. The thermal relativistic corrections to the emission rate are simply obtained by replacing T in the Stefan-Boltzman law for the effective $T_{eff} = T(1 - \frac{T^2}{T_P^2})^{-1/2}$. For large masses $M >> M_o$, the corrections are *negligible*. However this is *no* longer the case for small masses $M \sim M_o$ (Planck size mini-black holes), consequently the mini-black holes evaporate much $faster$ than before, and their lifetimes are much shorter.

(iii) One can envision a universe populated by mini-black holes whose effective temperature is $T_{eff} = \gamma_T T_H$. The idea that primordial black holes might be a hypothetical source of dark matter/energy is *not* new. What is *novel* now is the very large enhancement effects resulting from the very large thermal relativistic dilation factors γ_T, for very small masses, close to the minimal mass M_o. Since their thermal dilation factors γ_T are very large, their contribution to the effective energy/mass of the universe will be very large. As they evaporate by shedding off their mass down to the minimal mass M_o, their *enhanced* radiation due to the very large effective temperature $T_{eff} = \gamma_T T_H$, and much $faster$ evaporation times, will yield a very large contribution to the energy of the universe.

(iv) Given $\beta_{eff} = (\gamma(T)T)^{-1}$, the thermal relativistic corrections to the Bose-Einstein density distribution is $\tilde{\rho} = (e^{\beta_{eff}E} - 1)^{-1}$. As $T \rightarrow T_P, \gamma_T \rightarrow \infty, \beta_{eff} \rightarrow 0, \tilde{\rho} \rightarrow \infty$. The divergence of $\tilde{\rho}$ at the Planck temperature should signal a (spacetime) phase transition, like turning a smooth spacetime into a fractal one. The fact that $\tilde{\rho}$ *deviates* from a purely thermal distribution due to the thermal relativistic corrections could shed some light into the resolution of the black hole information paradox.

(v) Role in Quantum Gravity. The presence of a minimal area $A_o = L_P^2/4\pi$, a zero minimal proper entropy, a minimal mass $M_o = M_P/8\pi$, maximal temperature T_P, and a phase transition at Planck scales, should have profound consequences for Quantum Gravity.

(vi) Based on the gravity/gauge correspondence one could ponder if the presence of a minimal mass might be related to the mass gap in Yang-Mills.

Having reviewed briefly the key results of [2] we shall analyze next the Thermal Relativistic analog of the Lorentz transformations and their physical implications.

2 Thermal Relativistic Analog of Lorentz Transformations

It is straightforward to verify that the thermal relativistic analog of the Lorentz transformations that leave invariant the proper entropy infinitesimal interval

$$(d\mathbf{s})^2 \ = \ (T_P dS)^2 \ - \ (dM)^2 \ = \ (T_P dS')^2 \ - \ (dM')^2 \tag{2.1}$$

are given by

$$T_P \ S' \ = \ \frac{T_P \ S \ - \ \frac{T}{T_P} \ M}{\sqrt{1-(T/T_P)^2}}, \quad M' \ = \ \frac{M \ - \ T \ S}{\sqrt{1-(T/T_P)^2}} \tag{2.2}$$

The inverse transformations are obtained by replacing primed variables for unprimed ones and taking $T \to -T$

$$T_P \ S \ = \ \frac{T_P \ S' \ + \ \frac{T}{T_P} \ M'}{\sqrt{1-(T/T_P)^2}}, \quad M \ = \ \frac{M' \ + \ T \ S}{\sqrt{1-(T/T_P)^2}} \tag{2.3}$$

The Fulling-Davies-Unruh effect [7] states that a uniformly accelerating observer experiences the vacuum state of a quantum field in Minkowski spacetime as a mixed state in thermodynamic equilibrium. Such mixed state is comprised of a thermal bath (warm gas) of Rindler particles whose temperature is proportional to the proper acceleration. By invoking the expression of Unruh's temperature in terms of the proper acceleration $T = \frac{a}{2\pi}$ one finds that the maximal Planck temperature $T_P = m_P$ corresponds to a maximal proper acceleration $a_P = 2\pi m_P$. Therefore, one may rewrite the thermal analog of the Lorentz dilation factor in terms of the uniform linear proper acceleration a of an observer, and the maximal proper acceleration $a_{max} = a_P = 2\pi m_P$, as follows

$$\gamma_T \ = \ \frac{1}{\sqrt{1-\frac{T^2}{T_P^2}}} \ = \ \frac{1}{\sqrt{1-\frac{a^2}{a_{max}^2}}} \tag{2.4}$$

Consequently, if one postulates that the thermal relativistic effects result from the *acceleration* of an observer, from the transformations displayed in eqs-(2.2,2.3), one arrives at the conclusion that the macroscopic thermodynamic entropy S and the mass M are observer-dependent. The maximal Planck temperature T_P is a thermal relativistic invariant. The physics of maximal acceleration has had a long history, see [9], [6], [10] and references therein.

In recent years there has been considerable progress in study of the microscopic fine-grained von Neumann entropy, and the macroscopic thermodynamical entropy, pertaining black hole thermodynamics [26]. It has been found that the microscopic fine-grained von Neumann entropy is observer-dependent. In quantum gravity, it has been argued that a proper accounting of the role played by an observer promotes the von Neumann algebra of observables in a given spacetime subregion from Type III to Type II via the use of the crossed product [26], [27]. While this allows for a mathematically precise definition of its entropy, the authors [24] have shown that this procedure depends on which observer is employed. They found that the entropies seen by distinct observers can drastically differ. The work of [24] makes extensive use of the formalism of quantum reference frames (QRF). The "observers'" considered in [24] and in the previous works (clocks) [26] are nothing but QRFs. Furthermore, a correspondence between quantum error correcting codes and quantum reference frames was provided by [25]. To sum up, if gravitational entropy is observer-dependent one would expect that mass is also observer-dependent.

There is, however, a caveat which arises from the above transformations (2.2,2.3) . A careful inspection of eqs-(2.2,2.3) reveals that one is forced to introduce the notion of *negative* temperature, *negative* mass and *negative* entropy if one wishes to establish a resemblance with special relativity in flat $2D$ Minkowski spacetime where the spacetime coordinates (x,t) and the uniform velocities of the inertial observers can take positive or negative values since the existence of future and past light-cone regions require $-\infty \leq x \leq +\infty; -\infty \leq t \leq +\infty, -c \leq v \leq c$.

Far from being forbidden, negative temperatures are inevitable in systems with bounded energy spectra [14] A review of statistical mechanics of systems with *negative* temperature can be found in [15]. These authors pointed out that the great majority of models investigated by statistical mechanics over almost one century and a half exhibit positive absolute temperature, because their entropy is a nondecreasing function of energy. Since more than half a century ago it has been realized that this may not be the case for some physical systems as incompressible fluids, nuclear magnetic chains, lasers, cold atoms and optical waveguides. The authors [15] showed that negative absolute temperatures are consistent with equilibrium as well as with non-equilibrium thermodynamics.

A study of *negative* entropy and information in Quantum Mechanics can be found in [16]. In the 1944 book "What is Life?", Schrodinger [17], theorized that life, contrary to the general tendency dictated by the second law of thermodynamics, which states that the entropy of an isolated system tends to increase, decreases or keeps constant its entropy by feeding on negative entropy. Life does not in any way conflict with or invalidate the second law of thermodynamics, because the principle that entropy can only increase or remain constant applies only to a *closed* system which is adiabatically isolated, meaning no heat can enter or leave, and the physical and chemical processes which make life possible do not occur in adiabatic isolation, i.e. living systems are open systems [17].

Unlike in classical (Shannon) information theory, quantum (von Neumann) conditional entropies can be negative when considering quantum entangled systems, a fact related to quantum non-separability. The possibility that negative (virtual) information can be carried by entangled particles suggests a consistent interpretation of quantum informational processes. For the thermodynamic meaning of negative entropy see [18]. Black Hole thermodynamics and negative entropy in de Sitter and Anti-de Sitter Einstein-Gauss-Bonnet gravity was studied by [19] where they investigated the charged Schwarzschild-Anti-deSitter (SAdS) BH thermodynamics in 5d Einstein-Gauss-Bonnet gravity with electromagnetic field. The interesting feature of higher derivative gravity is the possibility for negative (or zero) SdS (or SAdS) BH entropy which depends on the parameters of higher derivative terms. The authors [19] concluded that the appearance of negative entropy may indicate a new type instability where a transition between SdS (SAdS) BH with negative entropy to SAdS (SdS) BH with positive entropy may occur.

Negative mass in general relativity was initially explored by Bondi [20]. Later on, the positive energy theorem in general relativity (also known as the positive mass theorem) was proved by [22]. Its standard form, broadly speaking, asserts that the gravitational energy of an isolated system is nonnegative, and can only be zero when the system has no gravitating objects [22]. Currently, the closest known real representative of such exotic matter (negative mass) is a region of negative pressure density produced by the Casimir effect [22]. Such matter would violate one or more energy conditions, one of which, the dominant energy condition, was required to prove the positive energy theorem.

A unifying theory of dark energy and dark matter involving *negative* masses and matter creation within a modified ΛCDM framework was analyzed by [23]. Contemporary cosmological results are derived upon the reasonable assumption that the Universe only contains positive masses. By reconsidering this assumption, Farnes [23] constructed a toy model which suggests that both dark phenomena can be unified into a single negative mass fluid. The model is a modified ΛCDM cosmology, and indicates that continuously-created negative masses can resemble the cosmological constant and can flatten the rotation curves of galaxies. The model leads to a cyclic universe with a time-variable Hubble parameter, potentially providing compatibility with the current tension that is emerging in cosmological measurements.

Having discussed how gravitational entropy (and mass) can be observer-dependent, and examined some examples of negative entropy, mass and temperature, we proceed now with the construction of the thermal relativistic analog of the two-vector (E, p) (energy, momentum). It requires the introduction of a proper thermal mass $\mathcal{M}$ such that

$$\Pi \equiv (\Pi_S, \Pi_M) \ = \ \mathcal{M} \ (T_P \frac{dS}{d\mathbf{s}}, \ \frac{dM}{d\mathbf{s}}) \ = \ \mathcal{M} \ \gamma_T \ (1, \frac{T}{T_P}) \ \Rightarrow$$

$$\Pi^2 \ = \ (\Pi_S)^2 \ - \ (\Pi_M)^2 \ = \ \mathcal{M}^2, \ \ \mathcal{M} \neq M \tag{2.5}$$

The second line of (2.5) is the analog of the relativistic relation $E^2 - p^2 = m_o^2$ where m_o is the invariant proper mass (rest mass) of a massive particle. The proper thermal mass $\mathcal{M}$ introduced in eq-(2.5) is invariant under the thermal relativistic transformations (2.2,2.3) and must *not* be confused with M (which is not invariant under the transformations (2.2,2.3)). Hence, by introducing the proper thermal mass $\mathcal{M}$ one has established the correspondence of $\mathcal{M}\gamma_T(1, \frac{T}{T_P})$ with $m_o(\frac{dt}{d\tau}, \frac{dx}{d\tau}) = (E, p) = m_o\gamma(1, v)$ $(c = 1)$. The analog of a massless particle (photon) is given by the null thermal momentum condition $\Pi^2 = (\Pi_S)^2 - (\Pi_M)^2 = \mathcal{M}^2 = 0$ associated with the maximal Planck temperature $T = T_P$.

In section **1** we showed that after renaming $\tilde{S} \equiv (\mathbf{s}/T_P)$, in terms of the *proper* entropy $\mathbf{s}$, the first law of black hole thermal-relativistic dynamics is $dM = \gamma(T_H) T_H d\tilde{S}$, and in the case of the $4D$ Schwarzschild black hole we found the expression for the thermal relativistic invariant entropy $\tilde{S}$ displayed in eq-(1.10). Since the maximal Planck temperature $T = T_P$ is a thermal relativistic invariant quantity, like the maximal speed of light is in special relativity, one may $define$ the proper thermal mass $\mathcal{M}$ introduced in eq-(2.5) in terms of the thermal relativistic invariants $\tilde{S}, T_P$ as $\mathcal{M} = T_P \tilde{S} \Rightarrow \mathcal{M} = \mathbf{s}$.

Therefore, the proper thermal mass can be identified with the proper entropy $\mathcal{M} = \mathbf{s}$. In special relativity one has the equivalence of mass and energy. In thermal relativity one has the equivalence of proper thermal mass $\mathcal{M}$ and proper entropy $\mathbf{s}$. In section **1** we found a *minimum* value for $M_o = (M_P/8\pi)$ which is associated with the *minimum* value of the horizon area $A_o = \frac{L_P^2}{4\pi}$, and such that the corrected entropy $\tilde{S}(M_o) = \tilde{S}_o = 0$ associated with the minimum area (mass) is zero. Therefore, when $\mathcal{S}_o = 0 \Rightarrow \mathcal{M}_o = 0$.

To conclude this section we should mention that "addition/subtraction" of temperatures in thermal relativity resembles the addition of velocities in special relativity

$$T'' \ = \ T \oplus T' \ = \ \frac{T \ + \ T'}{1 \ + \ \frac{T \ T'}{T_P^2}}, \ \ T \ominus T' \ = \ \frac{T \ - \ T'}{1 \ - \ \frac{T \ T'}{T_P^2}} \tag{2.6}$$

such that the Planck temperature remains invariant, $T_P \oplus T_P = T_P$; $T_P \ominus T_P = T_P$. One may recast the temperature addition laws (2.6) in terms of the addition laws of *proper* acceleration after recurring to the Unruh relation $T = \frac{a}{2\pi}$ as follows

$$a'' \ = \ a \oplus a' \ = \ \frac{a \ + \ a'}{1 \ + \ \frac{a \ a'}{a_{max}^2}}, \ \ a \ominus a' \ = \ \frac{a \ - \ a'}{1 \ - \ \frac{a \ a'}{a_{max}^2}}, \ \ a_{max} = 2\pi T_P = 2\pi m_P \tag{2.6}$$

3 Concluding Remarks : Extending Thermal Relativity to Cotangent Spaces

Most of the work devoted to Quantum Gravity has been focused on the geometry of spacetime rather than phase space per se. The first indication that phase space should play a role in Quantum Gravity was raised by [8]. The principle behind Born's reciprocal relativity theory [11], [12] was based on the idea proposed long ago by [8] that coordinates and momenta should be unified on the same footing. Consequently, if there is a limiting speed (temporal derivative of the position coordinates) in Nature there should be a maximal force as well, since force is the temporal derivative of the momentum. The principle of maximal acceleration was advocated earlier on by [9]. A *maximal* speed limit (speed of light) must be accompanied with a *maximal* proper force (which is also compatible with a *maximal* and *minimal* length duality) [12].

The generalized velocity and force (acceleration) boosts (rotations) transformations of the $flat$ $8D$ Phase space coordinates , where $X^i, T, E, P^i; i = 1,2,3$ are **c** -valued (classical) variables which are *all* boosted (rotated) into each-other, were given by [11] based on the group $U(1,3)$ and which is the Born version of the Lorentz group $SO(1,3)$. The $U(1,3) = SU(1,3) \times U(1)$ group transformations leave invariant the symplectic 2-form $\Omega = -\, dT \wedge dE + \delta_{ij} dX^i \wedge dP^j; i,j = 1,2,3$ and also the following Born-Green line interval in the $flat$ $8D$ phase-space

$$(d\omega)^2 \;=\; c^2 (dT)^2 \;-\; (dX)^2 \;-\; (dY)^2 \;-\; (dZ)^2 \;+$$

$$\frac{1}{b^2}\, \left((dE)^2 \;-\; c^2 (dP_x)^2 \;-\; c^2 (dP_y)^2 \;-\; c^2 (dP_z)^2 \right) \tag{3.1}$$

The maximal proper force is set to be given by b. The symplectic group is relevant because $U(1,3) = Sp(8,R) \cap O(2,6)$; $U(3,1) = Sp(8,R) \cap O(6,2)$, and $U(2,2) = Sp(8,R) \cap O(4,4)$.

These transformations can be $simplified$ drastically when the velocity and force (acceleration) boosts are both parallel to the x-direction and leave the transverse directions Y, Z, P_y, P_z intact. There is now a subgroup $U(1,1) = SU(1,1) \times U(1) \subset U(1,3)$ which leaves invariant the following line interval

$$(d\omega)^2 \;=\; c^2 (dT)^2 \;-\; (dX)^2 \;+\; \frac{(dE)^2 \;-\; c^2 (dP)^2}{b^2} \;=$$

$$(d\tau)^2 \;\left(\;1 \;+\; \frac{(dE/d\tau)^2 \;-\; c^2 (dP/d\tau)^2}{b^2} \;\right) = (d\tau)^2 \;\left(\;1 \;-\; \frac{F^2}{F^2_{max}} \;\right), \quad P = P_x \tag{3.2}$$

where one has factored out the proper time infinitesimal $(d\tau)^2 = c^2 dT^2 - dX^2$ in (1.2). The proper force interval $(dE/d\tau)^2 - c^2 (dP/d\tau)^2 = -F^2 < 0$ is "spacelike" when the proper velocity interval $c^2 (dT/d\tau)^2 - (dX/d\tau)^2 > 0$ is timelike. The analog of the Lorentz relativistic factor in eq-(1.14) involves the ratios of two proper *forces*.

Most recently [13], a continuation of the Born Reciprocal Relativity Theory (BRRT) program in phase space showed that a natural temperature-dependence of mass occurs after recurring to the Fulling-Davies-Unruh effect. The temperature dependence of the mass $m(T)$ resembled the energy-scale dependence of mass and other physical parameters in the renormalization (group) program of QFT. It was found in a special case that the effective photon mass is no longer zero, which may have far reaching consequences in the resolution of the dark matter problem. The Fulling-Davies-Unruh effect in a $D = 1+1$-dim spacetime was analyzed entirely from the perspective of BRRT, and we explained how it may be interpreted in terms of a linear superposition of an infinite number of states resulting from the action of the group $U(1,1)$ on the Lorentz non-invariant vacuum $|\tilde{0}\rangle$ of the relativistic oscillator studied by Bars [28].

The formulation of thermal relativity was based on the analog of an infinitesimal interval in flat Minkowski spacetime. The extension to the analog of infinitesimal intervals in curved spacetimes involving metrics would resemble the geometro-thermodynamical approaches of [3], [4], [5]. Furthermore, it is desirable to formulate the thermal relativity analog of the geometry of cotangent spaces (phase spaces) and, in turn, to implement the principles of Born Reciprocal Relativity theory. This would require establishing the "dictionary"

$$(d\mathbf{s})^2 \left(1 - \frac{\frac{d\Pi_\mu}{d\mathbf{s}} \frac{d\Pi_\mu}{d\mathbf{s}}}{\mathcal{F}^2_{max}} \right) \leftrightarrow (d\tau)^2 \left(1 - \frac{\frac{dP_\mu}{d\tau} \frac{dP^\mu}{d\tau}}{F^2_{max}} \right), \quad \mu = 1,2 \tag{3.3}$$

between the infinitesimal intervals in the thermal cotangent space and the intervals in the ordinary cotangent space associated with a $2D$-dim spacetime. The components of the "thermodynamic proper force" are

$$\frac{d\Pi_\mu}{d\mathbf{s}} = \left(\frac{d\Pi_S}{d\mathbf{s}}, \frac{d\Pi_M}{d\mathbf{s}}\right) \tag{3.4}$$

and whose magnitude-squared is

$$\mathcal{F}^2 = \frac{d\Pi_\mu}{d\mathbf{s}} \frac{d\Pi^\mu}{d\mathbf{s}} = \left(\frac{d\Pi_S}{d\mathbf{s}}\right)^2 - \left(\frac{d\Pi_M}{d\mathbf{s}}\right)^2 \tag{3.5}$$

where Π_S, Π_M were defined in eq-(2.5). $\mathcal{F}_{max}$ is the maximal thermodynamic proper force analog of the maximal proper force $F_{max} = m_P a_{max} = 2\pi m_P^2$ [13]. The "dictionary" between the thermal cotangent space and the ordinary cotangent space allows us to write down the symmetry transformations which leave invariant the intervals in the left-hand side of eq-(3.3). This will be the subject of future investigation.

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References

[1] L. Zhao Communications in Theoretical Physics, Vo **56**, Issue 6, (2011) 1052-1056.

[2] C. Castro, "Thermal Relativity, Corrections to Black-Hole Entropy, Born's Reciprocal Relativity Theory and Quantum Gravity", Canadian Journal of Physics **97** (12) (2019).

C. Castro, "On Thermal Relativity, Modified Hawking Radiation, and the Generalized Uncertainty Principle " International Journal of Geometric Methods in Modern Physics, Volume **16**, Issue 10, id. 1950156-3590 (2019)

[3] F. Weinhold, "Metric Geometry of Equilibrium Thermodynamics" I, II, III, IV, V, J. Chem. Phys. **63** (1975) 2479; 2484; 2488; 2496; 65 (1976) 558.

[4] G. Ruppeiner, Phys. Rev. **A 20** (1979) 1608.

G. Ruppeiner, Rev. Mod. Phys. **67** (1995) 605; Erratum **68** (1996) 313.

[5] H. Quevedo, J. Math. Phys. **48** (2007) 013506

H. Quevedo, A. Sanchez, and A.Vazquez, "Invariant Geometry of the Ideal Gas" , arXiv: math-ph/0811.0222.

[6] E. Benedetto and A. Feoli, "Unruh temperature with maximal acceleration" Mod. Phys. Lett **A 30** (2015) 1550075

[7] L. C.B Crispino, A. Higuchi and G.E.A. Matsas, "The Unruh effect and its applications", arXiv : 0710.5373.

[8] M. Born, "A suggestion for unifying quantum theory and relativity", *Proc. Royal Society* **A 165** (1938) 291.

M. Born, "Reciprocity Theory of Elementary Particles", *Rev. Mod. Physics* **21** (1949) 463.

[9] E.R. Cainiello, "Is there a Maximal Acceleration", Il. Nuovo Cim., **32** (1981) 65.

H. Brandt, "Finslerian Fields in the Spacetime Tangent Bundle" *Chaos, Solitons and Fractals* **10** (2-3) (1999) 267.

H. Brandt, "Maximal proper acceleration relative to the vacuum" Lett. Nuovo Cimento **38** (1983) 522.

M. Toller, "The geometry of maximal acceleration" Int. J. Theor. Phys **29** (1990) 963.

[10] R. Gallego Torrome, "On the effect of the maximal proper acceleration in the inertia" Proc. R. Soc. **A 480** : (2024) 20230876.

R. Gallego Torrome, "On the Unruh effect for Hyperbolic Observers in Spacetimes with Maximal Acceleration", arXiv : 2410.18155.

[11] S. Low, " $U(3,1)$ Transformations with Invariant Symplectic and Orthogonal Metrics" , *Il Nuovo Cimento* **B 108** (1993) 841.

S. Low, "Representations of the canonical group, (the semi-direct product of the unitary and Weyl-Heisenberg groups), acting as a dynamical group on noncommutative extended phase space", J. Phys. **A** : Math. Gen., **35** (2002) 5711.

[12] C. Castro, "Is Dark Matter and Black-Hole Cosmology an Effect of Born's Reciprocal Relativity Theory ?" Canadian Journal of Physics. Published on the web 29 May 2018, https://doi.org/10.1139/cjp-2018-0097.

C. Castro, "Some consequences of Born's Reciprocal Relativity in Phase Spaces" *Foundations of Physics* **35**, no.6 (2005) 971.

C. Castro, "On Born's Deformed Reciprocal Complex Gravitational Theory and Noncommutative Gravity " Phys Letts **B 668** (2008) 442.

[13] C. Castro, "On Born Reciprocal Relativity Theory, the Relativistic Oscillator and the Fulling-Davies-Unruh effect", DOI: 10.13140/RG.2.2.12368.11527

[14] Daan Frenkel, Patrick B Warren, "Gibbs, Boltzmann, and negative temperatures", Am. J. Phys. **83**, 163 (2015).

[15] Marco Baldovin, Stefano Iubini, Roberto Livi, Angelo Vulpiani, "Statistical Mechanics of Systems with Negative Temperature". Physics Reports **vol 923** (2021) 1-50.

[16] N. J. Cerf, C. Adami, "Negative entropy and information in quantum mechanics" Phys.Rev.Lett. 79 (1997) 5194

[17] Wikipedia, "Entropy and Life", https://en.wikipedia.org/wiki/Entropy_and_life

[18] L. del Rio, J. Aberg, R. Renner, O. Dahlsten, and V. Vedral, "The thermodynamic meaning of negative entropy" Nature **474**, 61-63 (2011)

[19] M. Cvetic, S. Nojiri, S.D. Odintsov, "Black Hole Thermodynamics and Negative Entropy in de Sitter and Anti-de Sitter Einstein-Gauss-Bonnet gravity", Nucl. Phys. **B 628** (2002) 295-330.

[20] H. Bondi, "Negative mass in General Relativity" Rev. Mod. Phys. **29** (1957) 423.

[21] "Positive Energy Theorem", https://en.wikipedia.org/wiki/Positive_energy_theorem

R. Schoen and S-T Yau, Shing-Tung , "On the proof of the positive mass conjecture in general relativity". Communications in Mathematical Physics. **65** (1) (1979) 45–76.

R. Schoen and S-T. Yau, Shing-Tung, "Proof of the positive mass theorem. II". Communications in Mathematical Physics. 79 (2) (1981) 231–260.

E. Witten, "A new proof of the positive energy theorem". Communications in Mathematical Physics **80** (3) (1981) 381–402.

[22] "Negative mass" https://en.wikipedia.org/wiki/Negative_mass

[23] J.S Farnes, "A unifying theory of dark energy and dark matter: Negative masses and matter creation within a modified ΛCDM framework", Astronomy and Astrophysics **620** (2018) A92.

[24] J. De Vuyst, S. Eccles, P. A. Hoehn, and J. Kirklin, "Crossed products and quantum reference frames: on the observer-dependence of gravitational entropy" arXiv : 2412.15502.

J. De Vuyst, S. Eccles, P. A. Hoehn, and J. Kirklin,"Gravitational entropy is observer-dependent", arXiv : 2405.00114.

[25] S. Carrozza, A. Chatwin-Davies, P. A. Hoehn, F. M. Mele, "A correspondence between quantum error correcting codes and quantum reference frames" arXiv : 2412.15317

[26] E. Witten, "Introduction to Black Hole Thermodynamics", arXiv : 2412.16795.

V. Chandrasekaran, R. Longo, G. Penington, and E. Witten, "An Algebra of Observables for de Sitter Space", JHEP **2023** 82 (2023).

[27] M. Cirafici, "Gravitational Algebras and Non-equilibrium Physics", arXiv : 2412.17674.

[28] I. Bars, "Relativistic Harmonic Oscillator Revisited" arXiv : 0810.2075.